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Generalized J-Rings and Commutativity

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Abstract

A J-ring is a ring R with the property that for every x in R there exists an integer n(x)>1 such that $x^{n(x)} = x$, and a well-known theorem of Jacobson states that a J-ring is necessarily commutative. With this as motivation, we define a generalized J-ring to be a ring R with the property that for all x, y in R_0 there exists integers $n = n(x) > 1, m = m(y) > 1$ such that $x^n y - xy^m$ is nilpotent, where R_0 is a certain subset of R. The commutativity behavior of such rings is considered.

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Throughout, R is a ring, N is the set of nilpotents, C is the center, J is the Jacobson radical of R, and Z denotes the ring of integers. As usual [x,y] will denote the commutator xy-yx.

Definition 1. A ring R is called a *generalized* J-*ring* if
(1) For all x, y, in $R \setminus (N \cup J \cup C)$, there exist integers $n > 1, m > 1$ such that $x^n y - xy^m \in N$.

The class of generalized J-rings is quite large and includes all commutative rings, all nil rings, all rings in which J=R, and all J-rings. On the other hand, a generalized J-ring need not be commutative, as can be seen by taking

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} : 0,1 \in GF(2) \right\}; m = n = 2.$$

In Theorem 5, we give a characterization of *commutative* generalized J-rings. We now introduce some basic definitions.

A ring is called *periodic* if for every x in R, $x^m = x^n$ for some distinct positive integers m and n. The ring R is called *weakly periodic* if every x in R can be written as a sum of a nilpotent element a and a "potent" element b in the sense that $b^k = b$ with k>1. R is called *weakly periodic-like* if (here C denotes the center of R)

(2) For all $x \in R \setminus C$, $x = a + b$, a nilpotent, b potent ($b^k = b$, k>1).

We are now in a position to prove our main theorems.

Theorem 1 *Suppose* R *is a generalized J-ring with identity and with central idempotents. Then the set N of nilpotents is contained in the Jacobson radical* J *of* R.

Proof. Let $a \in N, x \in R$. We claim that

(3) ax is right quasi-regular (r.q.r.).

The proof is by contradiction. Suppose $a \in N, x \in R$, ax is not r.q.r. If $ax \in J$, then ax is r.q.r., contradiction. Thus, $ax \notin J$. If $ax \in N$, then ax is r.q.r., and hence again $ax \notin N$. Now, if $ax \in C$, then $(ax)^q = a^q x^q$ for all positive integers q, which implies $ax \in N$ (since $a \in N$) and hence ax is r.q.r., contradiction. So $ax \notin C$, and hence

(4) $ax \notin (N \cup J \cup C)$.

Next, consider $1+ax$. If $1+ax \in C$, then $ax \in C$ and hence again ax is r.q.r., contradiction. So $1+ax \notin C$. Now suppose $1+ax = a_0 \in N$. Then $ax = a_0 - 1 = u$, where u is a *unit* in R. Let k_0 be the exponent of nilpotency of a. Clearly $k_0 > 1$, and hence $a^{k_0} x = a^{k_0 - 1}(ax) = a^{k_0 - 1} u$, which implies $a^{k_0 - 1} u = 0$, and hence $a^{k_0 - 1} = 0$ (since u is a unit), contradiction. Therefore, $1+ax \notin N$. Finally, if $1+ax \in J$, then $1+ax$ is r.q.r., and hence for some $b \in R$, we have $(1+ax)+b-(1+ax)b = 0$. Again, let k_0 be the exponent of nilpotency of a. Since $a \neq 0$, $k_0 > 1$, and hence the above equation implies that

$0 = a^{k_0 - 1}[(1+ax)+b-(1+ax)b] = a^{k_0 - 1}$ (since $a^{k_0} = 0$), contradiction. So $1+ax \notin J$. The net result is:

(5) $1+ax \notin (N \cup J \cup C)$.

Combining (4), (5), (1), we see that

$$(1+ax)^n(ax)-(1+ax)(ax)^m \in N, \text{ for some } n>1, m>1, \text{ which implies that}$$

(6) $(ax)^q = (ax)^{q+1}h(ax)$ for some $h(\lambda) \in Z[\lambda]$.

Let $e = [(ax)h(ax)]^q$. Then

(7) $(ax)^q = (ax)^q e$, $e = [(ax)h(ax)]^q$, $e^2 = e$.
Thus,

$$e = ee = e[(ax)h(ax)]^q = eat = aet \quad (\text{since } e \in C).$$

So $e = aet = a^2et^2 = ... = a^k et^k$ for all $k \geq 1$, and hence $e = 0$ (since $a \in N$). Therefore, by (7), $ax \in N$, and thus ax is r.q.r., contradiction. This contradiction proves (3), and hence ax is r.q.r. for all $a \in N, x \in R$. Thus, $N \subseteq J$.

Theorem 2 *Suppose* R *is a generalized* J-*ring with identity and with central idempotents. Then, we have*

(i) R/J *is commutative.*
(ii) *If, further,* J *is commutative, then the commutator ideal of* R *is nil.*

Proof (i) By Theorem 1, $N \subseteq J$, and hence by (1)

(8) For all $x, y \in R \setminus (J \cup C)$, $x^n y - xy^m \in J$ for some integers $n>1, m>1$.

This reflects in R/J as follows:

(9) For all *noncentral* elements x, y *of* R/J, $x^n y - xy^m = 0$, $n>1, m>1$.

Let x be any *noncentral* element of R/J. Then, by (9),

(10) $x^n(1+x) - x(1+x)^m = 0$, $n>1, m>1$, (x any *noncentral* element of R/J).

Therefore, by (10), $x - x^2 f(x) \in C$ for some $f(\lambda) \in Z[\lambda]$, where x is *any* element of R/J, which implies by a theorem of Herstein [3] that R/J is commutative.

(ii) By part (i), $[x, y] \in J$ for all x, y *in* R. Since, by hypothesis, J is commutative, we have

(11) $[[x, y],[z, w]] = 0$ for all x, y, z, w *in* R.

Note that (11) is a polynomial identity which is satisfied by all elements of R. However, (11) is not satisfied by any 2×2 complete matrix ring over GF(p) for any prime p. (To see this, take $[x, y] = [E_{11}, E_{12}]$, $[z, w] = [E_{22}, E_{21}]$.) Hence, by a theorem of Bell [2], the commutator ideal of R is nil.

Corollary 1. *Suppose* R *is a reduced ring* (N={0}) *and suppose* R *is a generalized* J-*ring with identity. Suppose, further, that* J *is commutative. Then* R *is commutative.*

Proof. Let $e^2 = e \in R,\ x \in R$. Then $(ex - exe)^2 = 0 = (xe - exe)^2$, and hence all idempotents are central (since N={0}). Hence, by Theorem 2(ii), the commutator ideal of R is nil, which implies that R is commutative (since N=[0}).

Theorem 3. *Suppose* R *is a generalized* J-*ring with identity and with central idempotents. Suppose, further, that* $J \subseteq N \cup C$. *Then the commutator ideal of* R *is nil.*

Proof. By hypothesis,

(12) $J \subseteq N \cup C$.

We claim that

(13) $J \subseteq N \text{ or } J \subseteq C$.

Suppose not. Then J is not a subset of N and J is not a subset of C. Let $x \in J, x \notin N$, and let $y \in J, y \notin C$. By (12), $x \in C \text{ and } y \in N$. Let $x + y = u$, and hence $x = u - y$. Since $x \in C, u - y$ commutes with y, and hence u commutes with y. If $u \in N$, then u and y are commuting nilpotents, and hence $u - y \in N$, which implies that $x \in N$, contradiction. On the other hand, if $u \in C$, then $x + y \in C$ and $x \in C$, which implies that $y \in C$, contradiction. Therefore, $u \notin N \text{ and } u \notin C$; yet $u \in J \subseteq N \cup C$ (by (12)). This is a contradiction, and (13) is proved. Recall that, by Theorem 1, $N \subseteq J$, which when combined with (13) yields

(14) $N = J \text{ or } N \subseteq J \subseteq C.$

If $N = J$, then N is an ideal and $R / N = R / J$ is indeed commutative, by Theorem 2(i), which implies that the commutator ideal of R is nil, and the theorem is proved in this case. Next, consider the case $N \subseteq J \subseteq C.$ Then (1) now implies that

(15) For all $x, y \in R \setminus C,\ x^n y - xy^m \in N$ for some $n > 1, m > 1$.

Suppose $x \in R,\ x \notin C$. Then $1 + x \notin C$, and hence by (15), $x^n (1 + x) - x(1 + x)^m \in N \subseteq C$ (since $N \subseteq J \subseteq C$ is the present case). Therefore,

$x - x^2 f(x) \in C$ for some $f(\lambda) \in Z[\lambda]$, where x is *any* element of R. It follows, by a Theorem of Herstein [3], that R is commutative in the present case, and the theorem readily follows.

Theorem 4. *Suppose* R *is a weakly periodic-like ring and suppose* R *is a generalized* J-*ring with identity and with central idempotents. Then, we have*

(i) *The commutator ideal of* R *is nil.*

(ii) *For any* $x \in R \setminus C$, $x - x^m \in N$ *for some integer m>1.*

Proof. (i) In view of Theorem 3, it suffices to show that

(16) $J \subseteq N \cup C$.

Suppose $j \in J, j \notin C$. Then, since R is weakly periodic-like, $j = a + b,\ a \in N,\ b^m = b$ with m>1. Hence,

$$j - a = b = b^m = (j-a)^m, \text{ and thus } j - a = (j-a)^{m^q} \text{ for all } q \geq 1.$$

Since $a \in N, a^{m^q} = 0$ for some $q \geq 1$, and hence the above equation implies that $j - a \in J$. Therefore, $b \in J$, and thus b^{m-1} is an idempotent element of J, which implies $b^{m-1} = 0$, and hence $b = b^m = 0$. Thus, $j = a + b = a + 0 \in N$, which proves (16), and part (i) follows (see Theorem 3).

(ii) Let $x \in R,\ x \notin C$. Then, since R is weakly periodic-like, $x = a + b,\ a \in N,\ b^m = b$ *with* $m > 1$. Therefore,

(17) $x - a = b = b^m = (x-a)^m,\ m > 1,\ (a \in N).$

By part (i), N is an ideal, and hence by (17), $x - x^m \in N,\ m > 1$. This proves the theorem.

We are now in a position to prove our main theorem, which gives a characterization of *commutative* generalized J-rings.

Theorem 5. *Suppose* R *is a generalized* J-*ring and suppose* R *is weakly periodic-like. Suppose that* $(N \cap J)$ *is commutative and, furthermore, suppose that every element which squares to zero is central (*$a^2 = 0$ *implies* $a \in C$*). Then* R *is commutative (and conversely).*

Proof. To begin with, if $e^2 = e \in R$ and $x \in R$, then $(ex - exe)^2 = 0 = (xe - exe)^2$. Therefore, by hypothesis, $ex - exe \in C$ and $xe - exe \in C$, and hence all idempotents are central. We now distinguish two cases.

Case 1. $1 \in R$. In this case, since all idempotents are central, it follows by Theorem 1 that $N \subseteq J$, and hence $N = N \cap J$. Since, by hypothesis, $N \cap J$ is commutative,

(18) N is commutative.

Moreover, by Theorem 4 (i), since the idempotents are central, the commutator ideal of R is nil and hence N is an ideal. Combining this with (18), we conclude that N is a commutative ideal of R. This fact implies that $(ax - xa)^2 = 0$ for all $a \in N,\ x \in R$, and hence by hypothesis, $ax - xa$ is central. Thus,

(19) $ax - xa$ is central for all $a \in N,\ x \in R$.

Also, by Theorem 4(ii), we have

(20) For every $x \in R \setminus C,\ x - x^m \in N$ for some integer $m > 1$.

It was proved by the authors that any ring which satisfies (18), (19), (20) is commutative [1], and hence the ground ring R is commutative (if $1 \in R$).

We now consider the general case (where we no longer assume that R has an identity). Let P be the set of potent elements of R; that is,

(21) $P = \{x : x \in R,\ x^k = x \ \textit{for some}\, k > 1\}$.

We now distinguish two cases.

Case A: $P = \{0\}$. In this case, since R is weakly periodic-like, we see that $R = N \cup C$. The argument used in the proof of Theorem 3 (namely, (12) implies (13)) shows that $R = N \ \textit{or}\ R = C$. If $R = N$, then N is an ideal of R and hence $N \subseteq J$. So $N = N \cap J$ is commutative, which implies that $R = N \cup C$ is commutative.

Case B: $P \neq \{0\}$. In this case, we claim that

(22) All potent elements of R are central.

To prove this, let $b \in P, b \neq 0$, and suppose $b^k = b,\ k > 1$. Let $e = b^{k-1}$. Then e is a nonzero central idempotent element of R, and hence eR is a ring with identity. It is readily verified that eR is a ring which satisfies all the hypotheses imposed on the ground ring R (keep in mind that the Jacobson radical of eR is eJ(R)). Since eR also has an identity, it follows by *Case* 1 that

(23) eR is a commutative ring.

Let $y \in R$. Then, $e[b, y] = [eb, ey] = 0$. Recalling that $e = b^{k-1} \in C \,\textit{and}\, b^k = b$, we see that

$0 = e[b,y] = b^{k-1}[b,y] = b^k y - b^{k-1} yb = b^k y - yb^k = by - yb$, and hence $by = yb$ for all $y \in R$, which proves (22).

Our next goal is to prove that

(24) $N \subseteq J$.

(Incidentally, it should be pointed out that Theorem 1 cannot be applied here, since R is not assumed to have an identity.) To prove (24), let $a \in N, x \in R$. If $ax \in C$, then $(ax)^m = a^m x^m$ for all m, and hence $ax \in N$ (since $a \in N$), which implies that ax is r.q.r. (if $ax \in C$). Next, suppose $ax \notin C$. Then, since R is weakly periodic-like,

(25) $ax = a_0 + b_0$, $a_0 \in N$, b_0 potent ($b_0^{q_0} = b_0$, $q_0 > 1$).

In view of (22), b_0 is central and hence $[ax, a_0] = 0$. Combining this with (25), we see that

(26) $ax - a_0 = b_0 = b_0^{q_0} = (ax - a_0)^{q_0}$, $[ax, a_0] = 0$, $a_0 \in N, q_0 > 1$.

A close look at (26) shows that $ax - (ax)^{q_0}$ is a sum of pairwise commuting nilpotent elements, and hence such a sum is indeed nilpotent, which implies that

(27) $ax - (ax)^{q_0} \in N$.

In view of (27), we see that

$(ax)^q = (ax)^{q+1} h(ax)$ for some $h(\lambda) \in Z[\lambda]$.

The argument used at the end of the proof of Theorem 1 (beginning with (6)) shows that ax is r.q.r. in the present case. The net result is that ax is r.q.r. for all $a \in N$, $x \in R$, and hence $N \subseteq J$, which proves (24). Recall that, by hypothesis, ($N \cap J$) is commutative. Also by (24), $N = N \cap J$, and hence

(28) N is commutative.

To complete the proof, suppose $x \notin C, y \notin C$. Then, $x = a + b$, $a \in N, b \in P$ and $y = a' + b'$, $a' \in N$, $b' \in P$. Hence by (22) and (28), we have

$[x, y] = [a + b, a' + b'] = [a, a'] = 0$.

Thus, R is commutative, and the theorem is proved.

Jacobson’s Theorem, namely that a J-ring is commutative [4, p.217], is a corollary of Theorem 5. Another corollary of Theorem 5 is the special case in which the exponents n and m in Definition 1 are always chosen to be equal (see [5]).

We conclude with the following remark.

Remark: The hypothesis "$a^2 = 0$ *implies* $a \in C$" in Theorem 5 cannot be replaced by the weaker hypothesis "the idempotents are central". This can be seen by considering the following example [6]:

Example 1 : Let

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{bmatrix} : a, b, c \in GF(4) \right\}.$$

Keeping an eye on the diagonal entries, it can be shown that every element of R is nilpotent or invertible, and, moreover $x^8 = x^2$ for all x in R. Furthermore, the set N of nilpotents is an ideal and, in fact, $N^2 = (0)$. Also,

(i) For all $x \in R, x = (x - x^7) + x^7$ shows that every element of R is a sum of a nilpotent and a potent element.

(ii) For all $x, y \in R,\ x^7 y - xy^7 \in N$.

(iii) N is commutative

(iv) The idempotents of R are precisely $\{0,1\}$.

(v) However, "$a^2 = 0$ *implies* $a \in C$" is false.

Thus, all the hypotheses of Theorem 5 *with the exception stated in (v)* are satisfied. But R is not commutative, as can be seen by considering the elements of R, namely, E_{12} and $diag[u, u^2, u]$, where u is a generator of the multiplicative group of units of GF(4).

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