# AREA REQUIREMENT IN ANALYSIS

This requirement consists of an examination based on undergraduate material such as that found in the UCSB courses 118ABC and 122AB, and a one of the following graduate course groups.
Mat 201 ABC
Math 202 ABC
Math 201 AB and Math 202 AB
Students should consult their advisor to decide which of these three options would be best.

## Analysis exam: The exam will have two parts, a section on real analysis and a section on complex analysis. Each section will typically contain 5 to 7 problems from which the student will be asked to choose 3 or 4 to solve. The Graduate Program Assistant has copies of old exams available upon request. The faculty members reading the exams will want to see carefully written proofs that supply enough details to show mastery of the problem. The emphasis in both sections of the exam will be on the fundamental aspects of analysis rather that the esoteric.

Topics that are especially important will be in bold type.

## Real analysis section of the exam

The real line is the fundamental object, but the emphasis is on functions with real values. The notion of **limit** is what distinguishes analysis from other mathematical subjects, so every exam will have limits all over the place, sometimes in the form of **derivatives** or **integrals**. For this exam only the Riemann integral will be tested. Lebesgue integrals and their methods will not appear (nor will students be able to use them in their solutions.) Nonetheless, it is important to understand how the Riemann integral is a limit, even though it is not presented in the format of a metric space. Often there are two limit operations and the problems is about the relationship between two limit processes. For example, when can you pass a limit under an integral sign?

Derivatives will appear in their basic one-dimensional form, i.e. **derivatives of real-valued functions defined on intervals of the real line.** They will also appear in the study of real-valued functions of several variables. For this application it is essential that the student know some linear algebra because the derivative at a point will be a linear operator, rather than just a scalar. Mastery in this area requires the knowledge of some basic linear algebra.

The domain of these functions will vary with the circumstances, but will always be a metric space of some sort. For this reason, **metric spaces** will be emphasized on exams. Metric spaces were invented to bring together many notions of **convergence**. The sequences might be sequences of real numbers, or they might be convergence of sequences of functions (more general things are possible in metric spaces, but will not be appearing on this exam). In the latter case, the notion of **uniform convergence** is basic and regularly appears on exams (sometimes in the complex part as well). **Continuity**, including **uniform continuity**, is the condition that is most commonly studied for real-valued functions defined on metric spaces. Metric spaces may have two properties that are commonly tested, **compactness** and **connectedness**. Both of these properties also appear on the topology exam, so they are not as heavily tested on this exam as might otherwise be the case.

All of the topics on the exam are covered in Walter Rudin's *Principles of Mathematical Analysis*. The book contains more that is required. The exam excludes 7.19-7.33, 8.15-8.22, and all of Chapters 10 and 11. From Chapter 6 the exam will cover only ordinary Riemann integration, not the more general Riemann-Stieltjes integral. Many other analysis books contain most or all of the material.

## Important items for the real analysis section

I. The Real Line
- A. Completeness and least upper bound property
- B. **Compactness of closed intervals**
- C. Connectedness of intervals
- D. Cantor set as an example

II. Functions from intervals on the real line to the real line.
- A. Derivatives
  1. **Mean value theorem**
  2. Interpretation of derivatives for max, min, increasing, etc.
  3. Higher derivatives, Taylor's Theorem
  4. L'Hospital's Rule
- B. Riemann Integral
  1. Definition
  2. **Fundamental theorem of calculus**
  3. **Limits of integrals versus integrals of limits**
- C. Sequences and series of functions.
  1. **Uniform convergence**
  2. **Reversal of limits**
  3. **Exponential and trigonometric functions**
  4. Power series (more on this in complex than in real)

III. Functions from $R^n$ to $R^n$
- A. **Derivatives of functions of several variables**
  1. Partial Derivatives
  2. Jacobian
  3. Implicit function theorem
  4. Inverse function theorem
- B. Multiple integrals
  1. Definition and independence of order
  2. Differentiating under integral signs

IV. Metric spaces

A. **Completeness and Cauchy sequences**
B. **Various equivalent conditions for compactness**
C. **Convergence of functions, including uniform convergence**
D. **Reversal of limit processes**
E. Contraction mapping theorem

# Complex analysis section of the exam

As it is taught at UCSB and in many other schools, complex analysis does not have real analysis as a prerequisite. This restricts the amount that an instructor can emphasize limits per se in this course. This restriction is also reflected in the complex analysis part of the exam. This means that the problems tend to emphasize what the theorems say rather than how to prove the theorems in the subject, as is more the case in the real part where epsilon and delta are rarely far away. Despite this, there will occasionally be a question in this area that combines the real and the complex analysis areas. Harmonic functions are real functions of two variables, but are typically covered in the complex section of the exam because of their role and the real and imaginary parts of analytic functions.

The fundamental object of complex analysis is the analytic =holomorphic=differentiable) function defined in an open subset of the complex plane. The beauty of the subject appears in the many nice properties that these functions share such as being infinitely differentiable. As with the real part, we do not apologize if a little bit of linear algebra slips into this part as well. It's everywhere, so get used to it.

All the material covered in the exam is contained in Brown & Churchill, *Complex Variables and Applications*, 6th ed., sections 1-80. (A cheaper book, Murray Spiegel's, Schaum's Outline book, *Complex Variables*, contains these topics as well, and has many solved problems.)

## Important items for the complex analysis section

I. What are analytic functions?

A. Limit of difference quotients
B. **Cauchy-Riemann equations**
C. **Taylor series (and infinite differentiability)**
D. **Laurent series**
E. Integral definitions (e.g. Morera's Theorem)
F. Definition of harmonic functions and relationship to analytic functions

**II. Analytic functions as maps**

A. Preserving "angles" (conformal maps)
**B. Linear fractional transformations**
C. Special functions like exp, sin, cos, log, $z^n$
D. **Finding analytic functions that map one subset of the plane into another using the functions mentioned above are VERY common questions on exams.**

III. Integration
- **A. Path integrals**
- B. Cauchy's theorem
  1. Path independence
  2. Fundamental theorem of calculus
- **C. Cauchy's Integral theorem and applications**
  1. Cauchy inequality
  2. **Liouville's theorem**
  3. Fundamental theorem of algebra
  4. Gauss' mean value theorem
  5. Maximum modulus theorem
  6. Argument theorem
  7. **Rouche's theorem**

**IV. Residues**
- **A. Calculation of residues <u>almost always</u> appears on exams in some form.**
- **B. Applications to real integrals. Finding the values of (proper and improper) real integrals using residue methods are VERY common on exams.**