2024-2025 GRADUATE COURSE DESCRIPTIONS

MATH 201 A-B-C (FWS), Criag/Garcia-Cervera/H. Zhou, *Real Analysis*
Measure theory and integration. Point set topology. Principles of functional analysis. L^p spaces. The Riesz representation theorem. Topics in real and functional analysis.

MATH 202 A-B-C (FWS), Labutin/Putinar, *Complex Analysis*
Analytic functions. Complex integration. Cauchy's theorem. Series and product developments. Entire functions. Conformal mappings. Topics in complex analysis.

MATH 206 A (F), Chandrasekaran, *Matrix Analysis & Computation*
Graduate level-matrix theory with introduction to matrix computations. SVDs, pseudoinverses, variational characterization of eigenvalues, perturbation theory, direct and interative methods for matrix computations.

MATH 206 B (W), Petzold, *Numerical Simulation*
Linear multistep methods and Runge-Kutta methods for ordinary differential equations: stability, order and convergence. Stiffness. Differential algebraic equations. Numerical solution of boundary value problems.

MATH 206 C (S), Ceniceros, *Numerical Solution of Partial Differential Equations - Finite Difference Methods*
Finite difference methods for hyperbolic, parabolic and elliptic PDEs, with application to problems in science and engineering. Convergence, consistency, order and stability of finite difference methods. Dissipation and dispersion. Finite volume methods. Software design and adaptivity.

MATH 206 D (F), Atzberger*, Numerical Solution of Partial Differential Equations - Finite Element Methods*
Weighted residual and finite element methods for the solution of hyperbolic, parabolic and elliptical partial differential equations, with application to problems in science and engineering. Error estimates. Standard and discontinuous Galerkin methods.

MATH 220 A-B-C (FWS), Morrison/X. Zhao/Castella, *Modern Algebra*
Group theory, ring and module theory, field theory, Galois theory, other topics.

MATH 221 A (F), Bigelow, *Foundations of Topology*
Metric spaces, topological spaces, continuity, Hausdorff condition, compactness, connectedness, product spaces, quotient spaces. Other topics as time allows.

MATH 221 B (W), Bigelow, *Homotopy Theory*
Homotopy groups, exact sequences, fiber spaces, covering spaces, van Kampen Theorem.

MATH 221 C (S), McCammond, *Differential Topology*
Topological manifolds, differentiable manifolds, transversality, tangent bundles, Borsuk-Ulam theorem, orientation and intersection number, Lefschetz fixed point theorem, vector fields.

MATH 225 A (F), Agboola, ***Topics in Number Theory***

This will be a course on the basic arithmetic of local fields. This is a fundamental topic in number theory, and indeed many of the techniques that we shall develop play a significant role in other areas of mathematics. Some of the topics that may be discussed include:
- Elementary facts: rings of integers, Hensel's lemma,...
- Local division algebras. Reduced norms. Brauer groups. Galois descent and Galois cohomology.
- Formal groups. Heights, classification of formal groups in characteristic p. Lubin-Tate groups.
- Ramification groups. Upper and lower numbering.
- The Hasse-Arf theorem.
- Local class field theory.

It may be helpful to have a look at the following references:
- J. W. S. Cassels, A. Frohlich, 'Algebraic Number Theory', CUP, especially Chapters I, III, IV, and VI.
- J.-P. Serre, 'Local Fields', Springer.

Both of these books are excellent and contain much valuable and interesting material—of course far more than we shall cover! They belong on the shelf of any aspiring number theorist.

Pre-requisites: A solid knowledge of 220ABC, and a level of mathematical maturity appropriate for an advanced graduate course. Some previous exposure to algebraic number theory will also be useful, but is probably not essential.

Math 225 B (W), Y. Zhang, ***Topics in Analytic Number Theory***

MATH 227 B (W), Z. Wang, ***Advanced Topics in Geometric and Algebraic Topology***
Quantum invariants of 4-manifolds
The topology of smooth 4-manifolds is still mysterious, and quantum invariants of 4-manifolds are not yet useful to distinguish smooth 4-manifolds that are homeomorphic. We will cover the basic topology of 4-manifolds and introduce quantum invariants of 4-manifolds that potentially distinguish exotic smooth structures.

MATH 227 C (S), McCammond, ***Advanced Topics in Geometric and Algebraic Topology***
Finite Complex Reflection Groups
Abstract: Many of the properties of finite Coxeter groups, also known as finite (real) reflection groups, readily extend to the broader class of finite complex reflection groups. The finite real reflection groups are closely related to high-dimensional Platonic solids. The complex reflection groups, on the other hand, are closely connected to invariant theory. In this course we will discuss the structure of the classification (one triply indexed infinite family plus 34 exceptions) and tools used to study them. For example, the infinite family uses monomial matrices, and the 19 two complex dimensional examples are classified using the Hopf fibration. No prior knowledge of Coxeter group theory will be assumed.

MATH 232 A-B (FW), McCammond, ***Algebraic Topology***
Singular homology and cohomology, exact sequences, Hurewicz theorem, Poincare duality.

MATH 236 A (W), Morrison, ***Homological Algebra***
Algebraic construction of homology and cohomology theories, aimed at applications to topology, geometry, groups and rings. Special emphasis on hom and tensor functors; projective, injective and flat modules; exact sequences; chain complexes; derived functors, in particular, ext and tor.

MATH 236 B (S), X. Zhao, ***Homological Algebra***

We will introduce a categorical approach to work with complexes and cohomology. For an abelian category (e.g. category of modules over a ring, category of sheaves of abelian groups over a topological space, etc), we construct its derived category. Roughly speaking, the derived category consists of complexes of objects in the abelian category, and two complexes having the same cohomology are now considered isomorphic to each other. The general framework of triangulated categories and localizations will be introduced to construct such a category and to study its properties. The second part of the course introduces derived functor. We will use sheaves on topological spaces as one major example, the basic definitions will be covered in class.

Prerequisite: Math 220ABC and 236A, or consent of instructor.

MATH 240 A-B-C (FWS), Wink/Wei/Dai, ***Introduction to Differential Geometry and Riemannian Geometry***

Topics include geometry of surfaces, manifolds, differential forms, Lie groups, Riemannian manifolds, Levi-Civita connection and curvature, curvature and topology, Hodge theory. Additional topics such as bundles and characteristic classes, spin structures Dirac operator, comparison theorems in Riemannian geometry.

MATH 241 A (F), Ye, ***Topics in Differential Geometry***
Spin Geometry, Special Holonomy, and Scalar curvature

The topics include basic theory of spin geometry, some topics of geometry of special holonomy, and geometry of positive scalar curvature.

MATH 241 B (W), Dai, ***Topics in Differential Geometry***
Atiyah-Singer index theorems and application to scalar curvature

We will first introduce the Atiyah-Singer index theorem for Dirac operators on closed manifolds and move on to manifolds with boundary and even corners. We will then discuss its applications to various problems concerning the scalar curvature.

MATH 241 C (S), Wink, ***Topics in Differential Geometry***
XXXX

MATH 246 A-B-C (FWS) Jacobs, ***Partial Differential Equations***

Existence and stability of solutions, Floquet theory, Poincare-Bendixson theorem, invariant manifolds, existence and stability of periodic solutions, Bifurcation theory and normal forms, hyperbolic structure and chaos, Feigenbaum period-doubling cascade, Ruelle-Takens cascade.

MATH 260D (FW) Krishna, ***Topics in Commutative Algebra***

The two-quarter course will consist of the following topics:
(1) Review of modules over commutative rings.
(2) Localization theory for commutative rings.
(3) Prime spectrum and primary decomposition.
(4) Integral extension of commutative rings.
(5) Hilbert Nullstelensatz
(6) Noether Normalization
(7) Noetherian rings
(8) Dedekind and Valuation rings

(9) Completion theory
(10) Krull dimension of commutative rings.
(11) Theory of ring and module spectra (if time permits).

MATH 260EE (FWS) Atzberger, *Graduate Student Colloquium*
Topics in algebra, analysis, applied mathematics, combinatorial mathematics, functional analysis, geometry, statistics, topology, by means of lectures and informal conferences with members of faculty.

MATH 260F (F) H. Zhou, *Inverse Problems: Seeing the Unseen*
In inverse problems one attempts to determine the interior properties of a medium by applying various non-intrusive methods. The mathematical problems under study are often motivated by real world application purposes, including questions arising in medical imaging (e.g. CT, EIT, MRI), geophysics (seismology), mathematical physics, etc. In this topics course, we expect to introduce the mathematical analysis of some basic types of inverse problems, including inverse boundary value problems (e.g. the inverse conductivity problem), X-ray and Radon transforms, and their generalizations in non-trivial geometry. If time permits, we will also give a brief introduction to related inverse problems, such as travel time tomography, invisibility and cloaking. There are no prerequisites, but some basic knowledge of PDEs and Fourier analysis will be helpful for taking the course.

Math 260L (W) Hu, *Deep learning theory and algorithms for high dimensional stochastic systems and PDEs*
Stochastic systems, particularly controlled stochastic systems, describe the strategic interaction among (infinitely) many agents, each of whom makes their optimal decisions in a shared environment. Computing Nash equilibria is one of the core objectives in such a setting, but a significant bottleneck arises from the notorious curse of dimensionality. Meanwhile, PDE systems describe complex phenomena in various fields, such as physics, finance, and engineering, often involving a multitude of variables, and solving them also suffers from the same curse of dimensionality.

Deep learning techniques, particularly neural networks, offer a novel approach to tackle the computational challenges posed by these systems. By leveraging their capacity to approximate highly nonlinear and high-dimensional functions, deep learning models can efficiently capture the intricate dependencies within PDE or stochastic systems. This enables more accurate simulations, faster convergence, and improved scalability. With ongoing research and innovation in deep learning theory and algorithms, their application in solving high-dimensional systems continues to open new avenues for addressing previously intractable problems and advancing scientific understanding across multiple domains.

In this course, we will systematically introduce the theory and algorithms of deep learning to address high-dimensional (and infinite-dimensional) stochastic systems and PDEs, with discussions on applications in finance and economics.

Tentative list of topics:
1. Intro to neural networks, feedforward, convolutional, recurrent NN, the universal approximation theorem.
2. PDE system: PINN method and their probabilistic representations
3. Stochastic system: Deep BSDE method and Deep Fictitious Play method.
4. Model-free problems via reinforcement learning.

Main references are recent publications, including but not limited to:
1. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations, by M. Raissi, P. Perdikaris, and G.E. Karniadakis
2. DGM: A deep learning algorithm for solving partial differential equations, by Justin Sirignano and Konstantinos Spiliopoulos
3. Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations, by Weinan E, Jiequn Han and Arnulf Jentzen
4. Deep fictitious play for stochastic differential games, by Ruimeng Hu
5. Convergence of deep fictitious play for stochastic differential games, by Jiequn Han, Ruimeng Hu, Jihao Long
6. Fictitious play for mean-field games: Continuous-time analysis and applications, by Sarah Perrin, Julien Pérolat, Mathieu Laurière, Matthieu Geist, Romuald Elie, Olivier Pietquin
7. Convergence Analysis of Machine Learning Algorithms for the Numerical Solution of Mean Field Control and Games: II - The Finite Horizon Case, by René Carmona, Mathieu Laurière

And three books on deep learning, the theory of stochastic differential equations, and stochastic control theory:
1. Deep Learning (Adaptive Computation and Machine Learning series) by Ian Goodfellow, Yoshua Bengio, and Aaron Courville
2. Stochastic Differential Equations An Introduction with Applications, by Bernt Øksendal.
3. Continuous-time Stochastic Control and Optimization with Financial Applications, by Huyen Pham.

MATH 260NN (S) Atzberger, *Optimization Theory and Applications*
Many problems in decision making and analysis can be formulated as the minimization or maximization of objective functions, along with constraints on permissible solutions. The field of optimization is concerned with developing mathematical approaches to make precise these types of problems, their wellposedness, and how to develop practical numerical methods to approximate their solutions. This course will cover mathematical topics in optimization theory, including in the linear and non-linear settings. The course will also cover motivating applications from the sciences, engineering, and machine learning.

The parts of the course on background will use the books:
"An Introduction to Optimization", (5th edition), E. Chong, W. Lu, and S. Żak,
"Numerical Optimization," by Nocedal and Wright, and
"Convex Analysis and Nonlinear Optimization," J. Borwein and A. Lewis.

The special topics part of the course will be based on materials developed by the instructor and recent papers in the literature.

More details are given below.
Sample of topics:
 Introduction
o History, Motivations, and Recent Developments in Optimization
o Basics of Unconstrained Optimization
o Conditions for Local Minimizers and Maximizers.
O Background in Convex Analysis
 Optimality Conditions,
 Subgradients

    - Fenchel Duality and Lagrangian Duality
    - Set-Constrained Optimization
  - One-Dimensional Search Methods
    - Bracketing, Bisection Method, Newton's Method
    - Line Search Methods
    - Wolfe Conditions
  - Gradient Methods
    - Gradient Descent
    - Stability Considerations and Heuristics
    - Convex Functions: Results on Convergence and Rates
    - Non-Convex Functions: Convergence to Local Minimizers
  - Stochastic Methods
    - Stochastic Approximation of Objectives
    - Robbins-Monro Theorem
    - Stochastic Gradient Descent
    - ADAM, RMSProp, and Other Methods
  - Newton and Quasi-Newton Methods for Smooth Objectives
    - Methods using Hessians and Inverses
    - Approximating Hessians with Rank One Corrections
    - BFGS Algorithm
  - Global Search Algorithms
    - Nelder-Mead Simplex Algorithm
    - Simulated Annealing
    - Genetic Algorithms
  - Applications
    - Training Neural Networks for Classification Tasks
    - Solving Singular Linear Systems With Minimum Norm
    - Least-Squares with Regularizations, (L1 and others).
- Linear Programming
  - Motivating Examples: Economics, Engineering Controllers, Machine Learning
  - Geometric View, Convex Polyhedra, Properties of Solutions
  - Simplex Method
  - Duality
    - Lagrangian and Derivation of Dual Linear Problems
    - Matrix Games
    - Khachiyan's Algorithm,
  - Primal-Dual Methods, Interior Point Methods
- Nonlinear Constrained Optimization
  - Convex vs Non-Convex Objectives and Constraints
  - Inequality Constraints
    - Geometric View: Interior vs Boundary Solutions
    - Karush-Kuhn-Tucker Conditions
    - Second-Order Conditions
  - Equality Constraints
    - Geometric View: Tangent and Normal Spaces
    - Lagrange Conditions
    - Second-Order Conditions
  - Lagrangian Duality
    - Constraint Qualifications
    - Strong Duality Conditions

o Gradient Methods with Projections, Proximal Methods
o Penalty Methods, Log-Barrier Methods
o Primal-Dual Methods, Interior Point Methods
o Applications
 Solving Variational Principles, Constrained Mechanical Systems
 Deep Neural Networks, Backpropagation, Stochastic Gradient Methods
 Inverse Problems in Imaging and Engineering
 Other Applications in the Natural Sciences and Machine Learning

Bibliography:
1. An Introduction to Optimization, (5th edition), E. Chong, W. Lu, and S. Żak.
2. Numerical Optimization, (2nd edition), by J. Nocedal and S. J. Wright.
3. Convex Analysis and Nonlinear Optimization, J. Borwein and A. Lewis.
4. Convex Optimization, by S. Boyd and L. Vandenberghe.
5. Also will use materials developed by the instructor and papers from the recent literature.

MATH 260RR (W) Craig, *Applied Optimal Transport and Gradient Flows*
Description: Over the past six years, optimal transport has evolved from its roots in probability, analysis, and partial differential equations to become a central feature of applied mathematics. While the classic formulation of the optimal transport problem posed the question of "how can one rearrange a probability measure to look like another, with the least amount of effort?" modern variants have introduced what it means to optimally transport between measures of unequal total mass, by incorporating creation and destruction, as well as how to optimally transport measures on graphs.

These variants have led to an explosion of applications. Optimal transport on graphs has inspired works defining the discrete Ricci curvature and heat flow of a graph. Optimal transport with creation and destruction has led to novel methods for studying the training dynamics of neural networks, in the infinite width limit. A variant of optimal transport adapted to early universe cosmology, has enabled physicists to develop state of the art methods for reconstructing fluctuations of the primordial density field.

In this topics course, we will introduce the foundations of optimal transport and gradient flows and quickly move on to recent applications in particle physics, mathematical biology, control theory, and machine learning.

References:
[1] Gradient Flows in Metric Spaces and the Space of Probability Measures, Ambrosio, Gigli, and Savaré
[2] "Infinite-width limit of deep linear neural networks," Chizat, Colombo, Fernández-Real, Figalli
[3] "A fast semi-discrete optimal transport algorithm for a unique reconstruction of the early Universe," Levy, Mohayaee, von Hausegger
[4] "Entropic Ricci Curvature for Discrete Spaces," Jan Maas
[5] Topics in Optimal Transportation, Cedric Villani

MATH 501 (F), Garfield, *Teaching Assistant Training*
Consideration of ideas about the process of learning mathematics and discussion of approaches to teaching.