Precalculus Review Topics

Preparatory for the UCSB Algebra Diagnostic Test and Calculus Courses

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University of California, Santa Barbara

PRECALCULUS REVIEW TOPICS

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Preface

This workbook is meant to be a review of precalculus material. It is not designed to be a complete text, but, rather, a summary of ideas and techniques that are often misunderstood or forgotten by incoming calculus students. By reviewing this booklet a student who is a little weak on algebra or trigonometry can prepare himself or herself for a successful year in the Calculus 34 or 3 series here at U.C. Santa Barbara. There are many topics that I would have liked to have covered that were not, and still others that begged for a more in depth discussion. However, the information that calculus instructors expect their students to have a thorough understanding of is covered.

The material presented in this booklet is very dense, it will be very difficult (if not impossible) to read without some background. Careless errors are very easy to make in mathematics—learning to reread your solutions to check for errors and learning techniques to check your final answer are necessary skills for the successful completion of calculus. Good Luck!

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CHAPTER I

FUNDAMENTAL CONCEPTS

Preliminaries

Here is a list of symbols that will be used thoughout the booklet.

$=$	equals
$<$	less than
$>$	greater than
$\leq$	less than or equal
$\geq$	greater than or equal
$\pm$	plus or minus
$\mp$	minus or plus: when seen with $\pm$ it means $-$ corresponds to $+$ and $+$ corresponds to $-$
$\Rightarrow$	implies
$\Leftrightarrow$	is equivalent to

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Fractions and Factoring

One of the most common deficiencies that calculus students have is the inability to add and divide fractions. Hence, the first section is devoted to this basic but very important topic.

First, recall a few definitions. In the fraction, $\frac{a}{b}$, a is the numerator, and b is the denominator. In the sum $a+b$, a and b are called terms, while in the product, $a \cdot b$, a and b are called factors.

Rules for Fractions.

For $b \neq 0$ and $d \neq 0$

Add,	$\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{cb}{bd} = \frac{ad \pm cb}{bd}$,	Find a common denominator
Multiply,	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$,	Multiply numerators and denominators
Divide,	$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$,	Invert and multiply
Cancel,	$\frac{ad}{bd} = \frac{a}{b} \cdot \frac{d}{d} = \frac{a}{b} \cdot 1 = \frac{a}{b}$,	Cancel only factors, not terms

Note.

(1) *If the denominator is zero then the fraction is undefined.*

(2) *If $d \neq 0$ then $\frac{0}{d} = 0$.*

In order to cancel we need to remember how to factor. Recall that a quantity is factored if if is the product of two or more factors. This booklet will assume that you remember how to factor. These formulas should help:

$$
\begin{aligned}
x^2 - 2ax + a^2 &= (x-a)^2, && \text{Square of difference}\\
x^2 + 2ax + a^2 &= (x+a)^2, && \text{Square of sum}\\
x^2 - a^2 &= (x-a)(x+a), && \text{Difference of squares}\\
x^3 - a^3 &= (x-a)(x^2+ax+a^2), && \text{Difference of cubes}\\
x^3 + a^3 &= (x+a)(x^2-ax+a^2), && \text{Sum of cubes}
\end{aligned}
$$

Examples.

(1)

$$\frac{1}{4}+\frac{5}{3}=\frac{3}{12}+\frac{20}{12}=\frac{3+20}{12}=\frac{23}{12} \qquad \text{Find a common denominator}$$

(2)

$$\frac{1}{3}\div\frac{4}{5}=\frac{1}{3}\cdot\frac{5}{4}=\frac{5}{12} \qquad \text{Invert and multiply}$$

(3)

$$\frac{8}{3}\cdot\frac{3}{4}=\frac{24}{12}=2 \qquad \text{Multiply numerators and denominators}$$

(4)

$$\frac{\frac{50}{3}}{\frac{2}{15}}=\frac{50}{3}\cdot\frac{15}{2}=\frac{25\cdot 2\cdot 3\cdot 5}{3\cdot 2}=25\cdot 5=125 \qquad \text{Invert and multiply}$$

(5)

$$
\begin{aligned}
\frac{3x+2}{x+1}-\frac{1}{x-1} &= \frac{(3x+2)(x-1)}{(x+1)(x-1)}-\frac{1\cdot(x+1)}{(x-1)(x+1)} \qquad \text{Find a common denominator}\\
&= \frac{(3x+2)(x-1)-(x+1)}{(x+1)(x-1)}\\
&= \frac{3x^2-3x+2x-2-x-1}{(x+1)(x-1)}\\
&= \frac{3x^2-2x-3}{(x+1)(x-1)}
\end{aligned}
$$

(6)

$$\begin{aligned}\frac{1}{x+3}+\frac{3}{x+2} &= \frac{x+2}{(x+3)(x+2)}+\frac{3(x+3)}{(x+2)(x+3)} \qquad \text{Find a common denominator}\\ &= \frac{x+2+3x+9}{(x+2)(x+3)}\\ &= \frac{4x+11}{x^2+5x+6}\end{aligned}$$

(7)

$$\begin{aligned}\frac{x}{x+a}+\frac{a}{a-x} &= \frac{x(a-x)+a(x+a)}{(a-x)(x+a)} \qquad \text{Find a common denominator}\\ &= \frac{xa-x^2+xa+a^2}{(a-x)(x+a)}\\ &= \frac{a^2+2ax-x^2}{(a-x)(x+a)}\\ &= \frac{a^2+2ax-x^2}{a^2-x^2}\end{aligned}$$

Leaving the denominator in factored form often makes cancellations easier to see.

(8)

$$\begin{aligned}\frac{ax+3}{2a+1}\div\frac{a^2x^2+3ax}{4a^2-1} &= \frac{(ax+3)}{(2a+1)}\cdot\frac{(4a^2-1)}{(a^2x^2+3ax)} \qquad \text{Invert and multiply}\\ &= \frac{(ax+3)(2a-1)(2a+1)}{(2a+1)ax(ax+3)}\\ &= \frac{2a-1}{ax}\end{aligned}$$

(9)

$$\frac{x-a}{a-x}=\frac{-(a-x)}{a-x}=\frac{-1}{1}=-1$$

As you can see when working with fractions it is helpful to brush up on your factoring skills.

Exercises.

Evaluate and simplify; Assume that no denominator is zero.

1) $\dfrac{5}{16} \div \dfrac{1}{4}$

2) $\dfrac{2}{3} \cdot \dfrac{12}{5}$

3) $\dfrac{4}{5} + \dfrac{3}{2}$

4) $\dfrac{1}{3} - \dfrac{1}{2}$

5) $\dfrac{1}{3} - \dfrac{1}{2}$

6) $\dfrac{x+3}{x-2} - \dfrac{2}{x+2}$

7) $\dfrac{1}{a} + \dfrac{1}{b}$

8) $\dfrac{1}{a} - \dfrac{1}{b}$

9) $\dfrac{2x+3}{x^2-9} - \dfrac{15}{2(x-3)}$

10) $\dfrac{1+1/2}{2-1/2}$

11) $\dfrac{1-1/x}{1+1/x}$

12) $1 - \dfrac{1}{1-1/3}$

13) $1 - \dfrac{1}{1-\frac{1}{y-2}}$

14) $\dfrac{\dfrac{a^2-b^2}{ab^2}}{a+\dfrac{a^2}{b}}$

15) $\dfrac{5x+3}{x^2-4} + \dfrac{1-x}{x+2} + \dfrac{3}{2-x}$

16) $\dfrac{x+2}{x-2}$

17) $\dfrac{25yx+5}{3x-1} \div \dfrac{5xy+1}{9x^2-1}$

18) $\dfrac{3x^2+y^2}{3x} + \dfrac{y}{x^2+y^2}$

Absolute Values

Definition. *The Absolute Value of x, written*

$$|x| = \textit{The distance from } x \textit{ to zero} = \begin{cases} x & \textit{if } x \geq 0 \\ -x & \textit{if } x < 0. \end{cases}$$

Note that the Absolute Value of a number is always nonnegative, i.e. $|x| \geq 0$

Examples.

(1)

$$|5| = 5$$

(2)

$$|-5| = 5$$

(3)

$$|\pi - 4| = -(\pi - 4) = 4 - \pi \qquad \text{because } \pi < 4 \Rightarrow \pi - 4 < 0$$

(4) Evaluate $|x+5| - |x+6|$ when $-6 < x < -5$

$$\begin{array}{lllll} x < -5 & \Rightarrow & x+5 < 0 & \Rightarrow & |x+5| = -(x+5) \\ -6 < x & \Rightarrow & x+6 > 0 & \Rightarrow & |x+6| = x+6 \end{array}$$

Hence,

$$|x+5| - |x+6| = -(x+5) - (x+6) = -2x - 11$$

Absolute value gives us a convenient way to write the distance between two points on the real number line.

$$\{\text{The distance between a and b}\} = |a-b| = |b-a|$$

This is always a positive number when $a \neq b$.

Exercises. *Evaluate each expression*

1) $3+|-3|$ 2) $|-5+2|-|7|$ 3) $\dfrac{|30-6|}{|6-30|}$

4) $|7\cdot(-6)|-|7||6|$

5) $|6+2|-|3-2b|$ when $b=-2$ 6) $|x+1|+4|x+3|$ when $x<-3$

7) $|x-3|+|x+4|$ when $x<-4$

Integer Exponents

Definition. *For* $x \neq 0$,

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{\text{n-times}}$$
$$x^{-1} = \frac{1}{x}$$
$$x^0 = 1$$

Examples.

(1)
$$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$$

(2)
$$3^{-1} = \frac{1}{3}$$

(3)
$$(x+5)^2 = (x+5)(x+5)$$

(4)
$$(x+2)^{-1} = \frac{1}{x+2}$$

(5)
$$x^3 = x \cdot x \cdot x$$

(6)
$$(a+b)^{-2} = \frac{1}{(a+b)^2} = \frac{1}{(a+b)(a+b)}$$

Rules for Exponents.

$$
\begin{array}{llll}
i) & x^m x^n = x^{m+n} & ii) & (x^m)^n = x^{mn} \\
iii) & (\frac{x}{y})^n = \frac{x^n}{y^n} \text{If } y \neq 0 & iv) & x^{-n} = \frac{1}{x^n}\text{If } x \neq 0 \\
v) & (xy)^n = x^n y^n & vi) & \frac{x^m}{x^n} = x^{m-n}\text{If } x \neq 0
\end{array}
$$

Proof.
i)

$$x^m x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{\text{m-times}} \cdot \underbrace{x \cdot x \cdot \ldots \cdot x}_{\text{n-times}} = \underbrace{x \cdot x \cdot \ldots \cdot x}_{\text{m+n-times}} = x^{m+n}$$

ii), iii), and v) are left as exercises
iv)

$$
\begin{aligned}
x^{-n} &= (x^{-1})^n && \text{by (ii)} \\
&= (\frac{1}{x})^n && \text{by definition} \\
&= \frac{(1)^n}{x^n} && \text{by (iii)} \\
&= \frac{1}{x^n} && \text{because } \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{\text{n-times}} = 1
\end{aligned}
$$

vi)

$$\frac{x^m}{x^n} = \frac{\overbrace{x \cdot x \cdot \ldots \cdot x}^{\text{m-times}}}{\underbrace{x \cdot x \cdot \ldots \cdot x}_{\text{n-times}}} = \begin{cases} x^{m-n}, & \text{If } m > n \text{, for there are } m-n \text{ } x\text{'s in the numerator after cancellation} \\ \frac{1}{x^{n-m}}, & \text{If } m < n \text{, for there are } n-m \text{ } x\text{'s in the denominator after cancellation.} \end{cases}$$

Then all we need to notice is

$$\frac{1}{x^{n-m}} = \frac{1}{x^{-(m-n)}} = x^{-(-(m-n))} = x^{(m-n)}$$

as desired. We should recognize that in the second case our exponent will be negative, but this is perfectly O.K.

Examples.

(7)

$$x^2 \cdot x^3 = x \cdot x \cdot x \cdot x \cdot x = x^{2+3} = x^5 \qquad \text{Add exponents when multiplying; Property i)}$$

(8)

$$\begin{aligned} (x^2y^3)^4 &= (x^2)^4 \cdot (y^3)^4 && \text{Property v)} \\ &= x^8 \cdot y^{12} && \text{Multiply exponents when exponentiated; Property ii)} \end{aligned}$$

(9)

$$x^6 \cdot x^{-5} = x^{6+(-5)} = x \qquad \text{Add exponents when multiplying: Property i)}$$

(10)

$$\begin{aligned} \frac{x^3y^2}{x^2y^4} &= \frac{x^3}{x^2} \cdot \frac{y^2}{y^4} && \text{Multiplication of fractions} \\ &= x^{3-2} \cdot y^{2-4} && \text{Property vi)} \\ &= x \cdot y^{-2} \\ &= \frac{x}{y^2} && \text{Property iv)} \end{aligned}$$

(11)

$$3^x 3^x = 3^{x+x} = 3^{2x} \qquad \text{Property i)}$$

(12)

$$\frac{x^2+x}{x} = \frac{x(x+1)}{x} = x+1 \qquad \text{Cancellation property for fractions}$$

(13)

$$\begin{aligned} \frac{26}{75x^4} \cdot \frac{3x^2}{52y} \cdot 25y^2 &= \frac{26 \cdot 3 \cdot 25}{75 \cdot 52} \cdot \frac{x^2}{x^4}\frac{y^2}{y} && \text{Multiplication property for fractions} \\ &= \frac{1}{2} \cdot x^{-2} \cdot y && \text{Factorization of the constants} \\ &= \frac{1}{2} \cdot \frac{1}{x^2} \cdot y && \text{Property iv)} \\ &= \frac{y}{2x^2} && \text{Multiplication property for fractions} \end{aligned}$$

(14)

$$\begin{aligned}
\frac{(x+y)^3(x-y)}{x^2-y^2} &= \frac{(x-y)(x+y)^3}{(x-y)(x+y)} && \text{Difference of squares} \\
&= (x-y)^{1-1}(x+y)^{3-1} && \text{Property vi)} \\
&= (x-y)^0(x+y)^2 \\
&= 1\cdot(x+y)^2 && \text{Definition} \\
&= (x+y)^2
\end{aligned}$$

Exercises. *Simplify, leaving only nonnegative exponents. Assume that the denominators are nonzero.*

1)	$(x+1)^2(x+1)^{10}$	2)	$(a^2)^3-(a^3)^2$	3)	$(x+2)(x+2)^2(x+2)^3$
4)	$[(x+a)(x+a)^3]^2$	5)	$\frac{x^4}{x^6}$	6)	$\frac{x^6}{x^4}$
7)	$\frac{(x^2+1)^3}{(x^2+1)^5}$	8)	$((2^2)^2)^2$	9)	$(5x^4)^2$
10)	$(\frac{x^3y^9}{x^5y^{15}})^2$	11)	$(\frac{x^3y^{-2}z}{z^{-1}x^4y})^{-2}$	12)	$((\frac{1}{5})^{-1}+(\frac{1}{5})^{-3})^{-1}$
13)	$\frac{x^5}{x^2}\div\frac{x^2}{y^{-3}}$	14)	$((\frac{x^2+2x}{4xy})^0)^5$	15)	$\frac{x^2y}{2x}+\frac{y^2x}{2y}$
16)	$(\frac{1}{a}+\frac{1}{b})^{-1}$				

17) Prove rules (ii),(iii), and (v) using the techniques demonstrated in the proofs of the other rules.

Radicals and nth–roots

Definition. *The number a is the nth–root of b if $a^n = b$. If $n = 2$, then a is the square root of b, and if $n = 3$, then a is the cube root of b.*

Examples.

(1) Since $(-5)^2 = 5^2 = 25$, both 5 and -5 are square roots of 25.
(2) Since $2^3 = 8$, 2 is a cube root of 8.
(3) Since $(-3)^4 = 3^4 = 81$, 3 and -3 are 4th–roots of 81.
(4) Since $(-3)^3 = -27$, -3 is a cube root of -27

Note. *If n is even then nth–roots are not unique: one is positive and the other is negitive.*

If n is odd, then there is a unique real nth–root.

Definition. *The principle nth–root of b is the positive real nth–root of b when n is even, and the only real nth–root of b when n is odd. It is written $\sqrt[n]{b}$. We write the square root of b as $\sqrt{b}$. The symbol $\sqrt[n]{\cdot}$ is called a radical.*

Given any nonzero real number a and any *even* n, $a^n > 0$. Hence if $b < 0$ there are no real solutions to the equation $a^n = b$. Hence, we do not define $\sqrt[n]{b}$, when n is even and b is negitive.

$$\boxed{\text{n even and } b < 0 \Rightarrow \sqrt[n]{b} \text{ is undefined (or not real)}}$$

Notice that when solving quadratic equations in the later sections it will be convenient to write $\sqrt{-1} = i$ and for $x > 0, \sqrt{-x} = i\sqrt{x}$. However, i and $i\sqrt{x}$ do not lie on the real number line where all of our work is currently taking place. These numbers are called complex or imaginary numbers. We will see more of them later.

Examples.

(5) $\sqrt{9} = 3$
(6) $\sqrt[3]{27} = 3$
(7) $\sqrt[3]{-27} = -3$
(8) $\sqrt[4]{-16}$ is undefined

Rules for Radicals.

i)	$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$	*ii)*	$\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$
iii)	n even $\sqrt[n]{x^n} = \lvert x \rvert$	*iv)*	n odd $\sqrt[n]{x^n} = x$
v)	$(\sqrt[n]{x})^n = x$	*vi)*	$\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$ If $y \neq 0$

Proof. i) Let $\sqrt[n]{x} = a$ and $\sqrt[n]{y} = b$. Then $x = a^n$ and $y = b^n$ by definition of nth–roots. Hence

$$xy = a^n b^n = (ab)^n \qquad \text{by rule v) of exponents.}$$

We now have that $xy = (ab)^n$. So applying our definition once again we get

$$\sqrt[n]{xy} = ab$$

But $ab = \sqrt[n]{x}\,\sqrt[n]{y}$ from above. Thus, $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$. This is exactly what we wanted to be true.

The proofs of (ii), (iii), (iv), (v), and (vi) are left as exercises.

Be sure to avoid the following common errors.

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}, \qquad \sqrt[3]{c+d} \neq \sqrt[3]{c} + \sqrt[3]{d}$$
$$\sqrt{x^2+y^2} \neq x + y, \qquad \sqrt[3]{x^3+y^3} \neq x + y$$

Exercises. *Evaluate. If the expression does not represent a real number say so.*

1)	$\sqrt[5]{32}$	2)	$\sqrt[3]{-64}$	3)	$-\sqrt[4]{-16}$
4)	$-\sqrt[5]{-32}$	5)	$\sqrt[3]{(-10)^3}$	6)	$\sqrt[4]{(-8)^4}$
7)	$\sqrt{\dfrac{25}{4}}$	8)	$\sqrt{3}+\sqrt{12}-\sqrt{48}$	9)	$\sqrt[3]{2}+\sqrt[3]{16}$
10)	$\sqrt{36x^2} \quad x>0$	11)	$\sqrt[4]{16a^4} \quad a<0$	12)	$\sqrt[3]{\dfrac{16a^{12}b^2}{c^9}} \quad a,b,c>0$
13)	$\sqrt{\dfrac{(a+b)^3}{16a^2b^3}}, a,b>0$	14)	$\sqrt[3]{125x^9}$	15)	$\dfrac{3}{\sqrt{8}}-\sqrt{450}$

16) Prove rules (ii), (iii), (iv), (v), and (vi) for radicals.

Rational Exponents

Definition.

$$b^{1/n} = \sqrt[n]{b} \qquad \textit{when } n \textit{ is even and } b > 0$$
$$b^{1/n} = \sqrt[n]{b} \qquad \textit{for } \underline{\textit{any}}\ b \textit{ when } n \textit{ is odd}$$
$$b^{m/n} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m \qquad \textit{and is defined whenever } \sqrt[n]{b} \textit{ is defined.}$$

Theorem. *The rules for integer exponents hold for all rational (and even real) exponents.*

Rules for Exponents.

i)	$x^m x^n = x^{m+n}$	*ii)*	$(x^m)^n = x^{mn}$
iii)	$(\frac{x}{y})^n = \frac{x^n}{y^n}$ If $y \neq 0$	*iv)*	$x^{-n} = \frac{1}{x^n}$ If $x \neq 0$
v)	$(xy)^n = x^n y^n$	*vi)*	$\frac{x^m}{x^n} = x^{m-n}$ If $x \neq 0$

Examples.

(1)

$$4^{1/2} = \sqrt{4} = 2$$

(2)

$$8^{1/3} = \sqrt[3]{8} = 2$$

(3)
$$(-64)^{-1/3} = \frac{1}{(-64)^{1/3}} = \frac{1}{\sqrt[3]{-64}} = \frac{1}{-4} \qquad \text{since } (-4)^3 = -64$$

(4)
$$36^{-3/2} = \frac{1}{(\sqrt{36})^3} = \frac{1}{6^3} = \frac{1}{216}$$

(5)
$$(x^2+1)^{2/3}(x^2+1)^{4/3} = (x^2+1)^{2/3+4/3} = (x^2+1)^2$$

(6)
$$\frac{(x+1)^{5/4}}{(x+1)^{1/3}} = (x+1)^{5/4-1/3} = (x+1)^{15/12-4/12} = (x+1)^{11/12}$$

In the following examples we will write the radical expressions with rational exponents.

(7)
$$\sqrt[5]{(x+2)^2} = (x+2)^{2/5}$$

(8)
$$\begin{aligned}\sqrt[3]{\frac{\sqrt{x}\sqrt[5]{y}}{\sqrt[4]{z^8}}} &= (\frac{x^{1/2}y^{1/5}}{z^{8/4}})^{1/3} \\ &= \frac{x^{\frac{1}{2}\frac{1}{3}}y^{\frac{1}{5}\frac{1}{3}}}{z^{2\cdot\frac{1}{3}}} \\ &= \frac{x^{\frac{1}{6}}y^{\frac{1}{15}}}{z^{\frac{2}{3}}}\end{aligned}$$

Exercises. *Evaluate. Assume that all variables represent positive real numbers and that no denominators are zero. Write answers with only positive exponents.*

1) $(-2)^{-3}$	2) $(\frac{2}{7})^{-3}$	3) $(64a^{12})^{\frac{5}{6}}$
4) $(3m)^2(-2m)^3$	5) $(2y^{\frac{1}{4}}z)(3y^{\frac{3}{4}}z^{-\frac{1}{3}})$	6) $(\frac{p^3r^9}{27p^{-3}r^6})^{-\frac{1}{3}}$
7) $\frac{(3x^n)^3}{(x^2)^{n-1}}$	8) $(\frac{x^4y^3z}{16x^6yz^{-5}})^{-\frac{1}{4}}$	9) $(n+1)^{\frac{1}{2}}(1+n)^{\frac{3}{4}}$
10) $\frac{x^3+y^3}{x^2-y^2}\cdot\frac{x+y}{x^2-xy+y^2}$	11) $\frac{3}{m^3}-\frac{m}{(m-1)^2}$	12) $\frac{p^{-1}-q^{-1}}{(pq)^{-1}}$

Write the following with rational exponents.

13) $-\sqrt[3]{\frac{4}{5}}$

14) $\sqrt{98r^3s^4t^{10}}$

15) $\sqrt{\frac{5}{2p}}$

16) $-\sqrt[3]{\frac{9x^5y^6}{z^5w^2}}$

17) $\sqrt[4]{\frac{32x^6}{y^5}}$

18) $\frac{m}{\sqrt{p}} + \frac{p}{\sqrt{m}}$

19) $(\sqrt{2}+1)^2$

CHAPTER II

SOLVING EQUATIONS

Linear Equations

Definition. *Any equation that can be written in the* <u>*standard form*</u>

$$ax + b = 0$$

is a <u>*linear equation.*</u>

Rules for solving linear equations.
Do the same thing to both sides of the equation.

$$\begin{aligned} a = b \qquad &\Rightarrow \qquad a \pm c = b \pm c \\ &\Rightarrow \qquad ac = bc \\ &\Rightarrow \qquad \frac{a}{c} = \frac{b}{c} \text{ if } c \neq 0 \end{aligned}$$

Stratagy for solving linear equation.

(1) Clear denominators.
(2) Distribute.
(3) Move x's to left hand side and move constants to right hand side of the equation.
(4) Divide by the coefficient of x.
(5) Substitute into the original equation to see if the equation is true at this value.

Examples.

(1)

$$\begin{aligned} 6x+5&=0 \\ 6x+5-5&=-5 && \text{this step will not be shown in subsequent examples} \\ 6x&=-5 \\ x&=\frac{-5}{6} \end{aligned}$$

(2)

$$\begin{aligned} 5(2x-5)&=6-(x-5) && \text{Distribute} \\ 10x-25&=6-x+5 && \text{Collect terms} \\ 10x+x&=6+25+5 \\ 11x&=36 \\ x&=\frac{36}{11} \end{aligned}$$

FACT: An equation is undefined whenever it has a denominator that is equal to zero, just as a fraction is.

(3) Although

$$\frac{2x}{x-1}+\frac{5x+1}{5}=x$$

does not look like a linear equation, it actually is.

$$\begin{aligned}
\frac{2x}{x-1} + \frac{5x+1}{5} &= x \\
\frac{2x \cdot 5}{(x-1)5} + \frac{(5x+1)(x-1)}{5(x-1)} &= x && \text{find a common denominator} \\
\frac{10x + (5x+1)(x-1)}{5(x-1)} &= x \\
10x + (5x^2 - 4x - 1) &= 5x^2 - 5x && \text{Clear denominators} \\
10x - 4x + 5x &= 1 && \text{Collect terms and cancel} \\
11x &= 1 \\
x &= \frac{1}{11} && \text{Divide}
\end{aligned}$$

Notice that our answer is $\frac{1}{11} \neq 1$, and 1 is the only place where our equation is undefined. So our answer should be valid. Check it by substituting into the original equation.

Exercises. *Solve the following equations.*

1) $-3y + 2 = 15$

2) $5/6k - 2k + 1/3 = 2/3$

3) $\dfrac{3x-2}{5} = \dfrac{x+2}{10}$

4) $\dfrac{x}{3} - 7 = 6 - \dfrac{3x}{4}$

5) $\dfrac{5}{2p+3} + \dfrac{1}{p-6} = 0$

6) $\dfrac{2}{x+1} = \dfrac{5}{3x+5}$

7) $\dfrac{7}{3a+2} - \dfrac{5}{a-1} = \dfrac{6}{3a+2}$

8) $\dfrac{5k}{k+3} = 3 - \dfrac{20}{k+3}$

9) $\dfrac{3x}{2x-1} + \dfrac{3x+2}{3} = x$

Solve for x.

10) $2(x-a) + b = 3x + a$

11) $\dfrac{2a}{x-1} = a - 3$

Quadratic Equations

Definition. *A Quadratic equation is any equation that can be written in the standard form*

$$ax^2 + bx + c = 0.$$

Where a,b, and c are real numbers and $a \neq 0$.

Before we can solve such equations we need two properties.

The Zero Factor Property	$ab = 0$	$\Leftrightarrow$	$a = 0$ or $b = 0$

The Square Root Property	$x^2 = b$	$\Leftrightarrow$	$x = \pm\sqrt{b}$

To see that the second property makes sense, note that if $x^2 = 13$ then $x = \sqrt{13}$ or $x = -\sqrt{13}$. This follows directly from our definition of square root. Also, if $x = \sqrt{13}$ or $x = -\sqrt{13}$ then $x^2 = 13$.

You will also need to memorize the quadratic formula. (Yes, I hate the word memorize too, but for this formula it fits.) We will go through how the formula is derived but you should be able to use it without going through the proof first. We will see that it is just the generalization of the method of completing the square. The formula gives the solutions to a quadratic equation in standard form.

The Quadratic Formula	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Strategy for Solving Quadratic Equations.

(1) Get the equation in standard form using the techniques introduced for linear equations.
(2) Use one of the following techniques.
   a) Factoring and employing the Zero Factor Property.
   b) Completing the square and using the Square Root Property.
   c) Using the Quadratic Formula.

Factoring.

Examples. *Solve the following equations*

(1)

$$\begin{aligned} 6x^2 + 7x &= 3 \\ 6x^2 + 7x - 3 &= 0 && \text{Standard Form} \\ (3x-1)(2x+3) &= 0 && \text{Factoring} \\ &\Rightarrow \\ 3x - 1 = 0 \quad &\text{or} \quad 2x + 3 = 0 && \text{By the Zero Factor Property} \\ 3x = 1 \quad &\text{or} \quad 2x = -3 && \text{We now have linear equations to solve} \\ x = \frac{1}{3} \quad &\text{or} \quad x = \frac{-3}{2} && \text{Notice that there are two solutions} \end{aligned}$$

(2)

$$\begin{aligned} x^2 &= 2x \\ x^2 - 2x &= 0 && \text{Standard Form} \\ x(x-2) &= 0 && \text{Factor} \\ x = 0 \quad &\text{or} \quad x - 2 = 0 && \text{Zero Factor Property} \\ x = 0 \quad &\text{or} \quad x = 2 \end{aligned}$$

Do not divide by x in the original equation. You would be dividing by zero which is not allowed. You will lose one of your solutions, namely $x = 0$.

Completing the Square.

(5)

$$\begin{aligned} m^2 &= 25 \\ m &= \pm\sqrt{25} = \pm 5 && \text{By the Square Root Property} \end{aligned}$$

(6)

$$z^2 = -36$$

$$z = \pm\sqrt{-36} = \pm i\sqrt{36} = \pm 6i$$

Remembering our definition of i from chapter I, namely $i = \sqrt{-1}$

(7)

$$\begin{aligned} (x-4)^2 &= 17 \\ x - 4 &= \pm\sqrt{17} && \text{By the Square Root Property} \\ x &= 4 \pm \sqrt{17} \end{aligned}$$

To complete the square in general we must manipulate an equation that is in standared form into the form that is shown in example (5) or (7). We need to recall the formula for the square of a difference or sum.

$$x^2 \pm 2ax + a^2 = (x \pm a)^2$$

Notice that the coefficient of x^2 is 1, and the constant term is half the coefficient of x squared. The technique of completing the square capitalizes on these two facts.

Examples.

(8) Consider the quadratic equation

$$x^2 - 12x + 1 = 0$$

The coefficient of x is -12. We want the constant term to be half of this squared or $(-6)^2 = 36$. Adding 35 to both sides of the equation will maintain an equation with the same roots and give us the desired constant term. We get

$$x^2 - 12x + 36 = 35.$$

The left hand side is now a perfect square.

$$(x-6)^2 = 35$$

Using the Square root Property as in example (5), we find

$$x - 6 = \pm\sqrt{35}$$

or

$$x = 6 \pm \sqrt{35}$$

Notice that in order to apply this technique the coefficient of x^2 must be 1, so divide through by this coefficient before completing the square.

Completing the Square in General.

$ax^2 + bx + c = 0$	Determine a, b, and c
$ax^2 + bx = -c$	Subtract c
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Divide by a
$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}$	Add $(\frac{b}{2a})^2$
$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a \cdot a}$	Write the left hand side as the square of a sum
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	Take square root of both sides (don't forget the $\pm$)
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Subtract $\frac{b}{2a}$ from both sides

This is the Quadratic Formula. It is obtained by completing the square in general. This solution works regardless of the values of a, b, and c. This calculation may look complicated because of all of the variables. However, the steps taken are the same ones that you've been using all along to solve equations. Take a moment to memorize the quadratic formula.

$$\boxed{x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{The Quadratic Formula}}$$

(9) Solve $2x^2 = x - 4$.

$2x^2 - x + 4 = 0$	$a = 2, \quad b = -1, \quad c = 4$
$2x^2 - x = -4$	Subtract c
$x^2 - \frac{1}{2}x = -2$	Divide by a
$x^2 - \frac{1}{2}x + \frac{1}{16} = -2 + \frac{1}{16}$	Add $(\frac{b}{2a})^2$
$(x - \frac{1}{4})^2 = -\frac{31}{16}$	Write the left hand side as the the square of a sum
$x - \frac{1}{4} = \pm\sqrt{-\frac{31}{16}} = \pm i\sqrt{\frac{31}{16}}$	Take square root of both sides (don't forget the $\pm$)
$x = \frac{1}{4} \pm \frac{i\sqrt{31}}{4}$	Subtract $\frac{b}{2a}$ from both sides

Exercises. *Solve each of the following equations by completing the square, then check your answers by using the quadratic formula.*

1)	$x^2 + 6x + 9 = 0$	2)	$x^2 - 9x = 0$	3)	$(m + 2)^2 = 7$
4)	$x^2 + 4 = 3x$	5)	$\frac{x}{x-1} = \frac{3x}{x-2}$	6)	$3x^2 + 2 = x$
7)	$x^2 + 8x = -13$	8)	$px^2 = 12x - 8$		

CHAPTER III

GRAPHING

Definition of a Function and Function Notation

Definition. *A Function from a set X to a set Y is a rule that assigns to each x in X a unique y in Y.*

X is called the Domain or input

Y is called the Range or output

Elements of the domain, X, are called independent variables

Elements of the range, Y, are called dependent variables

The domain will be the largest set of real numbers which will lead to a range within the set of real numbers. We will see that this convention is necessary to ensure that we can graph our functions on the Cartesian Coordinate System.

Reasons to restrict domain.

i) A negative number appears inside an even root.

ii) Zero appears as a denominator.

Example.

(1) Find the domain of $y = \sqrt{3x+9}$.

The output will be real whenever $\sqrt{3x+9}$ is real. In other words when $3x + 9 \geq 0$, for otherwise we will get complex numbers in the range which

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is not allowed.

$$\begin{aligned} 3x + 9 &\geq 0 \\ 3x &\geq -9 \\ x &\geq -3 \end{aligned}$$

Thus, the domain is all real numbers greater than or equal to -3. The range is all real numbers greater than or equal to zero since for any $y \geq 0$ we can find an x so that $\sqrt{3x+9} = y$.

Here, x is the independent variable and y is the dependent variable. The value of y depends on the value of x. Traditionally this is how mathematicians label the variables.

(2) Find the domain of

$$y = \frac{x+3}{(x-2)(x+1)}$$

.

Since a fraction is not defined when the denominator is zero, this function will not have a defined output when $x = 2$ or $x = -1$. Therefore we include all real numbers in the domain except 2 and -1.

(3) The domain of $y = 3x + 2$ is all real numbers.

Function Notation.

We often use single letters to denote functions, e.g. f, g, h, or F. If f is a function and x is in its domain, we write $f(x)$ for the output in the range. The notation $f(x)$ is read "f of x" and means the "value of f at x".

(4) Suppose that f is the function given by the equation $f(x) = x^2 + 2x + 1$. Evaluate f at 3.

We are asked to find the output given by the function f when we input 3. For shorthand mathematicians write $f(3)$.

$$f(3) = (3)^2 + 2(3) + 1 = 16.$$

Just plug in 3 wherever you see x and evaluate.

(5) Let

$$f(x) = \frac{3}{x-3},$$

then

$$\begin{aligned} f(0) &= \frac{3}{0-3} = \frac{3}{-3} = -1 \\ f(1) &= \frac{3}{1-3} = \frac{3}{-2} = -\frac{3}{2} \\ f(x+2) &= \frac{3}{(x+2)-3} = \frac{3}{x-1} \end{aligned}$$

(6) Let $f(x) = x^2 + 2$ and $g(x) = \frac{3}{x}$, then

$$f(g(x)) = (g(x))^2 + 2 = (\frac{3}{x})^2 + 2 = \frac{9}{x^2} + 2$$
$$g(f(2)) = \frac{3}{f(2)} = \frac{3}{2^2+2} = \frac{3}{4+2} = \frac{3}{6} = \frac{1}{2}$$

We can even write equations that have functions within them.

$$\frac{g(x+k)-g(x)}{k} = \frac{\frac{3}{x+k} - \frac{3}{x}}{k}$$
$$= \frac{\frac{3x}{(x+k)x} - \frac{3(x+k)}{x(x+k)}}{k} \qquad \text{Find a common denominator}$$
$$= \frac{3x - 3(x+k)}{x(x+k)} \cdot \frac{1}{k} \qquad \text{Invert and multiply}$$
$$= \frac{3x - 3x - 3k}{kx(x+k)}$$
$$= \frac{-3k}{kx(x+k)}$$
$$= \frac{-3}{x(x+k)}$$

Exercises. *Find the domain of each function.*

1) $y = 3x + 2$ 2) $y = \frac{1}{3x+2}$ 3) $y = \sqrt{3x+2}$

4) $f(x) = \frac{1}{x^2}$ 5) $g(x) = \frac{1}{x^2+5x+4} - \sqrt{x-5}$

6) $k(x) = \sqrt{x^2+5x+4}$ 7) $y = \sqrt[3]{x+2}$ 8) $h(x) = \sqrt{\frac{x^2+9}{x^{-9}}}$

9) $f(x) = (5x+2)^{\frac{1}{2}}$ 10) $g(x) = \sqrt{|x|}$ 11) $h(x) = \frac{\sqrt{22-x}}{x-3.25}$

Let $f(x) = x^2 - 5$, $g(x) = 4 - 5x$, and $h(x) = \frac{x}{2-x}$. Evaluate the following.

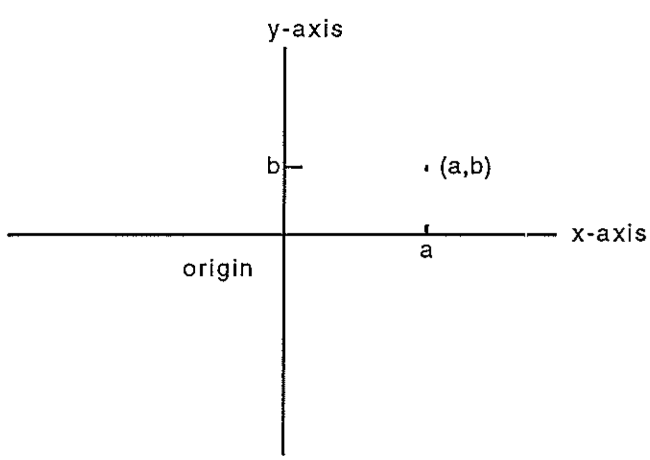
12) $f(3)$ 13) $g(\frac{1}{5})$ 14) $h(-8)$

15) $f(-2)$ 16) $f(g(2))$ 17) $h(x+k)$

18) $\frac{f(x+k)-f(x)}{k}$ 19) $\frac{g(x+k)-g(x)}{k}$ 20) $\frac{h(x+k)-h(x)}{k}$

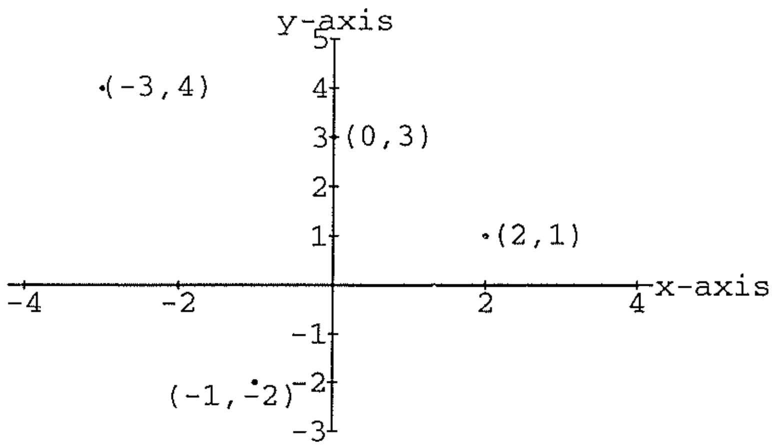
The Cartesian Coordinate System

Whenever a function maps a subset of the real numbers into the set of real numbers we can represent this function by a graph in the xy–plane. The horizontal axis is the x–axis and the vertical axis is the y–axis. The point where the x–axis and the y–axis intersect is the origin. Any function corresponds to a set of ordered pairs, (a,b), where a is the input and b is the output. Any point is the graph of an ordered pair (a,b), where a is the distance away from the origin on the x–axis and b is the distance away from the origin on the y–axis.



Examples.

(1) Graph the ordered pairs (-3,4), (2,1), (-1,-2), (0,3) in the xy–plane.



Definition. *The* <u>*graph of an equation*</u> *is the set of all points (x,y) that satisfy the equation.*

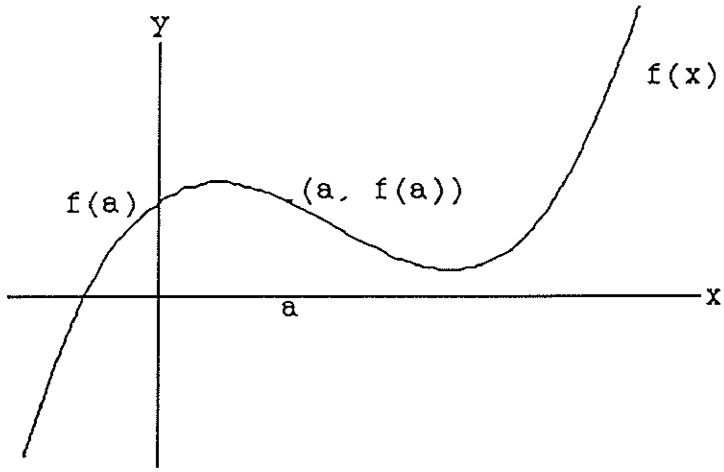
You may have seen some examples of equations and the graphs they describe in the past:

$x^2 + y^2 = 1$	A Circle
$\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$	An Ellipse
$\dfrac{x^2}{4} - \dfrac{y^2}{125} = 1$	A Hyperbola
$y = mx + b$	A Line
$y = x^2$	A Parabola

Only the last two of these are functions, and although all of them have importance we will be concentrating on only the equations that define functions.

Definition. *The* <u>*graph of a function*</u> *in the xy–plane is the set of all points (x,y) such that x is in the domain of f and y=f(x).*

Notice that the point (a,f(a)) lies a units from the origin on the x–axis and $f(a)$ units from the origin on the y–axis.

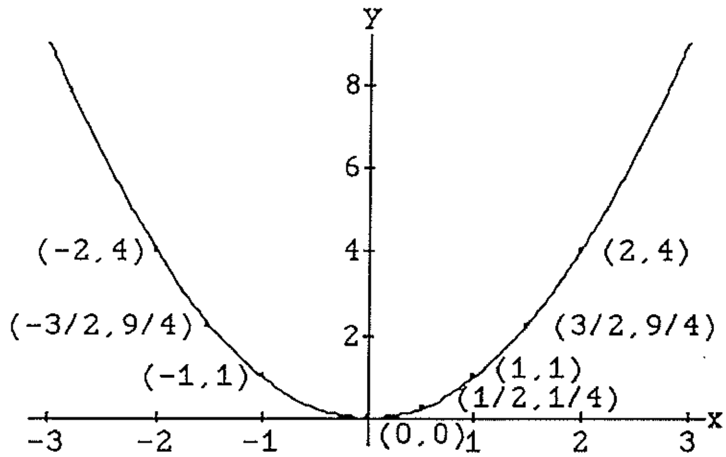


If we have a graph of an equation, we can tell it is the graph of a function if every vertical line intersects the graph in at most one point. This test is called the Vertical Line Test. Notice that the graph of a circle is not the graph of a function.

Examples.

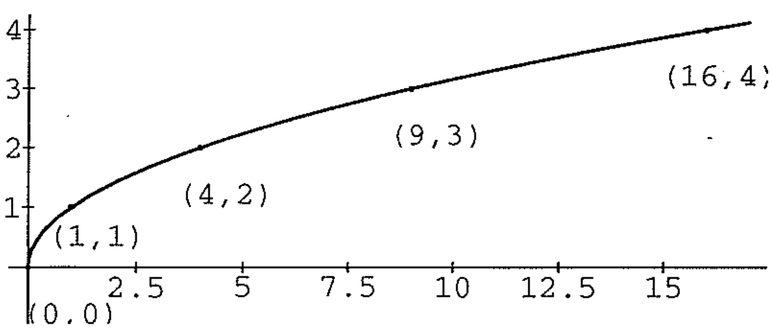
(2) Graph 6 points that will lie in the graph of the function, $y = x^2$. Use this information to guess what the shape of the entire graph is.

x	$y = x^2$
0	0
1	1
2	4
-1	1
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$



(3) Graph the function $f(x) = \sqrt{x}$.

x	$f(x) = \sqrt{x}$
0	0
1	1
4	2
9	3
16	4



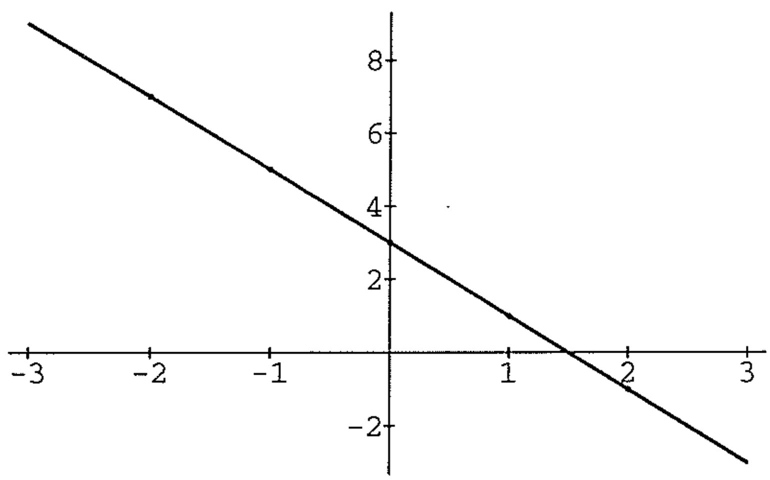
Notice that the domain is restricted to nonnegative real numbers.

In this example we chose numbers for which we could easily find the square root. You will find it helpful to choose your x–values so that calculations are easier. However, at times, just knowing the values at the integers does not give you enough information.

The smooth curve drawn through the graphed points is a guess at the shape of the entire graph of the function $f(x) = \sqrt{x}$.

(4) Graph $y = -2x + 3$

x	$f(x) = -2x + 3$
0	3
1	1
2	$-4 + 3 = -1$
-1	$2 + 3 = 5$
-2	$4 + 3 = 7$

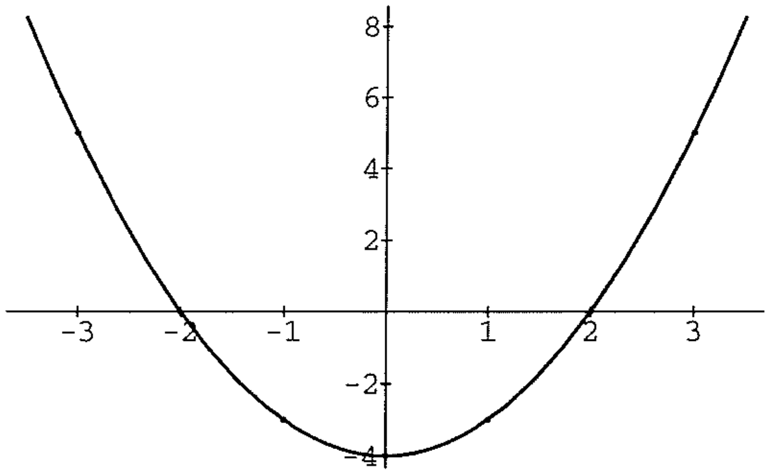


The x–intercepts of the graph of a function, f, are all values of x which make $f(x) = 0$. These numbers are also called the roots of f or the zeros of f. The y–intercept is f(0).

In examples 2 and 3, 0 is both the x–intercept and the y–intercept. In example 4 the x–intercept is $\frac{3}{2}$, for $-2(\frac{3}{2}) + 3 = 0$, and the y–intercept is f(0)=3.

(5) Graph $y = x^2 - 4$.

x	$f(x) = x^2 - 4$
0	-4
1	-3
2	0
-1	-3
-2	0
3	5
-3	5



x–intercepts

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

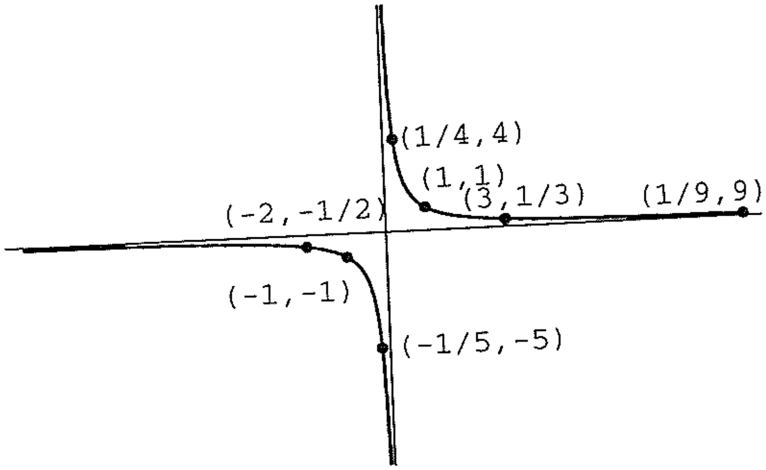
$x = 2 \qquad \text{or} \qquad x = -2$

y–intercept

$f(0) = -4$

(6) Graph $y = \frac{1}{x}$.

x	$f(x) = \dfrac{1}{x}$	x	$f(x) = \dfrac{1}{x}$
1	1	2	$\frac{1}{2}$
-1	-1	-2	$-\frac{1}{2}$
3	$\frac{1}{3}$	$\frac{1}{2}$	2
$\frac{1}{9}$	9	$-\frac{1}{5}$	-5

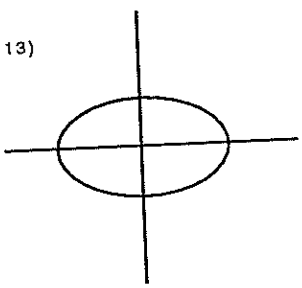


Exercises. *Graph the following. Determine the x and y–intercepts if they exist.*

1)	$y = x$	2)	$y = \|x\|$	3)	$y = 2x + 3$
4)	$y = -x + 2$	5)	$y = x^2 - 1$	6)	$f(x) = (x+2)^2 - 4$
7)	$y = \sqrt{x-3}$	8)	$y = \sqrt{x+2}$	9)	$g(x) = x^2 + 3$
10)	$h(x) = -\frac{1}{x}$	11)	$y = \frac{1}{x^2}$	12)	$y = x^3$

State whether or not the graph is the graph of a function.

13)



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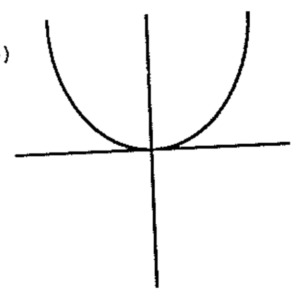

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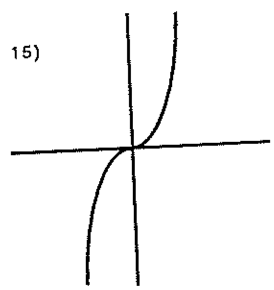

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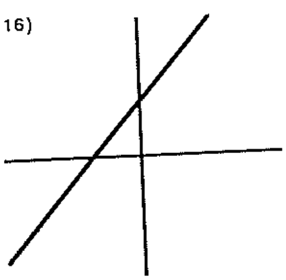

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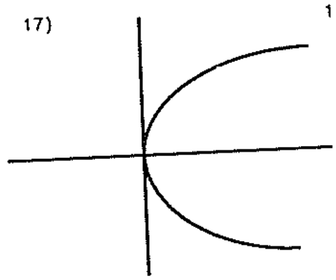

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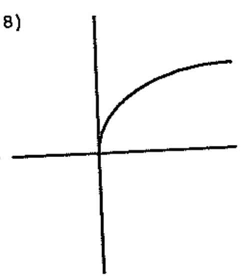


Graphing Lines

The linear function $f(x) = mx + b$ has for its graph a straight line. If (x_1, y_1) and (x_2, y_2) are any two distinct points on a nonvertical line the slope of the line is defined as

$$\text{the slope} = m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

The slope of a vertical line is undefined.
The slope of a horizontal line is zero.
If (x_1, y_1) is a point on a line with slope m the equation of the line is

$$\boxed{(y - y_1) = m(x - x_1) \qquad \text{Point–Slope Form}}$$

If b is the y–intercept of a line with slope m the equation for the line is

$$\boxed{y = mx + b \qquad \text{Slope–Intercept Form}}$$

The equation for any line can be written in the form

$$\boxed{ax + by + c = 0 \qquad \text{Standard Form}}$$

So we choose this form as the standard form for the equation of a line. Either a or b can be zero but not both.

Example:

(1) If (2,3) is on a line with slope -2 find the equation of the line in standard form and slope–intercept form.

$$\begin{aligned} (x_1, y_1) &= (2, 3) && \text{Point} \\ m &= -2 && \text{Slope} \\ (y - 3) &= -2(x - 2) && \text{Point–Slope Form} \\ y - 3 &= -2x + 4 && \\ y &= -2x + 7 && \text{Slope–Intercept Form} \\ 2x + y - 7 &= 0 && \text{Standard Form} \end{aligned}$$

(2) Find the point–slope form of the line through the points (5,2) and (-1,3).

Before we can find the point–slope form for the equation of a line we must first find the slope of the line. By definition, with $(x_1, y_1) = (5, 2)$ and $(x_2, y_2) = (-1, 3)$ we find that

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-2}{-1-5} = -\frac{1}{6}$$

Then,

$$(y-2) = -\frac{1}{6}(x-5)$$

is the point–slope form of the equation of the line through these two points.

Two lines are Parallel if they have the same slope.
Two lines are Perpendicular if the product of their slopes is -1.

(3) Find the equation of the line through the point (4,6) which is parallel to the line $x + 2y = 12$.

First write the equation of the line in y–intercept form; this will allow you to determine the slope of the line.

$$x + 2y = 12$$
$$2y = 12 - x$$
$$y = -\frac{1}{2}x + 6$$

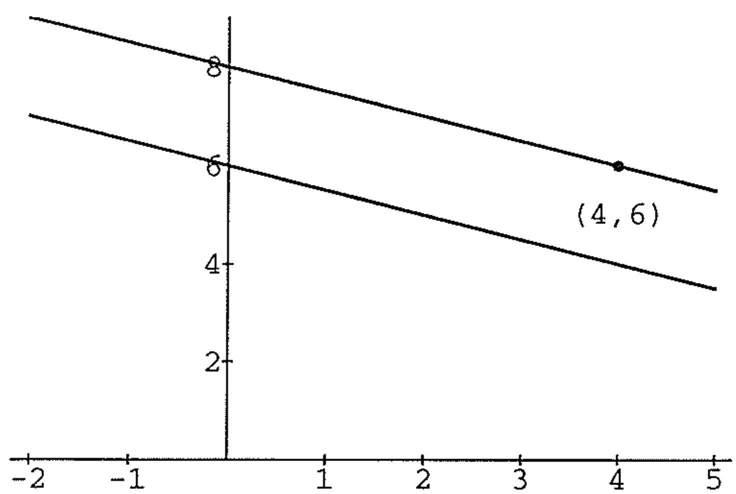
Since the line we wish to find is parallel to this line it has the same slope, namely $m = -\frac{1}{2}$.

Hence, using the point–slope form, we find that the equation of the line is

$$y - 6 = -\frac{1}{2}(x-4) \qquad \text{or} \qquad y = -\frac{1}{2}x + 8$$

So the equations of the two lines are

$$y = -\tfrac{1}{2}x + 6 \text{ AND } y = -\tfrac{1}{2}x + 8$$



(4) Find the equation of the line, l_2 through (-5,3) which is perpendicular to the line l_1, $y = 2x+4$. Graph these two lines and find their point of intersection.

The slope of l_1, $m_1 = 2$. let m_2 be the slope of l_2. Since l_1 is perpendicular to l_2,

$$2 \cdot m_2 = -1 \qquad \Rightarrow \qquad m_2 = -\frac{1}{2}$$

so the equation for l_2 is given by

$$\begin{aligned} y - 3 &= -\frac{1}{2}(x+5) \\ y &= -\frac{1}{2}x - \frac{5}{2} + 3 \\ y &= -\frac{1}{2}x + \frac{1}{2} \end{aligned}$$

using the point–slope form.

The lines intersect at the point with coordinates x and y that satisfy both equations simultaneously. Hence, we wish to find x so that both of these equations have the same value for y.

$$\begin{aligned} 2x + 4 &= -\frac{1}{2}x + \frac{1}{2} \\ 4x + 8 &= -x + 1 \\ 5x &= -7 \\ x &= -\frac{7}{5} \end{aligned}$$

For l_1 we find that

$$2(-\frac{7}{5}) + 4 = -\frac{14}{5} + \frac{20}{5} = \frac{6}{5}$$

.

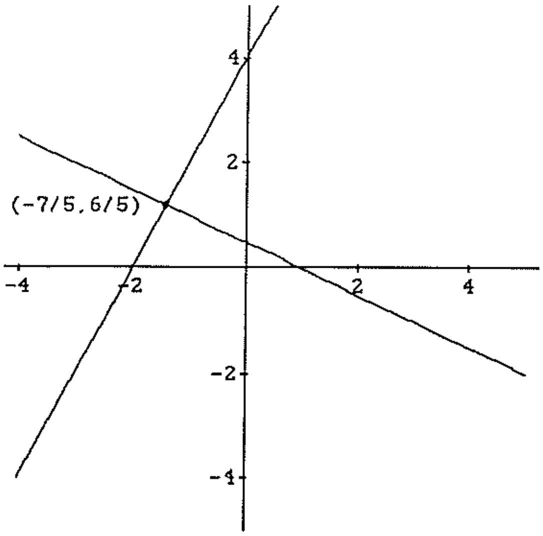
For l_2 we find that

$$-\frac{1}{2}(-\frac{7}{5}) + \frac{1}{2} = \frac{7}{10} + \frac{5}{10} = \frac{12}{10} = \frac{6}{5}$$

. Hence our math calculation above is correct. The lines intersect at the point $(x, y) = (-\frac{7}{5}, \frac{6}{5})$.

To graph these lines, we can draw a straight line through any two points that lie on them. For each line we have determined a y–intercept this will be one of our points. For both lines, we will choose the point $(-\frac{7}{5}, \frac{6}{5})$ as the second point because it lies on both of these lines. Just draw a straight line through the y–intercept and this point to graph each line.

$$y = -\tfrac{1}{2}x + \tfrac{1}{2} \text{ AND} y = 2x + 4$$



We can see from the geometry of the plane that two lines intersect in one point unless they are parallel or they are the same line. Hence, a system of two equations in two unknowns

$$\begin{aligned} ax + by + c &= 0 \\ dx + ey + k &= 0 \end{aligned}$$

has 1,0, or infinitely many solutions. A solution is any point that will satisfy both equations simultaneously.

Exercises. *Write the equation of each of the following lines in standard form.*

1) through $(1,3), m = -2$

2) through $(-4,3), m = \dfrac{3}{4}$

3) $f(0) = -1, m = \dfrac{3}{5}$

4) $f(-3) = 2, m = 0$

5) $f(0) = 4,$ undefined slope

6) through $(8,-9)$ and $(4,3)$

7) through $(-1,3)$ and $(3,4)$

8) x–intercept 3, y–intercept -2

9) Through (3,-2), parallel to $2x - y = 5$

10) $f(3) = -4,$ and f is perpendicular to $x + y = 4$

11) y–intercept 3, perpendicular to $x = 4$

Graph the following lines

12) $f(x) = 1 - x$ 13) $g(x) = 3x$ 14) $y = 3$

15) $x = 6$ 16) $f(x) = 2x - 1$ 17) $y = -\frac{1}{3}x$

18) $\frac{x}{2} + \frac{y}{3} = 1$

Find the point of intersection of the lines given. If they do not intersect or if they are the same line, state this.

19) $\begin{aligned} 5x + 10y &= 5 \\ x + y &= 4 \end{aligned}$ 20) $\begin{aligned} 2x - y &= 3 \\ 3x + y &= 7 \end{aligned}$ 21) $\begin{aligned} -4x + 2y &= 1 \\ -8x + 4y &= 2 \end{aligned}$

22) Find the line through (1,6) and perpendicular to $3x + 5y = 1$. Determine their point of intersection and graph both lines.

Graphing Quadratic Functions

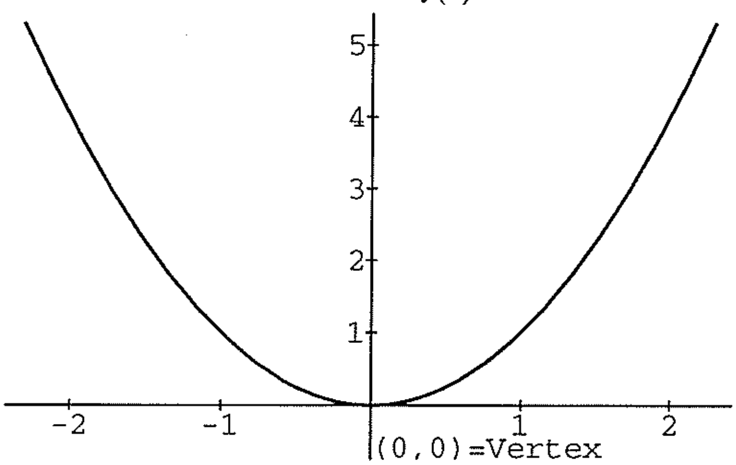
Definition. *A function f is a quadratic function if*

$$f(x) = ax^2 + bx + c$$

where $a, b,$ and c are real numbers and $a \neq 0$.

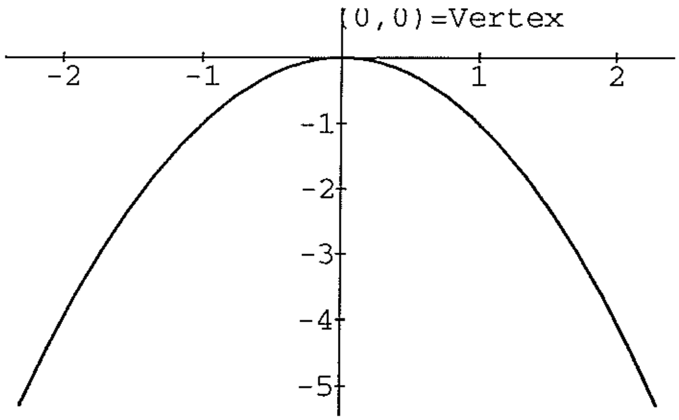
In the first section of this chapter we saw the graph of the simplest quadratic function, $f(x) = x^2$. We will work from this graph to determine the graphs of all quadratic functions. The graph of a quadratic function is called a parabola. A parabola will have a point that has the largest or smallest y–value. This value is called a maximum or respectively a minimum of the function. On a parabola, the point where this extremum occurs is called the vertex. The vertical line through this vertex is called the axis of symmetry of the parabola.

$f(x) = x^2$



The parabola $y = x^2$ has a minimum of 0 at the vertex (0,0).

$$f(x) = -x^2$$

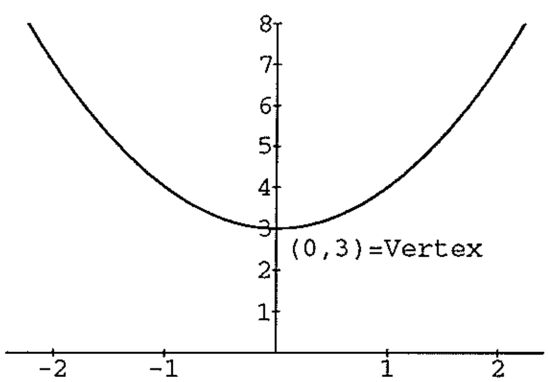


To graph of $y = -x^2$ just flip the graph of $y = x^2$ over the x-axis. This parabola has a maximum of 0 at the vertex (0,0). Plot a few points to convince yourself that this works.

To graph $y = x^2 + 3$, just move the graph of $y = x^2$ up 3 units.

x	$y = x^2 + 3$
0	3
1	4
2	7
−1	4
−2	7

$$f(x) = x^2 + 3$$

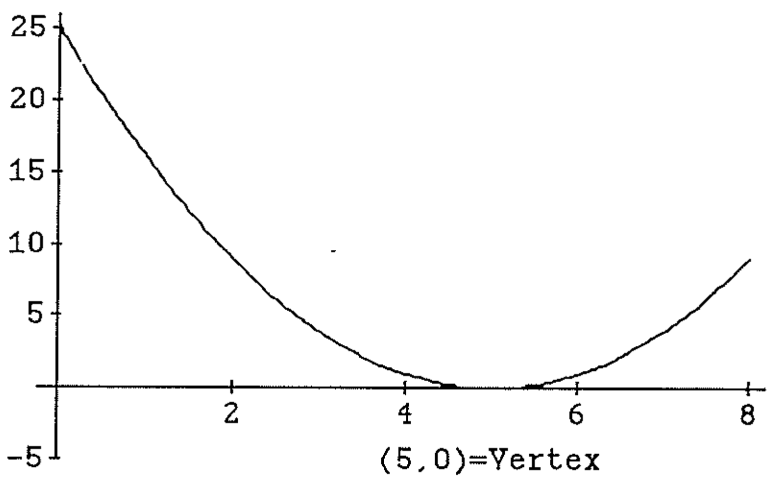


Here the minimum is 3 and it occurs at the vertex (0,3).

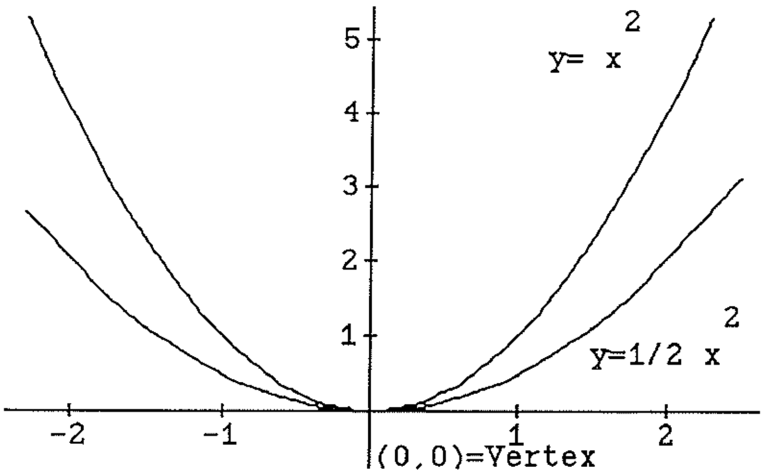
To graph $f(x) = (x-5)^2$ move the graph of $f(x) = x^2$ to the right 5 units.

x	$y = (x-5)^2$
0	25
1	16
2	9
3	4
4	1
5	0
6	1
7	4

$f(x) = (x-5)^2$



The graph $f(x) = ax^2$ is narrower than the graph of $f(x) = x^2$ if $a > 1$ and broader if $a < 1$. Here is the graph of $y = \frac{1}{2}x^2$ in comparison with $y = x^2$.



To graph the general quadratic function

$$g(x) = ax^2 + bx + c$$

(1) Complete the square

$$g(x) = a(x-h)^2 + k, \qquad \text{where } h = \frac{-b}{2a} \text{ and } k = g(h).$$

(2) The vertex is the point (h,k).
(3) The graph opens upward if $a > 0$ and downward if $a < 0$.
(4) The axis of symmetry is the vertical line x=h.
(5) The graph of $g(x)$ is narrower than the graph of $f(x) = x^2$ if $a > 1$ and broader if $a < 1$.
(6) The x–intercepts are the solutions to the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(7) The y–intercept is $y = g(0)$.

Example.

(1) Graph $f(x) = (x-3)^2 - 4$.

$$h = 3 \qquad \Rightarrow \qquad \text{Move } f(x) = x^2 \text{ to the right 3 units}$$

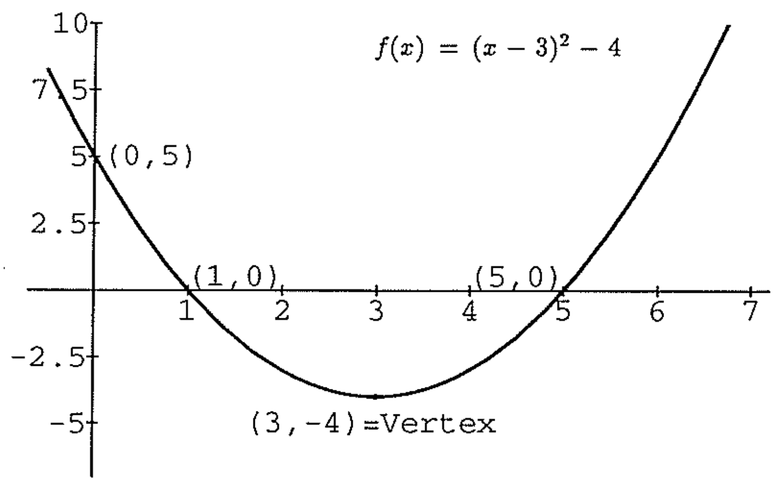
$$k = -4 \qquad \Rightarrow \qquad \text{Move } f(x) = x^2 \text{ down 4 units}$$

Vertex $= (h, k) = (3, -4)$

y–intercept $= f(0) = (0-3)^2 - 4 = 9 - 4 = 5$

x–intercepts = values of x that will make $f(x) = 0$.

$$f(x) = 0 = (x-3)^2 - 4$$
$$(x-3)^2 = 4$$
$$x - 3 = \pm 2$$
$$x = 3 \pm 2$$
$$x = 5 \text{ or } 1 = \text{ x–intercepts}$$



(2) Graph $f(x) = -2x^2 - 4x - 1$

Vertex

$$f(x) = -2x^2 - 4x - 1$$
$$h = -\frac{b}{2a} = -\frac{-4}{2(-2)} = -1$$
$$k = f(h) = f(-1) = -2(-1)^2 - 4(-1) - 1 = 1$$

Vertex is (-1,1)

Complete the Square

$$f(x) = -2(x+1)^2 + 1$$

Multiply out this form of the equation for $f(x)$ to be sure that it is the same as the original equation.

x–intercepts

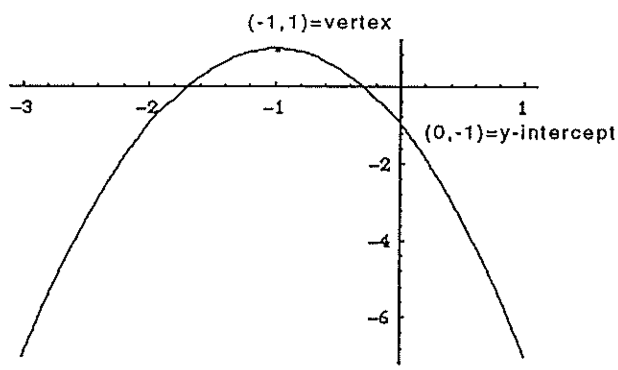
$$f(x) = 0 = -2(x+1)^2 + 1$$
$$(x+1)^2 = \frac{1}{2}$$
$$x + 1 = \pm\sqrt{\frac{1}{2}}$$
$$x = -1 \pm \sqrt{\frac{1}{2}} = \text{x–intercepts}$$

y–intercept

$$f(0) = -1 = \text{y–intercept}$$

Since $a < 0$ the parabola opens downward.

$$f(x) = -2x^2 - 4x - 1$$



Exercises. *Graph each of the following showing x–intercepts, y–intercept, and vertex.*

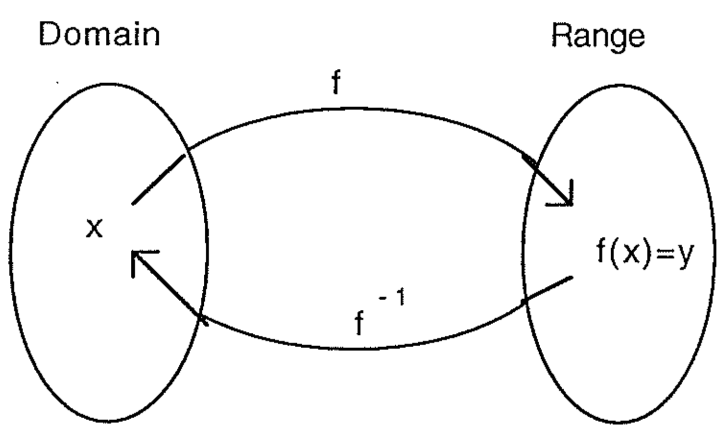
1)	$f(x) = x^2 - 3$	2)	$g(x) = -x^2 + 2$	3)	$h(x) = (x-2)^2$
4)	$y = (x+3)^2 - 4$	5)	$y = 9(x+2)^2 + 4$	6)	$y = x^2 - 2x + 3$
7)	$f(x) = x^2 - 2x + 8$	8)	$g(x) = 2x^2 - 3x - 2$	9)	$h(x) = 4x^2 - 8x + 3$

Inverse Functions

Definition. *A function is one–to–one if each element, y, in the range has only one preimage in the domain.*

If we have a graph we can determine if it is the graph of a one–to–one function by the horizontal line test. The Horizontal Line Test states that a graph is the graph of a one–to–one function if every horizontal line intersects the graph in at most one point. Nonhorizontal and nonvertical lines are graphs of one–to–one functions. Quadratic functions are not one–to–one. Do not confuse the Vertical Line Test with the Horizontal Line Test. The former tells us that the graph is a function, while the latter tells us that a graph is one–to–one.

Definition. *Let f be a one–to–one function. Then g is the Inverse Function for f if*
$(f \circ g)(x) = f(g(x)) = x$ for all x in the domain of g and
$(g \circ f)(x) = g(f(x)) = x$ for all x in the domain of f.
In this case $g = f^{-1}$, (read "f inverse.")



Example.

(1) Show that $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$ are inverse functions.

$$(f \circ g)(x) = (\sqrt[3]{x+1})^3 - 1 = x + 1 - 1 = x$$
$$(g \circ f)(x) = \sqrt[3]{(x^3 - 1) + 1} = \sqrt[3]{x^3} = x$$

Hence, by definition $g = f^{-1}$.

How to find f^{-1}.

(1) Check to see if f is one–to–one
(2) Solve $y = f(x)$ for x, obtaining $x = f^{-1}(y)$
(3) Exchange x and y
(4) Check that $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)$

The variables x and y are exchanged because the range of f is the domain of f^{-1}. When we think about the function f^{-1} we want to keep the convention that x represents a value in the domain and y represents a value in the range. This will allow us to graph f^{-1} as we graphed f.

(2) Find f^{-1} when

$$f(x) = \frac{5x+4}{3}$$

This is the equation of a nonvertical line, and being such, every horizontal line intersects it in only one point. Thus, by the horizontal line test f is a one–to–one function.

Solve for x

$$\begin{aligned} y &= \frac{5x+4}{3} \\ 3y &= 5x + 4 \\ 3y - 4 &= 5x \\ x &= \frac{3y-4}{5} \end{aligned}$$

Notice that, if you get two choices for x at this step, the function is <u>not</u> one–to–one. This would happen if f were a quadratic function.

Exchange x and y

$$y = \frac{3x-4}{5} = f^{-1}$$

Check

$$(f \circ f^{-1})(x) = \frac{5(\frac{3x-4}{5}) + 4}{3} = \frac{3x - 4 + 4}{3} = \frac{3x}{3} = x$$
$$(f^{-1} \circ f)(x) = \frac{3(\frac{5x+4}{3}) - 4}{5} = x$$

(3) Find f^{-1} for $f(x) = \sqrt{x+3}$.

$$\begin{aligned} y &= \sqrt{x+3} && \text{Solve for x} \\ y^2 &= x+3 \\ y^2 - 3 &= x \\ y &= x^2 - 3 = f^{-1}(x) && \text{Exchange x and y} \end{aligned}$$

We restrict the domain of f^{-1} to $x \geq 0$ since $f(x) \geq 0$ for all x in its domain (all $x \geq -3$). In other words, since the range of f only includes the nonnegative numbers its inverse function can only take the nonnegative numbers for its domain. The range of f is the domain of f^{-1}.

We find that the inverse function for the square root function is the squaring function restricted to the nonnegative numbers.

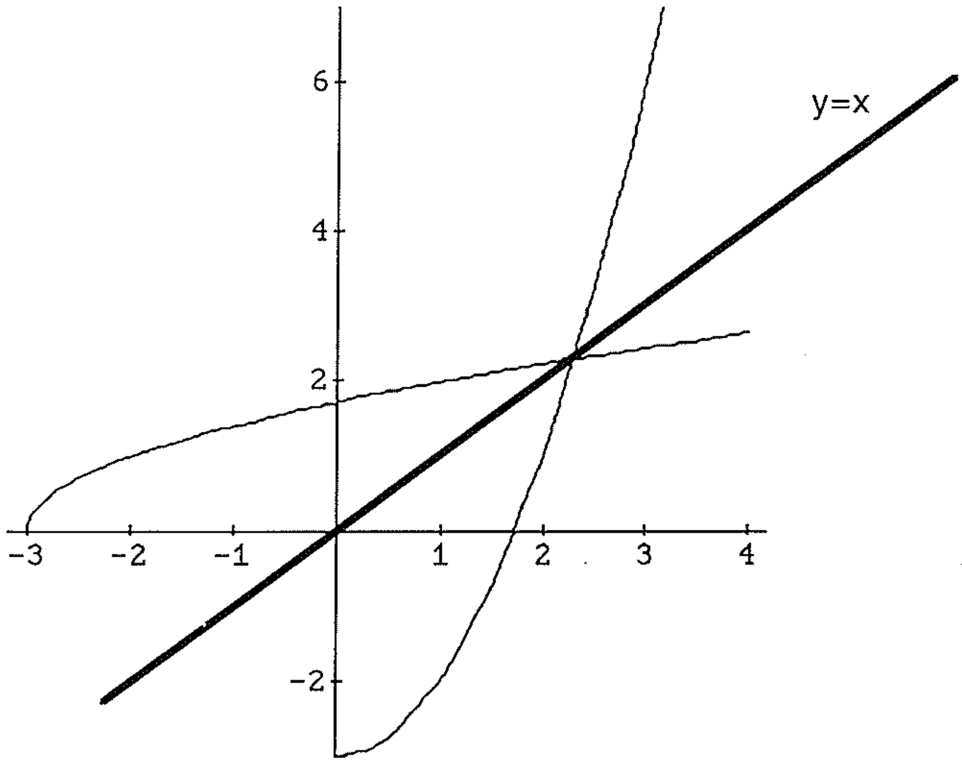
$$(f \circ f^{-1})(x) = \sqrt{x^2 - 3 + 3} = \sqrt{x^2} = |x| = x \qquad \text{when } x \geq 0$$

$$(f^{-1} \circ f)(x) = (\sqrt{x+3})^2 - 3 = x + 3 - 3 = x \qquad \text{Check}$$

The graphs of these two functions follow. Notice that if we fold the paper along the line $y = x$ the two graphs will lie on top of each other. This feature is called being symmetric about the line $y = x$.

$$\boxed{f \text{ and} f^{-1} \text{ are symmetric about the line } y = x}$$

$f(x) = \sqrt{x+3}$ AND $f^{-1}(x) = x^2 - 3$ WITH $x \geq 0$



Exercises. *For each of the following functions find the inverse or state that the function is not one–to–one. Graph each of the functions and their inverses if they exist.*

1)	$f(x) = 3x - 2$	2)	$f(x) = \dfrac{1}{x}$	3)	$f(x) = \dfrac{3-x}{2}$
4)	$f(x) = \dfrac{2}{1-x}$	5)	$g(x) = \sqrt{x-2}$	6)	$h(x) = x^2 + 2$
7)	$f(x) = x^3 - 3$	8)	$f(x) = \lvert x \rvert$	9)	$g(x) = \sqrt{x+5}$
10)	$f(x) = (x+2)^2 + 1$	11)	$h(x) = \dfrac{5}{x}$	12)	$f(x) = -\sqrt{x}$

CHAPTER IV

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Exponential Functions

Definition. *A function of the form*

$$f(x) = a^x \qquad a > 0, a \neq 1$$

is an *Exponential Function with Base a.*

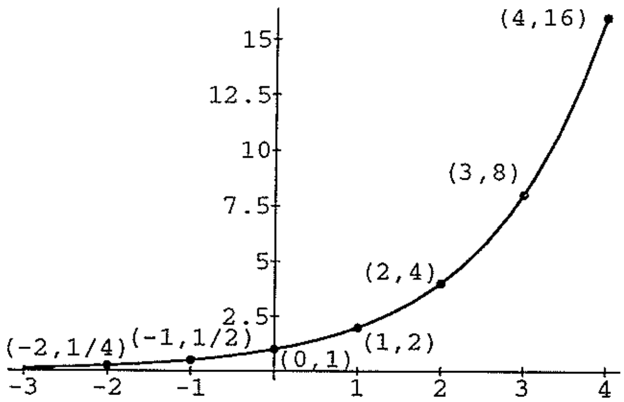
To graph exponential functions we plot some points and fill in the curve that they approximate. Like quadratic functions the form of the graph of all exponential functions is determined once we have the basic form of an exponential function.

Examples.

(1) Here we see the shape of the exponential function $y = 2^x$. Notice that the output or range of this function has only positive numbers.

x	$y = 2^x$	x	$y = 2^x$
0	$2^0 = 1$	3	$2^3 = 8$
1	$2^1 = 2$	4	$2^4 = 16$
2	$2^2 = 4$	-1	$2^{-1} = \frac{1}{2}$
$\frac{1}{2}$	$2^{\frac{1}{2}} = \sqrt{2}$	-2	$2^{-2} = \frac{1}{4}$

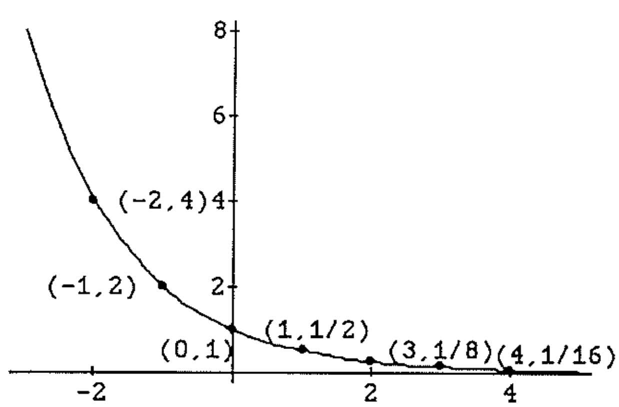
$$y = 2^x$$



(2) Here we just reflect the graph above through the y–axis to obtain the graph of $y = 2^{-x}$.

x	$y = 2^{-x}$	x	$y = 2^{-x}$
0	$2^0 = 1$	3	$2^{-3} = \frac{1}{8}$
1	$2^{-1} = \frac{1}{2}$	4	$2^{-4} = \frac{1}{16}$
2	$2^{-2} = \frac{1}{4}$	-1	$2^1 = 2$
$\frac{1}{2}$	$2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$	-2	$2^2 = 4$

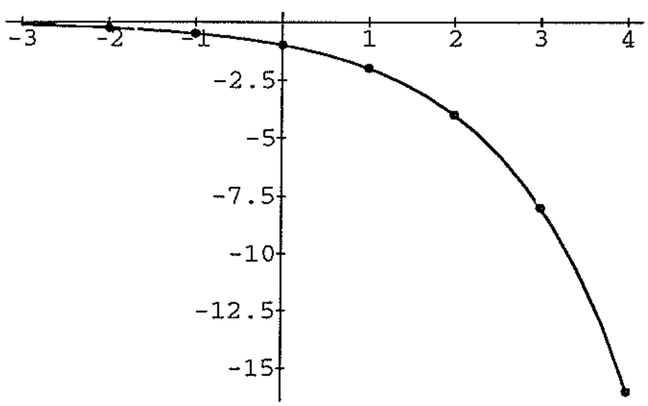
$$y = 2^{-x}$$



(3) To obtain the graph of $y = -2^x$ we flip (or reflect) the graph of $y = 2^x$ over the x–axis.

x	$y = -2^x$	x	$y = -2^x$
0	$-2^0 = -1$	3	$-2^3 = -8$
1	$-2^1 = -2$	4	$-2^4 = -16$
2	$-2^2 = -4$	-1	$-2^{-1} = -\frac{1}{2}$
$\frac{1}{2}$	$-2^{\frac{1}{2}} = -\sqrt{2}$	-2	$-2^{-2} = -\frac{1}{4}$

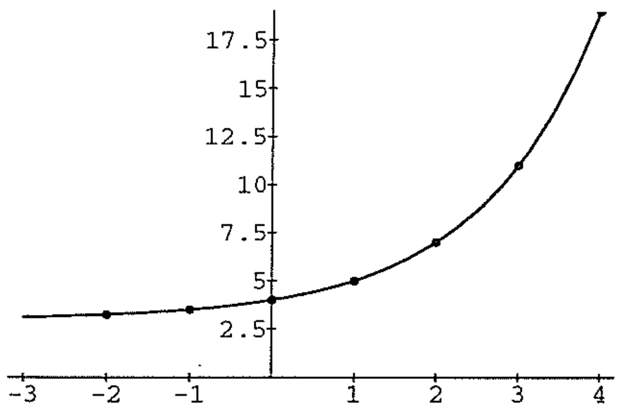
$y = -2^x$



(4) If we translate the whole graph of $y = 2^x$ up three units we get the graph of $y = 2^x + 3$

x	$y = 2^x + 3$	x	$y = 2^x + 3$
0	$2^0 + 3 = 4$	3	$2^3 + 3 = 11$
1	$2^1 + 3 = 5$	4	$2^4 + 3 = 19$
2	$2^2 + 3 = 7$	-1	$2^{-1} + 3 = \frac{1}{2} + 3 = \frac{7}{2}$
$\frac{1}{2}$	$2^{\frac{1}{2}} + 3 = \sqrt{2} + 3$	-2	$2^{-2} + 3 = \frac{1}{4} + 3 = \frac{13}{4}$

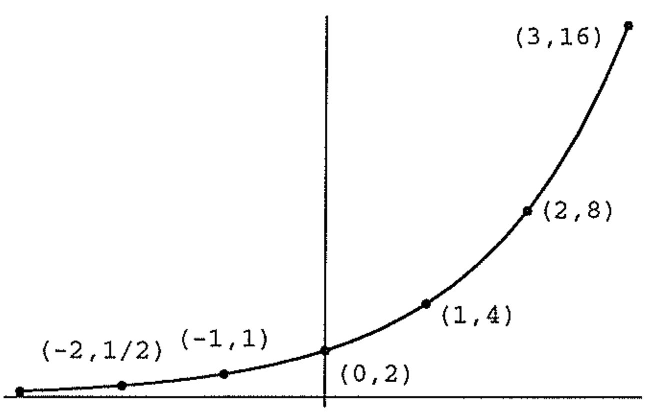
$$y = 2^x + 3$$



(5) Finally, if we move the graph of $y = 2^x$ to the left one unit we get the graph of $y = 2^{x+1}$

x	$y = 2^{x+1}$	x	$y = 2^{x+1}$
0	$2^{0+1} = 2$	3	$2^4 = 16$
1	$2^2 = 4$	4	$2^5 = 32$
2	$2^3 = 8$	-1	$2^{-1+1} = 2^0 = 1$
$\frac{1}{2}$	$2^{\frac{3}{2}} = \sqrt{2}^3$	-2	$2^{-1} = \frac{1}{2}$

$$y = 2^{x+1}$$



Notice that the same sort of translations and transformations occur that occurred with the quadratic functions. This pattern is true in general. Given a function y=f(x), the graphs of each of the following functions will be a transformation of the graph of $y = f(x)$ as listed. Assume that $a, b \geq 0$

$y = f(x) + a$	Up a units
$y = f(x) - a$	Down a units
$y = f(x + b)$	Left b units
$y = f(x - b)$	Right b units
$y = -f(x)$	Reflect over x–axis
$y = f(-x)$	Reflect over y–axis

Using these translations and transformations you can determine much about the shapes of the graphs of many functions. You just need to learn a few of the basic graphs such as lines, parabolas and exponential functions.

When solving equations with the variable in the exponent, find a common base and set the exponents equal to each other.

$$\boxed{a^x = a^y \quad \Longleftrightarrow \quad x = y}$$

(6)

$$\begin{aligned}
8^{3x+2} &= 2^3 \\
(2^3)^{3x+2} &= 2^3 \\
2^{3(3x+2)} &= 2^3 && \text{Find a common base} \\
3(3x+2) &= 3 && \text{Set exponents equal} \\
9x + 6 &= 3 \\
9x &= -3 \\
x &= -\frac{1}{3}
\end{aligned}$$

Since a common base is sometimes impossible to find we will introduce another technique using logarithms in the next section.

Exercises. *Graph each of the following functions.*

1)	$f(x) = 3^x$	2)	$f(x) = 3^{-x}$	3)	$y = 3^x + 2$
4)	$g(x) = 3^{x+4}$	5)	$f(x) = -3^{x-3}$	6)	$y = 2^{x-1} + 5$
7)	$y = -4^x$	8)	$f(x) = 2^{-x} + 2$	9)	$f(x) = 3^{-x+4}$

Solve each of the following equations

10)	$3^{2x+1} = 3^4$	11)	$3^x = 9^2$	12)	$(\frac{1}{3})^2 = 3^{2x+1}$
13)	$(\frac{2}{3})^4 x = \frac{9}{4}$	14)	$125 = 5^{x+3}$	15)	$64^{x+2} = 4^{3x+4}$

Logarithmic Functions

Recall that if f is a one–to–one function we can find an inverse function f^{-1} for f so that the compositions $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$. You already have a good understanding of common inverse functions in mathematics: addition and subtraction, multiplication and division, powers and roots. Please check the following table:

$f(x)$	f^{-1}
$f(x) = x + 5$	$f^{-1} = x - 5$
$f(x) = 5x$	$f^{-1} = \dfrac{x}{5}$
$f(x) = x^5$	$f^{-1} = \sqrt[5]{x}$
$f(x) = 5^x$	$f^{-1} = ?$

We see that the inverses are exactly what we expect them to be. However, we do not yet have notation for the inverse of the function $f(x) = 5^x$. Hence we make the following definition.

Definition. *If $a > 0$, $a \neq 1$, and $x > 0$, then $f(x) = \log_a x$ is* <u>*The Logarithmic Function with Base a*</u> *and satisfies the conditions*

$$\boxed{\log_a a^x = x \qquad \text{and} \qquad a^{\log_a x} = x}$$

Thus if $g(x) = a^x$ and $f(x) = \log_a x$ then f and g are inverse functions.

Directly from this definition we obtain the formula,

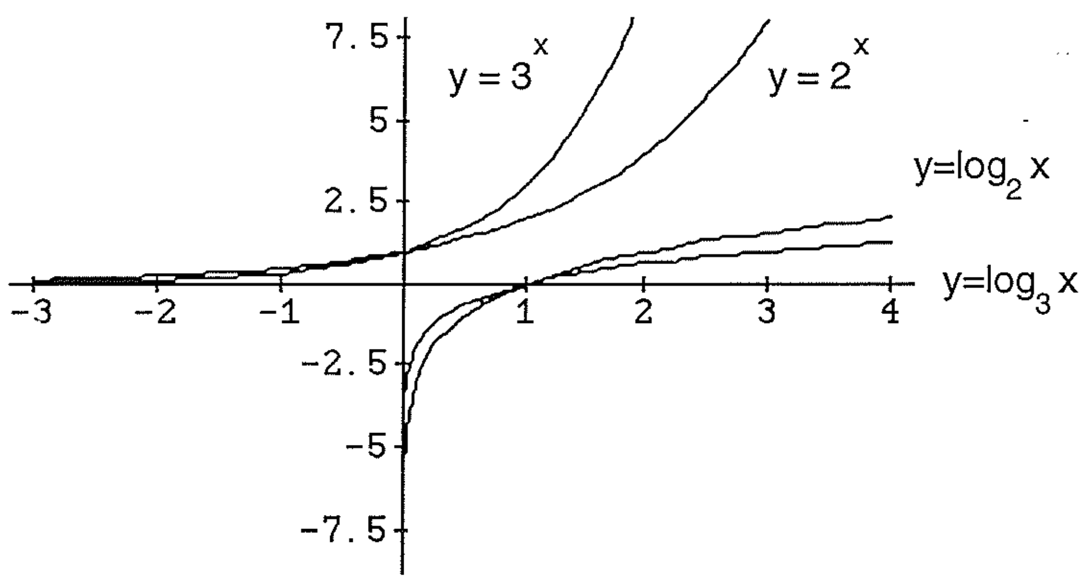
$$\boxed{\underbrace{\log_a x = m}_{\text{logarithmic form}} \iff a^{\log_a x} = a^m \iff \underbrace{x = a^m}_{\text{exponential form}}}$$

The left hand side is called the logarithmic form of the equation and the right hand side is called the exponential form of the equation. To get from the right hand side of the equation to the left hand side we exponentiate both sides of the logarithmic form of the equation. The theme remains the same; do the same thing to both sides of an equation and it remains an equation.

We immediately restrict the domain of the logarithmic function to the positive real numbers because it is the inverse function for the exponential function which only has positive numbers in its range. The domain of the inverse of a function is the range of the original function. (The set of positive numbers does not include zero; logarithmic functions are not defined at zero.)

We can determine the graph of a logarithmic function either by plotting points, as we have done in the previous sections, or we can recognize that it is the inverse of the exponential function and reflect the graph of the exponential function over the line $y = x$ to obtain the graph of the logarithmic function. Here are the graphs of the functions $y = 2^x, y = 3^x, y = log_2 x$, and $log_3 x$.

$y = 2^x, y = 3^x, y = \log_2 x$, AND $y = \log_3 x$



Examples.

(1)

$$\log_5 5^x = x$$

(2)

$$\log_3 3^{2x^2+1} = 2x^2 + 1$$

(3)

$$10^{\log_{10}(3x^2+1)} = 3x^2 + 1$$

To solve equations with variables in the exponent take the appropriate logarithm of both sides. (From here on I will abbreviate logarithm with log.)

To solve equations with logs of variables, exponentiate both sides.

Solve the following equations.

(4) In this example we take the log of both sides to get rid of the exponent. This is the same idea as multipling through by the denominator to get rid of fractions. Remember as long as you do the same thing to both sides of an equation it remains an equation.

$$\begin{aligned} 3^{2x+1} &= 9 \\ \log_3 3^{2x+1} &= \log_3 9 && \text{Take logs of both sides} \\ 2x + 1 &= \log_3 9 = \log_3 3^2 = 2 && \text{Definition} \\ 2x &= 2 - 1 = 1 \\ x &= \frac{1}{2} \end{aligned}$$

(5) Here we exponentiate both side to get rid of the log.

$$\begin{aligned} \log_7(3x + 2) &= 2 \\ 7^{\log_7(3x+2)} &= 7^2 && \text{Exponentiate both sides} \\ 3x + 2 &= 49 && \text{Definition} \\ 3x &= 47 \\ x &= \frac{47}{3} \end{aligned}$$

Notice that we can jump directly from the original equation to the third equation by just changing the equation from logarithmic form to exponential form. Carefully convince yourself that this is the case and then analyze how the second step implies the third. This is the step that makes understanding logarithms so difficult; primarily because it requires students to fundamentally understand the inverse relationship between exponentiation and logarithms.

Exercises. *Evaluate each of the following expressions. Assume that all variables are positive.*

1)	$\log_4 16$	2)	$\log_2 \sqrt{2}$	3)	$\log_3 \frac{1}{9}$
4)	$3^{\log_3 256}$	5)	$\log_2 \frac{\sqrt[3]{4}}{2}$	6)	$\log_5 5^{3x^5-6x+2}$

Solve the following equations. Assume that all variables are positive

7)	$2^{x^2+4x} = 16$	8)	$3^{x^2-3x} = 1$	9)	$2^{4x+8} = \frac{1}{2}$
10)	$\log_5 (2x+5) = 2$	11)	$\log_3 (6x^2 - x + 2) = 2$		

Graph the following pairs of functions on the same coordinate axis.

12) $y = 2^x$
$y = \log_2 x$

13) $y = 4^x$
$y = \log_4 x$

Logarithmic Equations

As with exponents we have certain rules for logs. These rules come directly from the definition of log and the corresponding rules for exponents. Look back on the rules for exponents and see if you can see a correspondence.

Rules for logarithms. All variables represent positive real numbers.

i)	$\log_b b = 1$	*ii)*	$\log_b 1 = 0$
iii)	$\log_b(xy) = \log_b x + \log_b y$	*iv)*	$\log_b x^t = t \log_b x$
v)	$\log_b(\frac{x}{y}) = \log_b x - \log_b y$		

Proof.
i) Since $b^1 = b$ we have $\log_b b^1 = 1$ by definition.
ii) Since $b^0 = 1$ we have $\log_b 1 = \log_b b^0 = 0$ by definition.
iii) Let $m = \log_b x$ and $n = \log_b y$ then,

$$\begin{aligned} x = b^m \quad \text{and} \quad y &= b^n && \text{Exponential form} \\ \log_b(xy) &= log_b(b^m b^n) && \text{Substitution using equations above} \\ &= \log_b(b^{m+n}) && \text{Rule i) for exponents} \\ &= m + n && \text{Definition of log} \\ &= \log_b x + \log_b y && \text{Definition of m and n above} \end{aligned}$$

We see that the rules for logarithms come directly from the rules for exponents and the definition of log.
iv)

$$\begin{aligned} \log_b x^t &= \log_b \underbrace{x \cdot x \cdot \ldots \cdot x}_{\text{t times}} \\ &= \log_b x + \log_b x + \cdots + \log_b x && \text{by iii)} \\ &= t \cdot \log_b x \end{aligned}$$

Although this proof only works when "t" is a positive integer, iv) is nonetheless true for all real numbers t. (Please see following examples when "t" is not a positive integer.)

v) The proof for v) is similar to the proof of iii) and is left as an exercise.

A common error for students who are just starting to understand logarithms is to apply rules that aren't rules at all. Here are a few examples:

$$\frac{\log_b x}{\log_b y} \neq \log_b x - \log_b y$$

$$\frac{\log_b x}{\log_b y} \neq \frac{x}{y}$$

Please be careful to only apply rules that are actually rules.

Examples. *Assume that all variables are positive.*

(1)

$$\log_3 \frac{9}{2} = \log_3 9 - \log_3 2 = \log_3 3^2 - \log_3 2 = 2 - \log_3 2$$

(2)

$$\log_{10} 30000 = \log_{10}(3 \cdot 10000) = \log_{10} 3 + log_{10}10000 = \log_{10} 3 + 4$$

(3)

$$\begin{aligned}\log_a \frac{m^9 n^3}{q^6} &= \log_a m^9 + \log_a n^3 - \log_a q^6 \\ &= 9 \log_a m + 3 \log_a n - 6 \log_s q\end{aligned}$$

Solve the following equations.

(4)

$$\begin{aligned}\log_2 8 &= x \\ \log_2 2^3 &= x \\ 3 &= x\end{aligned}$$

(5)

$$\begin{aligned}\log_x 36 &= -2 && \text{Logarithmic form} \\ x^{\log_x 36} &= x^{-2} && \text{Exponentiate both sides} \\ 36 = x^{-2} &= \frac{1}{x^2} && \text{Exponential form} \\ x^2 &= \frac{1}{36} \\ x &= \pm\sqrt{\frac{1}{36}} = \pm\frac{1}{6}\end{aligned}$$

But, $x > 0$ so $\frac{1}{6}$ is the only correct answer.

Generally we cannot determine the log with arbitrary base of some arbitrary number without laborious hand calculations. Hence, scientists have chosen two standard values to use as bases, 10 and $e \simeq 2.712$. The reason for using the number e will become clear when you take calculus. For our purposes 10 will suffice as a general base.

$$\boxed{\log_{10} x = \log x}$$

In the past, mathematicians and scientists used slide rules and tables of logarithms to determine the base 10 log of any number. Most precalculus and calculus texts have tables giving $\log_{10} x$, abbreviated $\log x$, for decimal values of x between 1 and 10. Fortunately, scientific calculators have a "log" key that will determine the base 10 log of any number. It would be nice for you to own such a calculator in your upcoming calculus courses so it would not be a bad investment now. As for this booklet, it will be sufficient for you to leave your answers in logarithmic form. The Algebra Diagnostic Exam given here at U.C. Santa Barbara does not allow calculators.

(6)

$$\begin{aligned}
5^{x+1} &= 6 \\
\log 5^{x+1} &= \log 6 && \text{Take logs of both sides} \\
(x+1)\log 5 &= \log 6 && \text{Rule iv)} \\
x+1 &= \frac{\log 6}{\log 5} \\
x &= \frac{\log 6}{\log 5} - 1 = \frac{0.7781513}{0.69897} - 1 = 0.1132828
\end{aligned}$$

This will be the last time that I show the decimal representations. It is instructive to notice that for numbers between one and ten the base ten log of that number is between zero and one.

(7)

$$\begin{aligned}
10^{x^2} &= 200 \\
\log 10^{x^2} &= \log 200 && \text{Take logs of both sides} \\
x^2 &= \log 200 = \log(2 \cdot 100) = \log 2 + \log 100 = 2 + \log 2 \\
x &= \pm\sqrt{2 + \log 2}
\end{aligned}$$

(8)

$$\begin{aligned}
5^x &= 12 \\
\log 5^x &= \log 12 && \text{Take logs of both sides} \\
x \log 5 &= \log 12 \\
x &= \frac{\log 12}{\log 5}
\end{aligned}$$

(9) We use the same principles to solve equations using logs that we did to solve equations with less complicated coefficients. Writing $\log 6$ is just a another way to write the real number 0.7781513.

$$\begin{aligned}
6^{2x+1} &= 4^{x+4} & \\
\log 6^{2x+1} &= \log 4^{x+4} & \text{Take logs of both sides} \\
(2x+1)\log 6 &= (x+4)\log 4 & \text{Rule iv)} \\
2x\log 6 + \log 6 &= x\log 4 + 4\log 4 & \text{Distribute} \\
2x\log 6 - x\log 4 &= 4\log 4 - \log 6 & \text{Combine like terms} \\
x(2\log 6 - \log 4) &= 4\log 4 - \log 6 & \text{Factor} \\
x &= \frac{4\log 4 - \log 6}{(2\log 6 - \log 4)} & \text{Divide} \\
x &= \frac{\log \frac{128}{3}}{\log \frac{36}{4}} & \\
x &= \frac{\log \frac{128}{3}}{\log 9} &
\end{aligned}$$

(10) In this example, before we can exponentiate to get rid of the logs we must combine them into one log, for our rules say nothing about what to do with a sum in the exponent.

$$\begin{aligned}
\log_a(x+2) - \log_a(x-4) &= \log_a x \\
\log_a\left(\frac{x+2}{x-4}\right) &= \log_a x \\
\frac{x+2}{x-4} &= x \\
x+2 &= x(x-4) \\
0 &= x^2 - 4x - x - 2 \\
0 &= x^2 - 5x - 2 \\
x &= \frac{5 \pm \sqrt{25 - 4\cdot(-2)}}{2} = \frac{5}{2} \pm \frac{\sqrt{33}}{2}
\end{aligned}$$

Exercises. *Write the following as a sum or difference of logarithms, when possible.*

1) $\log_k \frac{mn^2}{p}$ 2) $\log_2 \frac{2\sqrt{5}}{3}$ 3) $\log_4(2x+5y)$

Write the following sums and differences of logs as a single log when possible.

4) $\log 3 + 3\log 2 - 7\log 8$ 5) $\log_a m - 3\log_a 2 + 5\log_a p$

6) $\log_2 8 - \log_2 7 - \log_2 6$

Solve the following equations.

7) $3^x = 10$ 8) $10^k - 1 = 4$ 9) $4^{3m-1} = 12^{m+2}$

10) $2^{x^2-1} = 12$ 11) $\log(x+2) + \log(2x+1) = \log x$

12) $\log_2(\log_2 x) = 1$ 13) $5^{x \log_5 2} = 10$ 14) $logx^2 = (logx)^2$

Evaluate the following expressions.

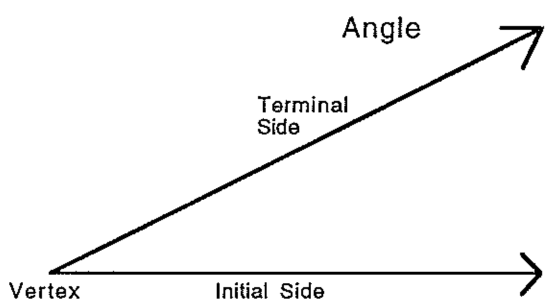
15) $10^{2 \log 3}$ 16) $\log_5 125^2$ 17) $3^{2 \log_3 9 - \log_3 27}$

CHAPTER V

TRIGONOMETRY

Angles and Angle Measurement

An <u>angle</u> is a figure determined by two <u>rays</u> emanating from a single point called the <u>vertex</u>.

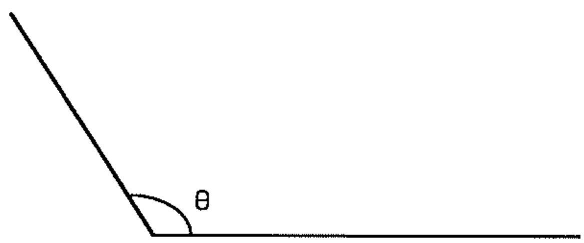


When we wish to measure an angle we place it so that one side lies on the positive x–axis. This edge is called the <u>initial side</u>. We create an angle with positive measure by rotating the <u>terminal side</u> counterclockwise from the initial side. An angle with negative measure is created by rotating the terminal side clockwise to create the angle. We can measure angles in <u>revolutions</u>, <u>degrees</u>, or <u>radians</u>.

$$\boxed{1 \text{ revolution} = 360^\circ = 2\pi \text{ radians}}$$

We often use the Greek letters, Theta, θ; Phi, ϕ; Alpha, α; and Beta, β to denote the measure of an angle.

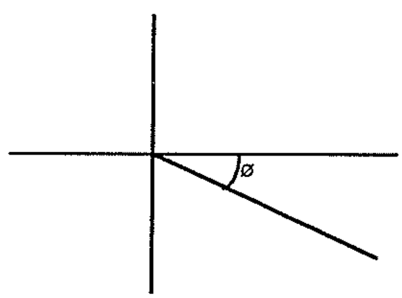
120°



$$\theta = \frac{1}{3} rev = 120° = \frac{2\pi}{3} radians$$

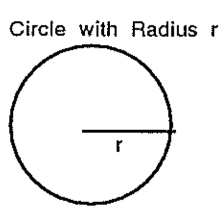
The units for radians are usually dropped and "°" is the symbol for degrees.

−30°



$$\phi = -30° = -\frac{\pi}{6}$$

Before we can continue we need to review some formulas from geometry. In what follows we will keep the conventions:



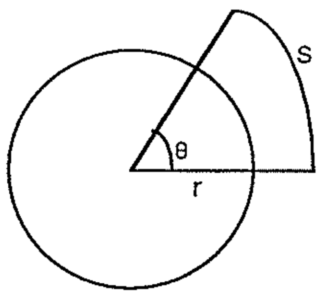
r denotes the radius of a circle.
θ denotes the measure of an angle in radians.
s denotes the arc length described by an angle around the edge of a circle.

Then we get the following formulas:

$$\text{Circumference} = 2\pi r = \text{ length of the edge of a circle}$$

$$\text{Area of a circle} = \pi r^2$$

$$\theta = \frac{s}{r} \qquad \text{This formula only works if } \theta \text{ is in radians.}$$



Definition. *A* <u>*Unit*</u> <u>*Circle*</u> *is a circle with radius one.*

A radian is a convenient unit of measure on a unit circle because in this case $\theta = s$; that is, the radian measure of an angle is equal to the length of the arc described by that angle.

When converting from radians to degrees and vice versa, a fraction with the degree measure of the angle in the numerator and the radian measure of the angle in the denominator will have the same value regardless of our choice of original angle.

$$\frac{360^\circ}{2\pi} = \frac{180^o}{\pi} = \frac{90^o}{\pi/2} = \frac{\text{measure of any angle in degrees}}{\text{measure of same angle in radians}}$$

The ratio using a full revolution, a half revolution, or any angle will always be the same.

Examples.

(1) Convert 45° to radians.
Set up the ratios:

$$\frac{360^o}{2\pi} = \frac{45^o}{\theta}$$

Solve for θ by cross multipying (i.e. multiply through by the denominators) to get

$$360 \cdot \theta = 2\pi \cdot 45$$

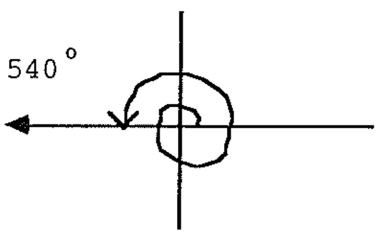
$$\theta = \frac{90\pi}{360} = \frac{\pi}{4}$$

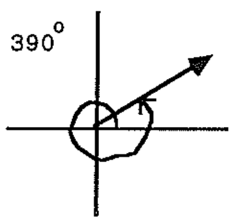
I choose the ratio $\frac{360}{2\pi}$ because I find it easy to remember that there are 360° in one revolution as well as 2π radians in one revolution. You can use any one of the ratios.

(2) Convert $\frac{\pi}{6}$ to degrees:

$$\begin{aligned}\frac{360}{2\pi} &= \frac{\theta}{\pi/6}\\ 360\cdot\frac{\pi}{6} &= 2\pi\theta\\ \frac{360\pi}{6\cdot 2\pi} &= \theta\\ \theta &= 30^\circ\end{aligned}$$

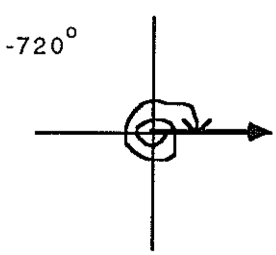
Although we generally think of angles as being less than 360°, we can generalize the concept of angle measurement beyond the 360° or 2π radian boundary, even into negative numbers of arbitrary size. For instance, we can say that a car that has spun out, making one and a half turns, has done a 540°. Mathematicians depict such an angle like this.

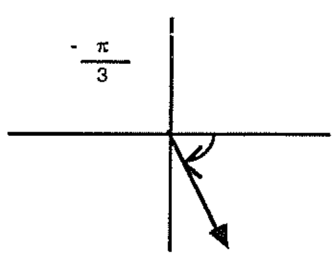


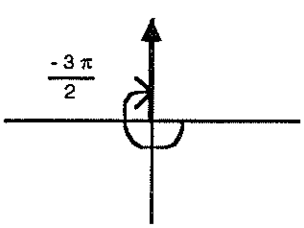


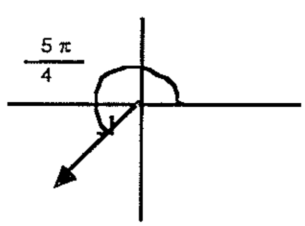
Notice that the 390° angle's terminal side rests in the same place as a 30° angle's terminal side and a −330° angle's terminal side.

Here are some other angles.









Exercises. *Convert to radian measure. Express your answer in terms of π. Sketch each angle.*

1)	$30°$	2)	$120°$	3)	$-60°$
4)	$720°$	5)	$-72°$	6)	$-50°$
7)	$-45°$	8)	$90°$	9)	$-240°$

Convert to degrees. Sketch each angle.

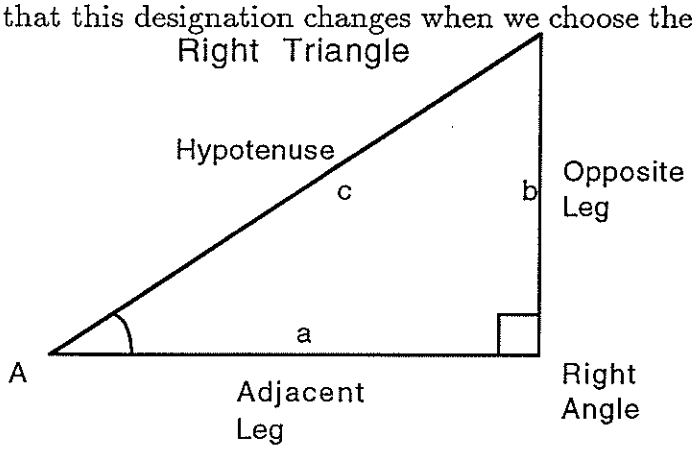
10)	$\frac{\pi}{4}$	11)	$\frac{3\pi}{2}$	12)	$\frac{5\pi}{6}$
13)	$\frac{5\pi}{4}$	14)	π	15)	$\frac{7\pi}{6}$

Find three angles that have the same terminal side location as the angle with the given measure.

16)	$45°$	17)	$180°$	18)	$-\frac{\pi}{2}$

Trigonometric Functions of Acute Angles

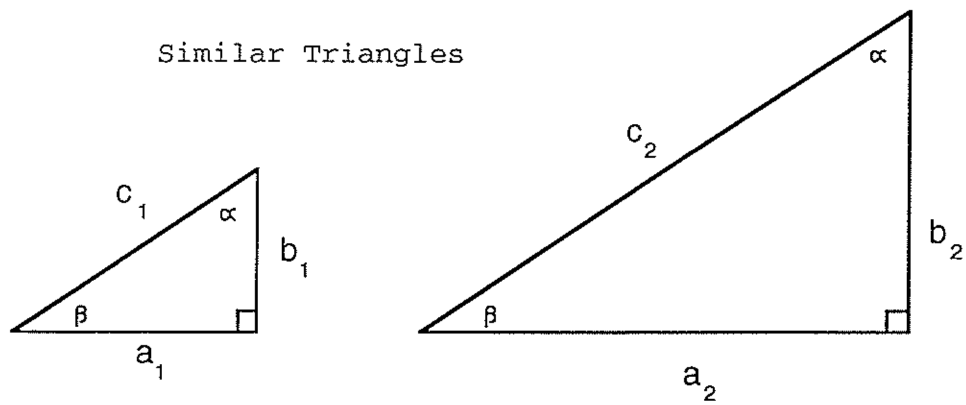
An Acute angle is an angle that has measure less than 90°. Before we can define the trigonometric functions we need to recall a little about the anatomy of a triangle. In any triangle the sum of the angle measures is 180° or π radians. If one of the angles is a right angle (a 90° angle), the triangle is a right triangle. We denote a right angle with a box. The longest side of a right triangle is always opposite the right angle and is called the hypotenuse. The other two sides are called legs. If we choose an angle, say angle A, we call the closest leg the adjacent leg and the side opposite the angle the opposite leg. Notice that this designation changes when we choose the other acute angle as our reference angle.



Pythagorean Theorem. *The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.*

$$a^2 + b^2 = c^2$$

We say that two triangles are similar if their angles have the same measures.

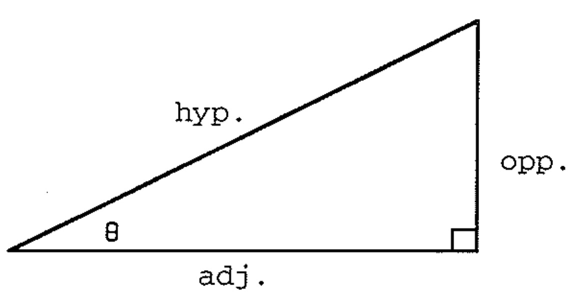


In this case there is a constant, k, so that.

$$ka_1 = a_2 \qquad \frac{a_1}{b_1} = \frac{a_2}{b_2}$$
$$kb_1 = b_2 \qquad \text{and} \qquad \frac{a_1}{c_1} = \frac{a_2}{c_2}$$
$$kc_1 = c_2 \qquad \frac{b_1}{c_1} = \frac{b_2}{c_2}$$

First Definition of Trigonometric Functions. We are now ready to make our first definition of the trigonometric functions. These definitions define the trigonometric functions as ratios of the lengths of the sides of right triangles.

Definition of the trigonometric functions as Ratios of the lengths of sides of triangles



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

One way to remember these is the acronym "SOH CAH TOA"
Sine is Opposite over Hypotenuse
Cosine is Adjacent over Hypotenuse
Tangent is Opposite over Adjacent

The other three trigonometric functions are just the reciprocals of these three and can be determined easily once we know the value of their reciprocal functions.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

These formulas must be memorized. The rest of the information in this chapter is dependent on your familiarity with these definitions.

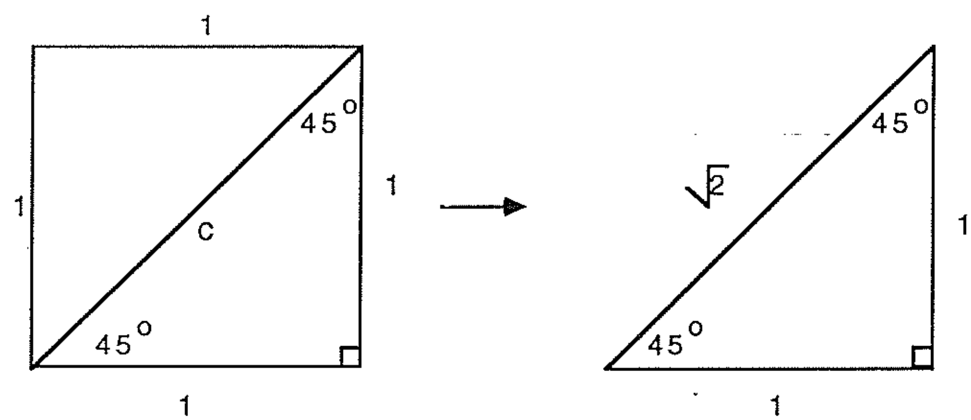
Notice that

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

We will apply these definitions to two important triangles.

The $45° - 45° - 90°$*-triangle.*

Start with a unit square and cut it along the diagonal. Since this cuts the angles forming the corners in half we are left with a $45° - 45° - 90°$–triangle. Both of the legs have length one and we'll let c denote the length of the hypotenuse.



Use the Pythagorean Theorem to determine the length of the hypotenuse.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + 1^2 &= c^2 \\ 2 &= c^2 \\ c &= \sqrt{2} \end{aligned}$$

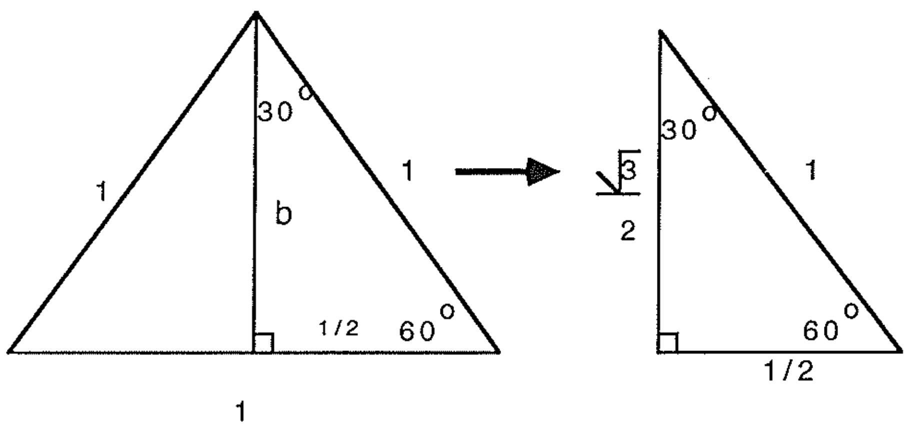
From this triangle we can read off,

$$\begin{aligned} \sin 45° &= \frac{opp}{hyp} = \frac{1}{\sqrt{2}} & \csc 45° &= \sqrt{2} \\ \cos 45° &= \frac{adj}{hyp} = \frac{1}{\sqrt{2}} & \sec 45° &= \sqrt{2} \\ \tan 45° &= \frac{opp}{adj} = \frac{1}{1} = 1 & \cot 45° &= 1 \end{aligned}$$

Any other $45°-45°-90°$-triangle will be similar to this triangle. It will produce the same ratios because the side lengths will be some constant, k, times the side lengths we have. For instance, a triangle with side lengths 1, $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$ will also be a $45° - 45° - 90°$−triangle and using SOH CAH TOA to calculate the trig. functions will produce the same answers.

The $30° - 60° - 90°$*−triangle.*

Start with an equilateral triangle. Since all of the angles are 60°,if we cut it down the middle, one half is then a $30° - 60° - 90°$−triangle. The leg on the bottom is 1/2 unit, the hypotenuse has length one and the length of the other leg, b, is determined using the Pythagorean Theorem.



$$(\frac{1}{2})^2 + b^2 = 1^2$$
$$b^2 = 1 - \frac{1}{4}$$
$$b^2 = \frac{3}{4}$$
$$b = \frac{\sqrt{3}}{2}$$

From this triangle or any triangle that is similar we can determine the values of the trig. functions for a 30° and a 60° angle.

$$\sin 30° = \frac{opp}{hyp} = \frac{1/2}{1} = \frac{1}{2} \qquad \csc 30° = 2$$
$$\cos 30° = \frac{adj}{hyp} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2} \qquad \sec 30° = \frac{2}{\sqrt{3}}$$
$$\tan 30° = \frac{opp}{adj} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} \qquad \cot 30° = \sqrt{3}$$

$$\sin 60^\circ = \frac{opp}{hyp} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2} \qquad \csc 60^\circ = \frac{2}{\sqrt{3}}$$

$$\cos 60^\circ = \frac{adj}{hyp} = \frac{1/2}{1} = \frac{1}{2} \qquad \sec 60^\circ = 2$$

$$\tan 60^\circ = \frac{opp}{adj} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \qquad \cot 60^\circ = \frac{1}{\sqrt{3}}$$

Another $30^\circ - 60^\circ - 90^\circ$–triangle you will often see has a hypotenuse with length 2 and legs with lengths 1 and $\sqrt{3}$.

Exercises.

A right triangle has legs with lengths 3 and 4.

1) Draw a sketch of this triangle.
2) Determine the length of the hypotenuse.
3) If angle A is opposite the side of length 3 and $\theta =$ measure of angle A. Determine

$a)$ $\sin\theta$ $b)$ $\cos\theta$ $c)$ $\tan\theta$

4) If angle B is opposite the side of length 4 and $\phi =$ measure of angle B. Determine

$a)$ $\sin\phi$ $b)$ $\cos\phi$ $c)$ $\tan\phi$

The value of one of the trig. functions is given. Determine the values of the other five trigonometric functions assuming that the angle $0 \le \theta < 90^o$.

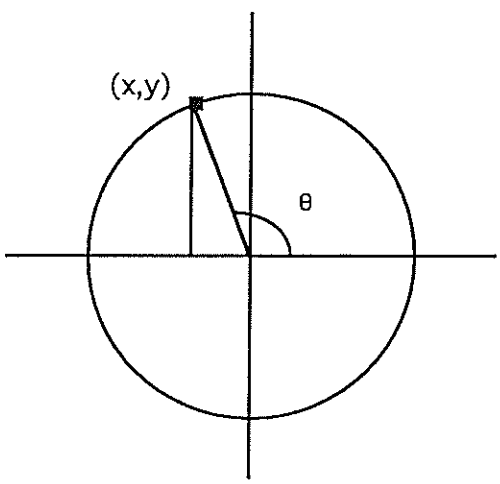
5) $\sin\theta = \dfrac{12}{13}$ 6) $\cos\theta = \dfrac{2}{5}$ 7) $\tan\theta = \sqrt{3}$

8) $\cot\theta = 1$ 9) $\sec\theta = \dfrac{2}{\sqrt{2}}$ 10) $\csc\theta = \dfrac{2\sqrt{3}}{3}$

11) Find a formula that gives $\cot\theta$ in terms of $\sin\theta$ and $\cos\theta$.

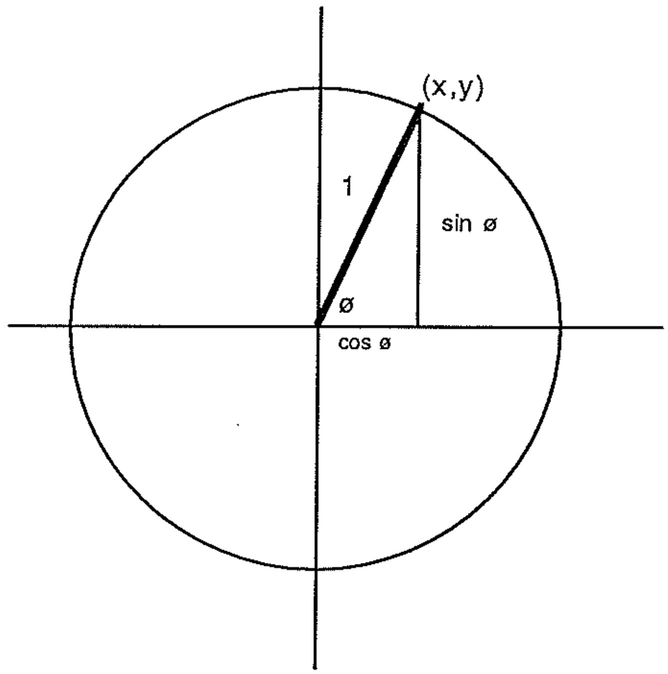
Trigonometric Functions Defined for All Angles

The second definition of the trigonometric functions comes from the unit circle. Any angle, θ, with vertex at the origin and initial side placed on the positive x–axis defines a unique point, with coordinates (x,y), on the unit circle. We define $x = \cos\theta$ and $y = \sin\theta$.



cos θ = x
sin θ= y

We form a triangle by dropping a vertical line from the point $(x, y) = (\cos\theta, \sin\theta)$ to the x–axis. The hypotenuse is a radius for the circle so it has length one. For angles less than 90^o we see that the unit circle definition agrees with the definition of trigonometric functions as ratios of the lengths of the sides of triangles. Notice that once sine and cosine are determined we can figure out the values of the rest of the trigonometric functions.



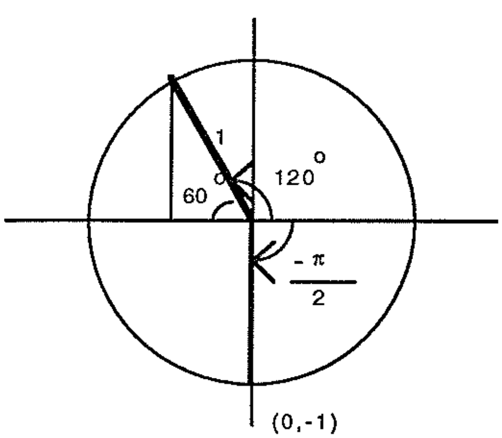
$$\sin \text{ø} = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

$$\cos \text{ø} = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

Hence the unit circle definition is just a generalization of the ratio definition. This allows us to define the trigonometric functions for all angles.

Example.

(1) Determine $\sin 120^o$, $\cos 120^o$, $\tan 120^o$, $\sin(-\frac{\pi}{2})$, $\cos(-\frac{\pi}{2})$.
First, draw a unit circle and place the angles in standard position.



Second, determine the coordinates of the points where the rays intersect the unit circle. To do this drop a vertical line from each of the points to the x–axis. The vertical line from $-\frac{\pi}{2}$ to the x–axis is a radius and therefore has length one. Thus the point associated with $-\frac{\pi}{2}$ is (0,-1) so we find that

$$\cos(-\frac{\pi}{2}) = 0$$

and

$$\sin(-\frac{\pi}{2}) = -1.$$

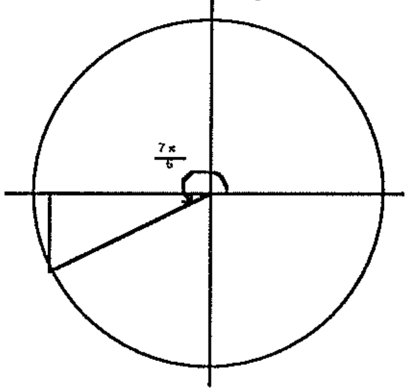
For the 120^o angle we again drop a vertical line to the x–axis. Since there are 180^o in a half circle, we see that the angle at the origin in the triangle formed is a 60^o angle. We then recognize that the triangle formed is a $30^o - 60^o - 90^o$–triangle with a hypotenuse of length one. This is one of our two important triangles so we realize that the side lengths are 1/2 and $\frac{\sqrt{3}}{2}$ with the longer side opposite the larger angle. We find that the angle 120^o corresponds to the point $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Hence,

$$\cos(120^o) = -\frac{1}{2}$$

$$\sin(120^o) = \frac{\sqrt{3}}{2}$$

$$\tan(120^o) = \frac{\sin(120^o)}{\cos 120^o} = -\sqrt{3}.$$

(2) Find $\sin \frac{19\pi}{6}$ and $\sin \frac{7\pi}{6}$.



Notice that these angles correspond to the same point. The triangle formed by dropping a vertical line to the x–axis is again a $30^o - 60^o - 90^o$–triangle. This time the 30^o angle is at the origin. The point corresponding to both $\frac{7\pi}{6}$ and $\frac{19\pi}{6}$ is $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ so

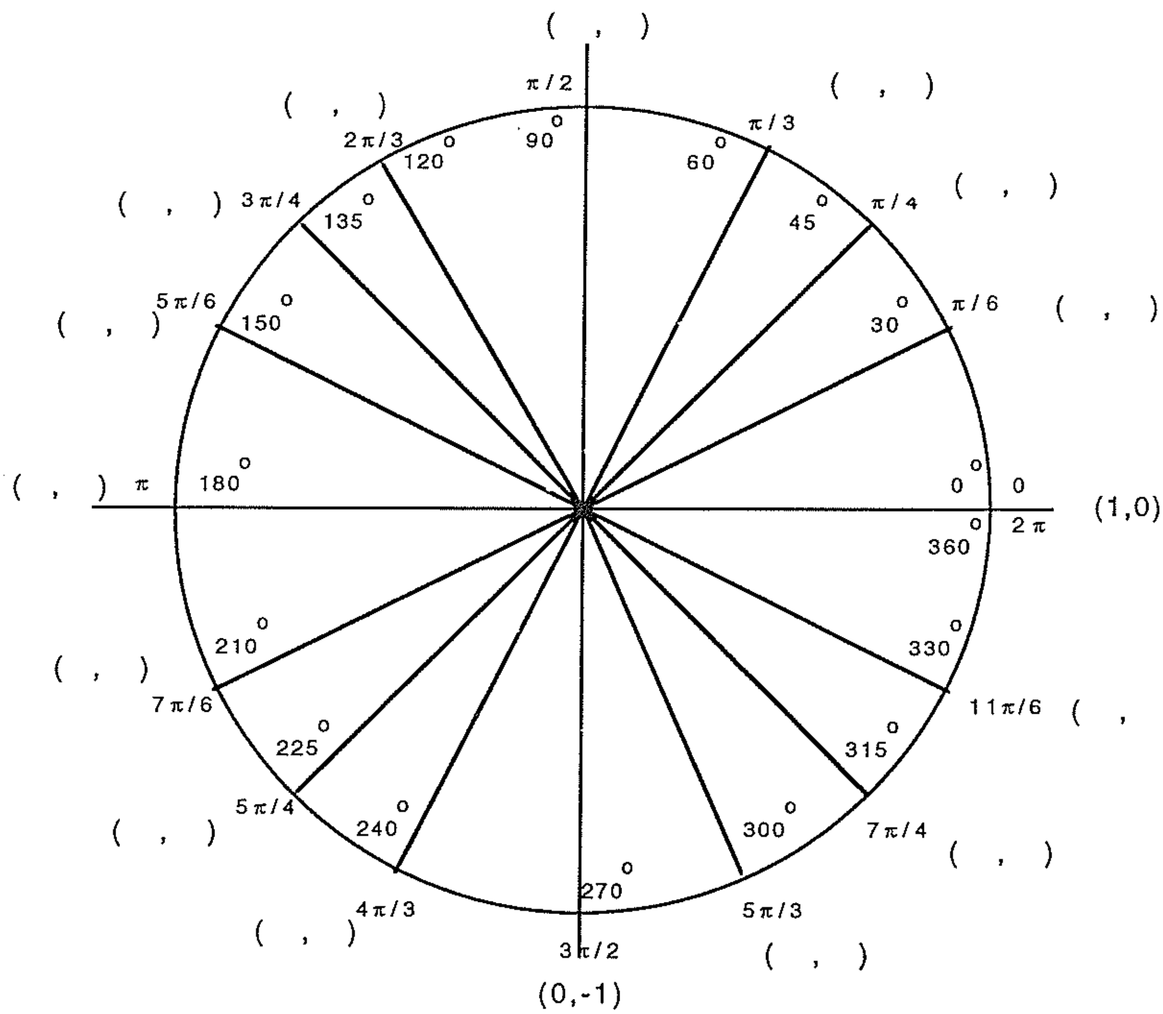
$$\sin(\frac{7\pi}{6}) = \sin(\frac{19\pi}{6}) = -\frac{1}{2}.$$

In general, for any integer k,

$$\boxed{\sin(\theta + 2k\pi) = \sin\theta \qquad \text{and} \qquad \cos(\theta + 2k\pi) = \cos\theta}$$

Exercises.

1) Find the coordinates of each of the points shown below and place them on a unit circle like the one provided.



2) Evaluate the value of the sine function at each of the above angles.
3) Evaluate the value of the cosine function at each of the above angles.
4) Evaluate the value of the tangent function at each of the above angles.

5) Find an angle that is negative and has the same point associated to it on the unit circle as that of the following angles.
a) 45^o b) $\frac{3\pi}{2}$ c) $\frac{5\pi}{3}$ d) $\frac{7\pi}{6}$

Evaluate each of the following.

6)	$\sin(-\pi)$	7)	$\cos(-\frac{\pi}{6})$	8)	$\tan(-\frac{5\pi}{4})$
9)	$\sin(6\pi)$	10)	$\cos(\frac{20\pi}{3})$	11)	$\tan(-270^o)$

Find three angles that will satisfy each of the following equations

12)	$\sin\theta = \frac{1}{2}$	13)	$\cos\theta = -\frac{\sqrt{3}}{2}$	14)	$\tan\theta = \sqrt{3}$

Algebra of trigonometric functions

Notice that on the unit circle all of the points satisfy the equation

$$x^2 + y^2 = 1$$

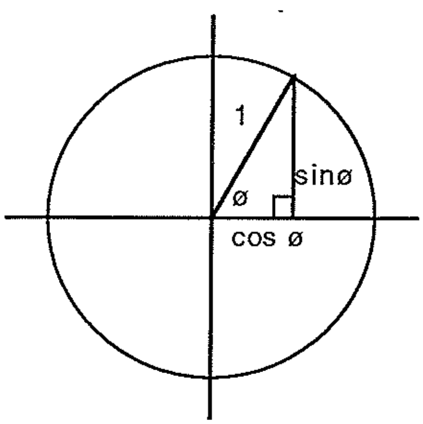
. But $x = \cos\theta$ and $y = \sin\theta$ so we get the equation

$$\cos^2\theta + \sin^2\theta = 1$$

.

Notice that $\sin^2\theta = (\sin\theta)^2$.

We can also get this equation from the Pythagorean Theorem by noticing that the lengths of the sides of the triangle determined by an angle ϕ are $\sin\phi$ and $\cos\phi$, as indicated in the following figure:



$$\sin^2 \phi + \cos^2 \phi = 1$$

We can define all of the trigonometric functions in terms of just sine and cosine.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

These formulas follow directly from the definitions and should be memorized. We will use these formulas to find the other two Pythagorean Identities. Take the first identity and first divide by $\sin^2\theta$ and then simplify.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} &= \frac{1}{\sin^2\theta} \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

Finally to get the third identity divide through by $\cos^2\theta$ instead.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\ \tan^2\theta + 1 &= \sec^2\theta \end{aligned}$$

Since the second and third identities are easily derived from the first, memorize only $\sin^2\theta + \cos^2\theta = 1$ and the method for determining the other two.

Examples.

(1) Suppose that $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \frac{5}{13}$. Determine $\cos\theta$ and $\tan\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

So

$$(\frac{5}{13})^2 + \cos^2\theta = 1$$

$$25 + 169\cos^2\theta = 169$$

$$169\cos^2\theta = 169 - 25 = 144$$

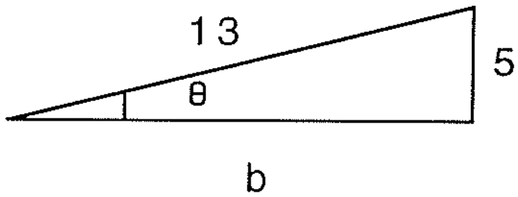
$$\cos^2\theta = \frac{144}{169}$$

$$\cos\theta = \pm\frac{12}{13}$$

for $\theta < \frac{\pi}{2}, \cos\theta > 0$ so $\cos\theta = \frac{12}{13}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$$

We also could have approached this problem by drawing a triangle, placed with the angle θ at the origin and the initial side on the positive x–axis. Since $\sin\theta = \frac{opp}{hyp} = \frac{5}{13}$ and $0 < \theta < \frac{\pi}{2}$, we draw,



Using the Pythagorean Theorem we find,

$$5^2 + b^2 = 13^2$$

$$b^2 = 13^2 - 5^2 = 144$$

$$b = 12$$

Thus we can apply our ratio definition to get

$$\cos\theta = \frac{adj}{hyp} = \frac{12}{13} \qquad \text{and} \qquad \tan\theta = \frac{opp}{adj} = \frac{5}{12}$$

Some people think algebraically and others think geometrically. Try both and pick the one that is best for you.

(2) Add

$$\frac{1}{\sin\theta} + \cos\theta = \frac{1}{\sin\theta} + \frac{\sin\theta\cos\theta}{\sin\theta} = \frac{1 + \sin\theta\cos\theta}{\sin\theta}$$

(3) Simplify

$$\frac{\sec\theta + 1}{\sin\theta + \tan\theta}.$$

$$
\begin{aligned}
\frac{\sec\theta+1}{\sin\theta+\tan\theta} &= \frac{\frac{1}{\cos\theta}+1}{\sin\theta+\frac{\sin\theta}{\cos\theta}} && \text{Change to sin and cos}\\
&= \frac{\frac{1}{\cos\theta}+\frac{\cos\theta}{\cos\theta}}{\frac{\sin\theta\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}} && \text{Find a common denominator}\\
&= \frac{\frac{1+\cos\theta}{\cos\theta}}{\frac{\sin\theta\cos\theta+\sin\theta}{\cos\theta}} && \text{Add}\\
&= \frac{1+\cos\theta}{\cos\theta}\cdot\frac{\cos\theta}{sin\theta\cos\theta+\sin\theta} && \text{Invert and multiply}\\
&= \frac{1+\cos\theta}{sin\theta\cos\theta+\sin\theta} && \text{Cancel}\\
&= \frac{1+\cos\theta}{\sin\theta(\cos\theta+1)} && \text{Factor}\\
&= \frac{1}{\sin\theta} && \text{Cancel}\\
&= \csc\theta && \text{Simplify}
\end{aligned}
$$

Exercises.

1) If $\sin\theta = -\frac{3}{5}$ and $-\frac{\pi}{2} < \theta < 0$, find $\cos\theta$ and $\tan\theta$.
2) If $cos\theta = -\frac{1}{2}$ and $\frac{\pi}{2} < \theta < \pi$, find $\sin\theta$ and $\tan\theta$.
3) If $\tan\theta = \frac{1}{3}$ and $0 < \theta < \frac{\pi}{2}$, find $\sin\theta$ and $\cos\theta$

Simplify each expression. (See Example (3))

4) $\dfrac{\tan\theta-1}{1-\frac{\sin\theta}{\cos\theta}}$ 5) $\sin\theta\cos\theta\sec\theta\csc\theta$

6) $\dfrac{\tan\theta}{\sec\theta-1}+\dfrac{\tan\theta}{\sec\theta+1}$ 7) $\dfrac{\cot^2\theta}{\csc^2\theta}+\dfrac{\tan^2\theta}{\sec^2\theta}$

8) $\dfrac{2+\frac{1}{\cos\theta}}{\frac{1}{\cos^2\theta}}$ 9) $\dfrac{\tan\phi+\tan\phi\sin\phi-\cos\phi\sin\phi}{\sin\phi\tan\phi}$

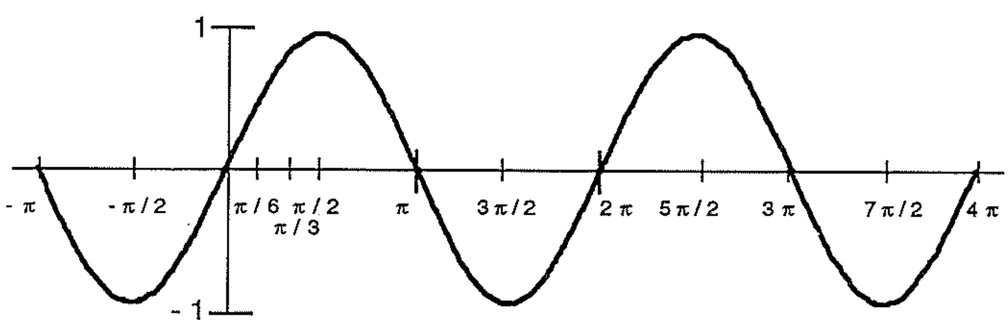
Graphing the Sine and Cosine Functions

Remember that sine and cosine are just the names of the functions that send angles to real numbers. When we disregard the units, the radian measure of an angle is also a real number. We learned how to graph functions that map real numbers to real numbers on the xy–plane. We did this by plotting points and working with translations. (Calculus gives much better methods for determining the shape of a graph.)

Let us begin by graphing $y = \sin x$. Make a table of points that will lie on the graph.

x	$y = \sin x$	x	$y = \sin x$	x	$y = \sin x$
0	0	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{2}$	1	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	π	0	$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{3\pi}{2}$	-1	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	2π	0		

Plot the points $(x, y) = (x, \sin x)$ on the xy–plane.



There are some important features of this graph. First, since $\sin(x+2\pi)=\sin x$, the graph looks the same between 2π and 4π as it does between 0 and 2π.

Definition. *A function f is periodic if there is a positive number P so that*

$$f(x+P)=f(x)$$

for all x in the domain of f. The number P is called the period of f

Second, the values of the sine function are never more than one in absolute value. We say that this function has Amplitude one.

Definition. *If f is periodic and M =maximum for f and m =minimum for f, then*

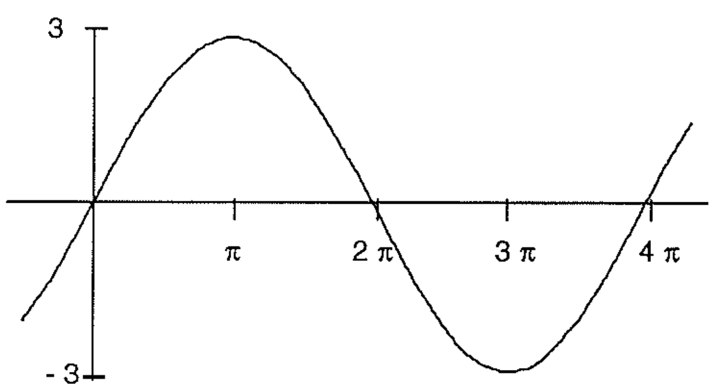
$$\textit{Amplitude} = \frac{M-m}{2}$$

Example.

(1) Graph $y=3\sin\frac{1}{2}x$.
Make a table and plot the points.

x	$y=3\sin\frac{1}{2}x$
0	0
$\frac{\pi}{2}$	$3\sin\frac{\pi}{4}=\frac{3}{\sqrt{2}}$
π	$3\sin\frac{\pi}{2}=3$
2π	$3\sin\pi=0$
3π	-3
4π	0

As you can see, choosing which numbers to plot is very important. You will get an algorithm later, but intuitively we're just picking the numbers that will give us nice outputs.



Notice that for this graph the

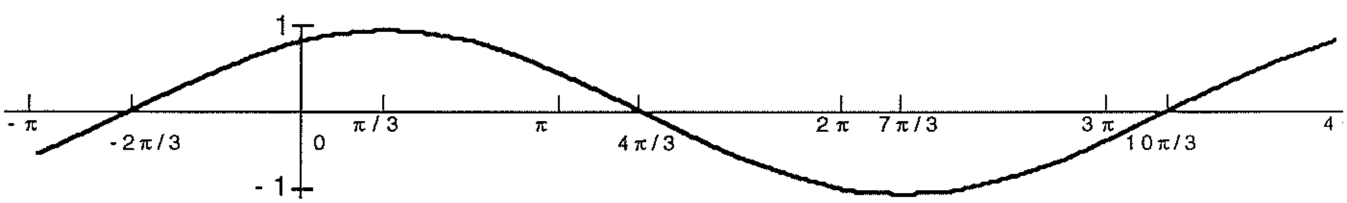
$$\text{Amplitude} = \frac{3-(-3)}{2} = 3$$

and

$$\text{Period} = 4\pi$$

(2) Graph $y = \sin(\frac{1}{2}x + \frac{\pi}{3})$.
Make a table,

x	$y = \sin(\frac{1}{2}x + \frac{\pi}{3}) = \sin(\frac{1}{2}(x + \frac{2\pi}{3}))$
0	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$-\frac{2\pi}{3}$	$\sin(0) = 0$
$\frac{\pi}{3}$	$\sin(\frac{1}{2}(\frac{\pi}{3} + \frac{2\pi}{3})) = \sin(\frac{1}{2}(\pi)) = 1$
$\frac{2\pi}{3}$	$\sin(\frac{1}{2}(\frac{4\pi}{3})) = \sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$
$\frac{4\pi}{3}$	$\sin(\pi) = 0$
$\frac{7\pi}{3}$	$\sin\frac{3\pi}{2} = -1$
$\frac{10\pi}{3}$	$\sin(2\pi) = 0$



Definition. *A translation of any sine curve to the right or left is called a* <u>*Phase Shift*</u>.

In the above example we have

$$\text{Amplitude} = 1$$
$$\text{Period} = 4\pi$$
$$\text{Phase Shift} = -\frac{2\pi}{3}$$

We can graph a general sine curve by putting it in the standard form and determining the values of the amplitude, period and phase shift. Standard form is

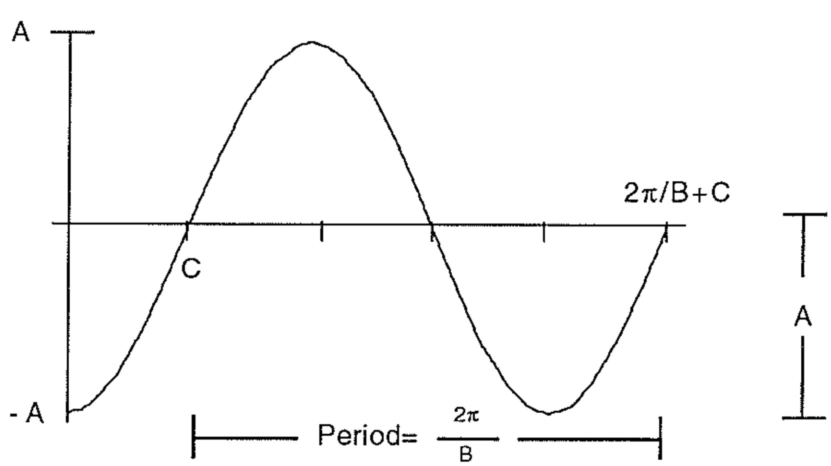
$$y = A\sin(B(x - C))$$

Then,

$$\begin{aligned}\text{Amplitude} &= |A| \\ \text{Period} &= \frac{2\pi}{B} \\ \text{Phase Shift} &= C\end{aligned}$$

If A is positive the curve has the same form as the sine curve. If A is negative then the curve is reflected over the x–axis. With this information we can make a graph of any sine curve without needing to plot any points. However, it is always a wise idea to plot at least a couple of points to be sure that you've done things correctly.

GRAPH OF $y = A\sin(B(x - C))$ FOR $A > 0$



(3) Graph $y = 4\sin(3x - \frac{\pi}{2})$

Be careful, the equation is not in standard form. We must first factor out the 3 from the $\frac{\pi}{2}$ to get $\frac{\pi}{6}$ as our phase shift.

$$y = 4\sin(3(x - \frac{\pi}{6})) \qquad \text{Standard Form}$$

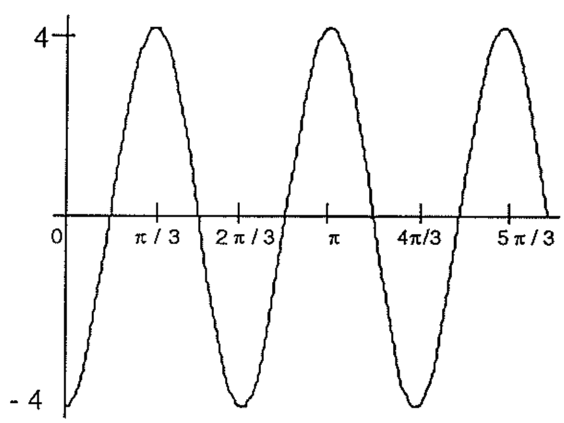
Then

$$\begin{aligned}
&\text{Amplitude} = |A| = |4| = 4\\
&\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{3}\\
&\text{Phase Shift} = C = +\frac{\pi}{6}\\
&A > 0 \qquad \Rightarrow \qquad \text{The curve is right side up}
\end{aligned}$$

To Determine what the graph looks like from these numbers, follow these steps.
First, determine $\frac{1}{4}\cdot$ period $= \frac{1}{4} \cdot \frac{2\pi}{3} = \frac{\pi}{6}$.
Second, mark off intervals of length $\frac{1}{4}\cdot$ period $= \frac{\pi}{6}$, starting at $C = \frac{\pi}{6}$.
Third, plot the points on the graph for these x–values.

0 at $C = \frac{\pi}{6}$
A at 1/4 period$+C = \frac{2\pi}{6} = \frac{\pi}{3}$
0 at 1/2 period$+C = \frac{3\pi}{6} = \frac{\pi}{2}$
-A at 3/4 period$+C = \frac{4\pi}{6} = \frac{2\pi}{3}$
0 at 1 period$+C = \frac{5\pi}{6}$

Fourth, draw a sine curve connecting these points.
This curve turns out to look like this.



The same ideas can be used to determine the graph of any cosine function. The exercises walk you through this development.

Exercises.

(1) Make a table showing the values of $y = \cos x$ for the following values: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{3\pi}{2}, 2\pi$.
(2) Plot these points on the xy–plane, and then connect them with a smooth curve resembling the sine curve.
(3) What are the amplitude and period of $y = \cos x$?

The standard form for a cosine equation is $y = A\cos(B(x - C))$ with amplitude, period, and phase shift having the same values as they do for a sine equation. Determine the amplitude, period and phase shift for the following equations.

4)	$y = 3\cos x$	5)	$y = 2\cos 5x$	6)	$y = \cos(2(x + \frac{\pi}{2}))$
7)	$y = \sin(x + 2)$	8)	$y = 3\sin(2x + \frac{2\pi}{3})$	9)	$y = 3\cos\frac{1}{3}x$

The steps used to graph a sine function can be used to graph a cosine function. However, we obtain the value A at C, 0 at 1/4 period + C, -A at 1/2 period + C, 0 at 3/4 period + C, and A at 1 period + C. Use this fact to graph the following equations

10)	$y = 3\sin x$	11)	$y = 3\cos x$	12)	$y = \sin\frac{1}{3}x$
13)	$y = \cos\frac{1}{4}x$	14)	$y = -\sin x$	15)	$y = -3\cos x$
16)	$y = \sin(x + \frac{\pi}{2})$	17)	$y = \cos(x - \frac{\pi}{2})$	18)	$y = -2\sin(3(x + \frac{\pi}{2}))$
19)	$y = cos(x + 1)$	20)	$y = 4\cos(3x - \frac{\pi}{4})$	21)	$y = 3\cos(2(x + \frac{\pi}{3}))$
22)	$y = 1 + \sin x$	23)	$y = -2\sin(\pi x + \pi)$	24)	$y = \sin(x + 2\pi)$

Inverse Trigonometric Functions

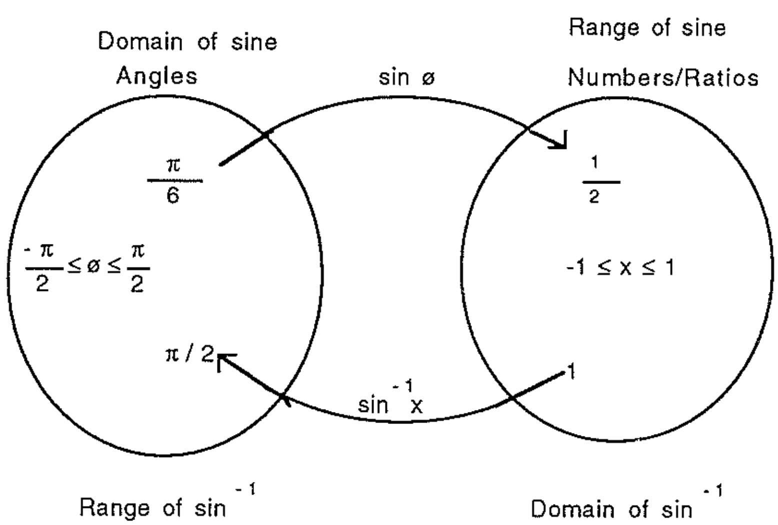
We now have the trigonometric functions which have angles in their domain and ratios of numbers in their range. We'd like to have inverses for these functions, but sine, cosine, and tangent are not one–to–one functions.

$$1/2 = \sin(30°) = \sin(150°) = \sin(390°) = \ldots$$

In order to define the inverse trigonometric functions we will have to restrict the domain of the trigonometric functions so that each of the numbers in the range will only have one preimage. We can then make the following definitions.

Definition.
$\sin^{-1} x$ *is the angle* θ, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, *so that* $\sin\theta = x$
$\cos^{-1} x$ *is the angle* θ, $0 \leq \theta \leq \pi$, *so that* $\cos\theta = x$
$\tan^{-1} x$ *is the angle* θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, *so that* $\tan\theta = x$

Notice that the domain of both $\sin^{-1} x$ and $\cos^{-1} x$ is just those numbers between -1 and 1 because this is all that is in the range of the sine and cosine functions.



Examples.

(1) Find $\sin^{-1}(1)$.

$$\sin\theta = 1 \text{ when } \theta = \frac{\pi}{2} \text{ so } \sin^{-1}(1) = \frac{\pi}{2}.$$

(2) Find $\sin^{-1}(\frac{\sqrt{3}}{2})$.
$\sin\theta = \frac{\sqrt{3}}{2}$ when $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$, but $\frac{2\pi}{3} > \frac{\pi}{2}$ so we must choose $\frac{\pi}{3}$ and then $\sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$.

(3) Find $\sin^{-1}(16)$.
We can't. $\sin\theta \neq 16$ for any value of θ because $-1 \leq \sin\theta \leq 1$ for all values of θ. $\sin^{-1}(16)$ is undefined.

(4) *Find* $\cos^{-1}(0.215)$.
Here we do not have one of our two important triangles to fall back on so we must use a calculator. Plug in the number and press "inv" and then "cos" and the number 77.58 will appear on the screen if the calculator is in degrees mode and the number 1.354 will appear if the calculator is in radians mode.

(5)

$$\cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$$

(6)

$$\cos^{-1}(-1) = \pi$$

Since the inverse trigonometric functions are inverse functionsfor the trigonometric functions we ge that $f(f^{-1})(x) = x$ for each of these functions with some restrictions on the domain.

$$\begin{array}{ll}
\sin(\sin^{-1}(x)) = x \ \text{ for } -1 \leq x \leq 1 & \sin^{-1}(sin(x)) = x \ \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
\cos(\cos^{-1}(x)) = x \ \text{ for } -1 \leq x \leq 1 & \cos^{-1}(cos(x)) = x \ \text{ for } 0 \leq x \leq \pi \\
\tan(\tan^{-1}(x)) = x \text{ for all } x & \tan^{-1}(tan(x)) = x \ \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{array}$$

The equations on the left hand side of the above display often get students in trouble. Suppose we are asked for

$$\sin^{-1}(\sin(\frac{3\pi}{2}))$$

Since $sin(\frac{3\pi}{2}) = -1$ we get

$$\sin^{-1}(\sin(\frac{3\pi}{2})) = \sin^{-1}(-1) = -\frac{\pi}{2}$$

We must choose an angle with measure between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Notice that this is not $\frac{3\pi}{2}$.

$$\sin^{-1}(\sin(\frac{3\pi}{2})) \neq \frac{3\pi}{2}$$

Note that,

$$\boxed{\sin^{-1}(\sin\theta)) = \phi \text{ provided that } \sin\phi = \sin\theta \text{ and } -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}.}$$

(7)

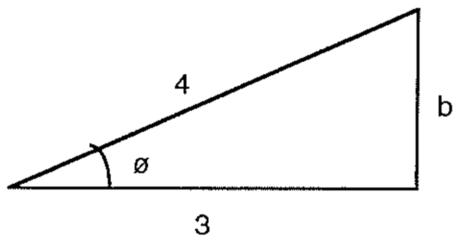
$$\sin^{-1}(\sin\pi) = \sin^{-1}(0) = 0$$

(8)

$$\sin^{-1}(\sin\frac{2\pi}{3}) = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$$

(9) Find $\tan(\cos^{-1}(\frac{3}{4}))$

Draw the appropriate triangle, labeling the angle ϕ.



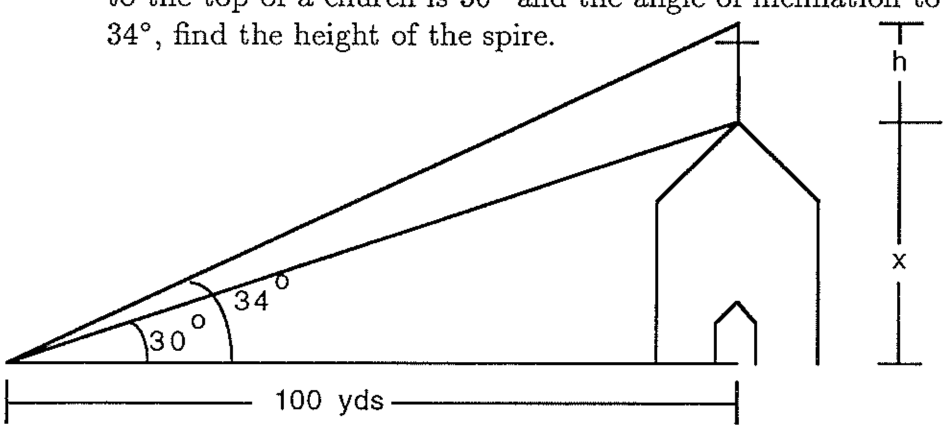
Determine the length of the opposite side by using the Pythagorean Theorem.

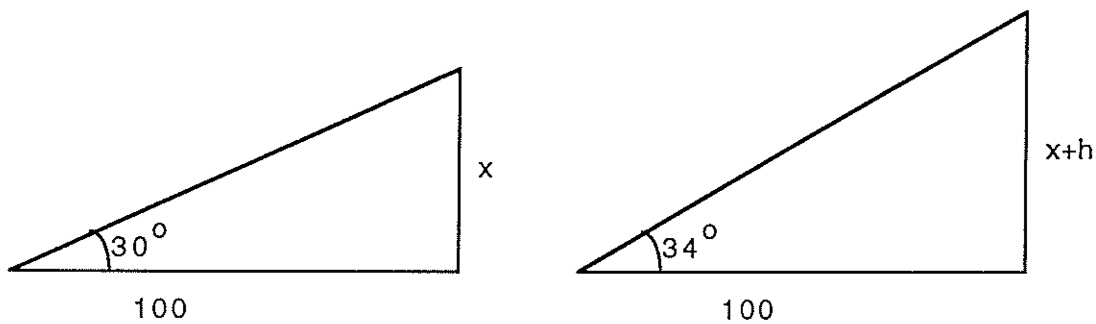
$$\begin{aligned} 3^2 + b^2 &= 4^2 \\ b^2 &= 16 - 9 \\ b^2 &= 7 \\ b &= \sqrt{7} \end{aligned}$$

Now apply "SOH CAH TOA" to get

$$\tan\phi = \frac{opp}{adj} = \frac{\sqrt{7}}{3}.$$

(10) Engineers have instruments to measure the angle of inclination from the ground to the top of something off in the distance. If at 100 yards, the angle of inclination to the top of a church is 30° and the angle of inclination to the top of the spire is 34°, find the height of the spire.





We have two triangles to consider.

We know the length of the adjacent side and we want to find the length of the opposite side so we will work with the tangent function. We find,

$$\tan(30^\circ) = \frac{x}{100} \qquad \text{and} \qquad \tan(34^\circ) = \frac{x+h}{100}$$

so we have

$$100\tan(30^\circ) = x \qquad \text{and} \qquad 100\tan(34^\circ) = x + h$$

Then substituting $100\tan(30^\circ)$ for x in the second equation we get

$$100\tan(34^\circ) = 100\tan(30^\circ) + h$$

Giving us the height of the spire as:

$$h = 100\tan(34^\circ) - 100\tan(30^\circ) = 67.45 - 57.735 = 9.715\text{yards}$$

If you don't have a calculator handy just leave your answer in terms of tangent.

Exercises. *Evaluate, leaving your answers in radians and in terms of π.*

1)	$\sin^{-1}(-\frac{\sqrt{3}}{2})$	2)	$\tan^{-1}(1)$	3)	$\cos^{-1}(\frac{1}{2})$
4)	$\tan^{-1}(\frac{\sqrt{3}}{3})$	5)	$\sin^{-1}(-1)$	6)	$\cos^{-1}(\frac{1}{\sqrt{2}})$
7)	$\sin(\cos^{-1}(\frac{1}{4}))$	8)	$\cos(\cos^{-1}(\frac{1}{2}))$	9)	$\sin^{-1}(\sin(\frac{3\pi}{4}))$
10)	$\cos(\cos^{-1}(-1))$	11)	$\tan(\tan^{-1}(-1))$	12)	$\cos(\sin^{-1}(\frac{3}{5}))$
13)	$\sin(\tan^{-1}(u))$				

14) If we are looking at a flagpole from 50ft away and the angle of inclination to the bottom of the flag is 45° and the angle of inclination to the top of the pole is 46°, find the height of the flag.

APPENDIX

Important Formulas

The Zero Factor Property	$ab = 0$	$\Leftrightarrow$	$a = 0$ or $b = 0$
The Square Root Property	$x^2 = b$	$\Leftrightarrow$	$x = \pm\sqrt{b}$

$$\begin{aligned} x^2 - 2ax + a^2 &= (x-a)^2, && \text{Square of difference} \\ x^2 + 2ax + a^2 &= (x+a)^2, && \text{Square of sum} \\ x^2 - a^2 &= (x-a)(x+a), && \text{Difference of squares} \\ x^3 - a^3 &= (x-a)(x^2+ax+a^2), && \text{Difference of cubes} \\ x^3 + a^3 &= (x+a)(x^2-ax+a^2), && \text{Sum of cubes} \end{aligned}$$

$$|x| = \text{The distance from x to zero} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Rules for Exponents.

$i)$	$x^m x^n = x^{m+n}$	$ii)$	$(x^m)^n = x^{mn}$
$iii)$	$(\frac{x}{y})^n = \frac{x^n}{y^n}$ If $y \neq 0$	$iv)$	$x^{-n} = \frac{1}{x^n}$ If $x \neq 0$
$v)$	$(xy)^n = x^n y^n$	$vi)$	$\frac{x^m}{x^n} = x^{m-n}$ If $x \neq 0$

Rules for Radicals.

$i)$	$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$	$ii)$	$\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$
$iii)$	n even $\sqrt[n]{x^n} = \|x\|$	$iv)$	n odd $\sqrt[n]{x^n} = x$
$v)$	$(\sqrt[n]{x})^n = x$	$vi)$	$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$ If $y \neq 0$

Equations for lines.

$(y - y_1) = m(x - x_1)$	Point–Slope Form
$y = mx + b$	Slope–Intercept Form
$ax + by + c = 0$	Standard Form

Two lines are Parallel if they have the same slope.
Two lines are Perpendicular if the product of their slopes is -1.

To graph the general quadratic function $g(x) = ax^2 + bx + c$.

(1) Complete the square $g(x) = a(x-h)^2 + k$, where $h = \frac{-b}{2a}$ and $k = g(h)$.
(2) The vertex is the point (h,k).
(3) The graph opens upward if $a > 0$ and downward if $a < 0$.
(4) The axis of symmetry is the vertical line x=h.
(5) The graph of $g(x)$ is narrower than the graph of $f(x) = x^2$ if $a > 1$ and broader if $a < 1$.
(6) The x–intercepts are the solutions to the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

.

(7) The y–intercept is $y = g(0)$.

Rules for logarithms. All variables represent positive real numbers.

$$\begin{array}{llll} i) & \log_b b = 1 & ii) & \log_b 1 = 0 \\ iii) & \log_b(xy) = \log_b x + \log_b y & iv) & \log_b x^t = t\log_b x \\ v) & \log_b(\frac{x}{y}) = \log_b x - \log_b y & & \end{array}$$

$$\log_a a^x = x \qquad \text{and} \qquad a^{\log_a x} = x$$

$$\underbrace{\log_a x = m}_{\text{logarithmic form}} \iff a^{\log_a x} = a^m \iff \underbrace{x = a^m}_{\text{exponential form}}$$

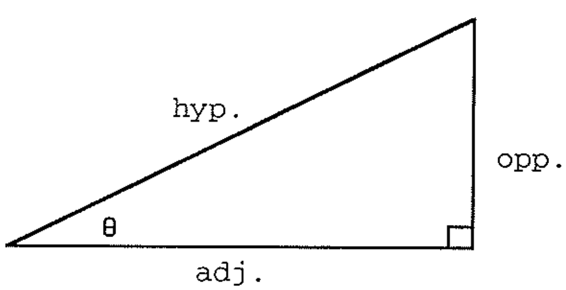
Trigonometry.
1 revolution $= 360° = 2\pi$ radians

Circumference $= 2\pi r =$ length of the edge of a circle

Area of a circle $= \pi r^2$

$\theta = \dfrac{s}{r}$ This formula only works if θ is in radians.

DEFINITION OF THE TRIGONOMETRIC FUNCTIONS AS RATIOS OF THE LENGTHS OF SIDES OF TRIANGLES



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Reciprocal Identities.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Pythagorean Identities.
$\cos^2\theta + \sin^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Graphing $y = A\sin(B(x - C))$.

$$\text{Amplitude} = |A|$$
$$\text{Period} = \frac{2\pi}{B}$$
$$\text{Phase Shift} = C$$

Inverse Trigonometric Functions.
$\sin^{-1} x$ is the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so that $\sin\theta = x$
$\cos^{-1} x$ is the angle θ, $0 \le \theta \le \pi$, so that $\cos\theta = x$
$\tan^{-1} x$ is the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $\tan\theta = x$