Final Exam
Math 5A
12:00-3:00, Wednesday, Dec. 9, 2009
Prof. Rick Ye

Your Name:
Your TA's name:
Day and Time of Your Discussion Section:
Your Perm Number:
Your Signature:

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1. Be sure to write down all the steps.

2. Write cleanly and clearly!

3. No calculator is allowed. A double-sided notecard up to the regular letter size is allowed.

4. Each problem is worth 10 points.

1. a) (6 points) Find a basis for the kernel of the following linear transformation

$$T(\mathbf{v}) = A\mathbf{v}, A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 3 & 4 \\ 1 & 0 & 7 \end{pmatrix} \tag{0.1}$$

b) (4 points) Also find a basis for the image of this linear transformation.

Solution a) We perform row operations

$$\begin{pmatrix} 1 & -3 & 3 \\ 0 & 3 & 4 \\ 1 & 0 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 3 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.2}$$

So we obtain equations $x - 3y + 3z = 0, 3y + 4z = 0$. Solving them we obtain the following formula for vectors in the kernel

$$\vec{\mathbf{v}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -\frac{4}{3}z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -\frac{4}{3} \\ 1 \end{pmatrix}. \tag{0.3}$$

Thus a basis of the kernel consists of the vector $\begin{pmatrix} 1 \\ -\frac{4}{3} \\ 1 \end{pmatrix}$.

b) From the above computations we see that a basis of the image is given by the first and second column vectors of the matrix A.

2. Consider

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \tag{0.4}$$

a) (8 points) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix D. Find D.
b) (2 points) A 3×3 matrix has only 2 distinct eigenvalues λ_1 and λ_2. What conditions about their eigenspaces do you need in order to be able to diagonalize this matrix?

Solution a) Since A is upper triangular, its eigenvalues are given by the diagonal entries $1, 2, -1$. We find eigenvectors for each eigenvalue. For the top eigenvalue 1 it is easy to see that an eigenvector is given by $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. For the middle eigenvalue 2 we perform row operations on the matrix $A - 2I$

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{0.5}$$

So we obtain equations $x - 2y - z = 0, z = 0$. Choosing $y = 1$ we obtain an eigenvector $\vec{\mathbf{v}}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. For the eigenvalue -1 we consider the matrix $A + I$

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.6}$$

We obtain equations $x + y + \frac{1}{2}z = 0, y + \frac{1}{3}z = 0$. Choosing $z = 6$ we obtain an eigenvector $\vec{\mathbf{v}}_3 = \begin{pmatrix} -1 \\ -2 \\ 6 \end{pmatrix}$. Now the desired matrix P is given by

$$P = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 6 \end{pmatrix}. \tag{0.7}$$

On the other hand, we have

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{0.8}$$

b) The dimension of one of the two eigenspaces must be two.

3. a)(4 points) Consider the functions $\cos t, \sin t$ and e^t on the real line $\mathbf{R}$ (the set of all real numbers). They are linearly independent. Next consider the vector space $\mathbf{V} = \{a \cos t + b \sin t + ce^t : a, b, c \in \mathbf{R}\}$, i.e. $\mathbf{V}$ consists of all linear combinations of $\cos t, \sin t$ and e^t. Let the linear transformation T be defined as follows:

$$T(a \cos t + b \sin t + ce^t) = ae^t + b \cos t + b \sin t. \tag{0.9}$$

Find the kernel of T. Also find the dimension of its image.

b) (3 points) Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = |x|$ (this is the absolute value of x). Is it linear? Why?

c) (3 points) Let $\mathbf{M_{33}}$ denote the vector space of all 3×3 matrices. Consider the function $f : \mathbf{M_{33}} \to \mathbf{R}$ defined by $f(A) = det(A)$, i.e. the determinant of A. Is f linear? Why?

Solution a) Obviously, a vector $\vec{\mathbf{v}} = a \cos t + b \sin t + ce^t$ is in the kernel precisely when $a = 0, b = 0$. Hence $Ker(T) = \{ce^t : c \in \mathbf{R}\}$. On the other hand, the image is the span of e^t and $\cos t + \sin t$. These two vectors are linearly independent, hence the image has dimension 2.

b) There hold $f(-1) = |-1| = 1$ and $-f(1) = -1$. Hence $f(-1) \neq -f(1)$. It follows that f is not linear.

c) Consider the identity matrix I. There holds $f(2I) = 2^3$ and $2f(I) = 2$. So $f(2I) \neq 2f(I)$. Hence f is not linear.

4. Consider the basis

$$B = \{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \}. \tag{0.10}$$

a)(7 points) Convert the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ from the standard basis to the basis B, i.e. find its coodinates w.r.t. the basis B.
b) (3 points) Assume that a vector $\vec{\mathbf{w}}$ has coordinates $-1, 1$ and -1 w.r.t. the basis B. Convert it back to the standard basis, i.e. find x, y and z such that $\vec{\mathbf{w}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Solution a) Denote the 3 vectors in the basis B (in the given order) by $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. Set

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x\vec{\mathbf{v}}_1 + y\vec{\mathbf{v}}_2 + z\vec{\mathbf{v}}_3. \tag{0.11}$$

Then we have

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{0.12}$$

Now we perform row operations on the augmented matrix

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{pmatrix}. \end{aligned} \tag{0.13}$$

Hence we arrive at the equations $x + y + z = 1, y = 0, z = 1/2$. It follows that $x = 1/2, y = 0, z = 1/2$.
b) There holds

$$\vec{\mathbf{w}} = -\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}. \tag{0.14}$$

5. a)(6 pints) Find the eigenvalues of the following matrix. For each eigenvalue, find a basis for the corresponding eigenspace.

$$A = \begin{pmatrix} -1 & 3 & 1 \\ 2 & 0 & 1 \\ -4 & 0 & -3 \end{pmatrix} \tag{0.15}$$

b) (4 points) Find the solution of the following system of differential equations with the above A

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{0.16}$$

such that $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution a) We have

$$\begin{aligned} p_A(\lambda) &= \begin{vmatrix} \lambda+1 & -3 & -1 \\ -2 & \lambda & -1 \\ 4 & 0 & \lambda+3 \end{vmatrix} = 4(\lambda+3) + (\lambda+3)(\lambda(\lambda+1)-6) \\ &= (\lambda+3)(\lambda^2+\lambda-2) = (\lambda+3)(\lambda+2)(\lambda-1). \end{aligned} \tag{0.17}$$

Hence we have 3 eigenvalues $-3, -2$ and 1. Next we look for eigenvectors. First consider the eigenvalue -3. We perform row operations for the matrix $-3I - A$

$$\begin{pmatrix} -3+1 & -3 & -1 \\ -2 & -3 & -1 \\ 4 & 0 & -3+3 \end{pmatrix} = \begin{pmatrix} -2 & -3 & -1 \\ -2 & -3 & -1 \\ 4 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ -2 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We obtain the equations $x = 0, 3y + z = 0$. Choosing $y = 1$ we get an eigenvector $\vec{\mathbf{v}}_1 = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$.

For the eigenvalue -2 we consider the matrix $-2I - A$

$$\begin{pmatrix} -2+1 & -3 & -1 \\ -2 & -2 & -1 \\ 4 & 0 & -2+3 \end{pmatrix} = \begin{pmatrix} -1 & -3 & -1 \\ -2 & -2 & -1 \\ 4 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & -12 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So we obtain the equations $x + 3y + z = 0, 4y + z = 0$. Choosing $y = 1$ we deduce $z = -4$ and then $x = 1$. So we obtain an eigenvector $\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$.

For the eigenvalue 1 we consider the matrix $I - A$

$$\begin{pmatrix} 1+1 & -3 & -1 \\ -2 & 1 & -1 \\ 4 & 0 & 1+3 \end{pmatrix} = \begin{pmatrix} 2 & -3 & -1 \\ -2 & 1 & -1 \\ 4 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & -3 & -3 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.18}$$

So we obtain equations $x + z = 0, y + z = 0$. Choosing $z = -1$ we obtain an eigenvector $\vec{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

b) The general solution of the system is given by $\dot{\mathbf{x}} = a_1 e^{-3t}\vec{\mathbf{v}}_1 + a_2 e^{-2t}\vec{\mathbf{v}}_2 + a_3 e^t \vec{\mathbf{v}}_3$. There holds $\mathbf{x}(0) = a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2 + a_3\vec{\mathbf{v}}_3$. This leads to the equation

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -3 & -4 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{0.19}$$

We perform row operations

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -3 & -4 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 5 \end{pmatrix} \tag{0.20}$$

So we obtain equations

$$a_1 + a_2 + a_3 = 1, a_2 + a_3 = 1, 3a_3 = 5. \tag{0.21}$$

If follows that $a_3 = 5/3, a_2 = -2/3, a_1 = 0$. So the desired solution is given by

$$\mathbf{x} = -\frac{2}{3}e^{-2t}\vec{\mathbf{v}}_2 + \frac{5}{3}e^t\vec{\mathbf{v}}_3. \tag{0.22}$$

6. A mass-spring system has the following properties: the mass is 1, the damping constant is 2, the spring constant is 5, and the forcing is given by $f(t) = \sin(2t) + t$.
a) (5 points) Find the general solution of the second order differential equation for this mass-spring system.
b) (2 points) Assume that the mass is initially pulled to the position $x(0) = -4$, and given an initial velocity of -3. Find the solution $x(t)$ which satisfies these initial conditions.
c) (3 points) Convert the equation into a system of first order differential equations.

Solution a) The differential equation for the system is

$$\ddot{x} + 2\dot{x} + 5x = \sin(2t) + t. \tag{0.23}$$

The characteristic equation is then $\lambda^2 + 2\lambda + 5 = 0$, which yields

$$\lambda = \frac{1}{2}(-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 5}) = \frac{1}{2}(-2 \pm \sqrt{-16}) = -1 \pm 2i. \tag{0.24}$$

Hence we obtain two linearly independent real solutions $x_1 = e^{-t}\cos(2t)$ and $x_2 = e^{-t}\sin(2t)$ for the homogeneous equation $\ddot{x} + 2\dot{x} + 5x = 0$.

Next we find a particular solution for the above nonhomogeneous equation (0.23). We handle $\sin(2t)$ and t separately. First we set $y_1 = A\cos(2t) + B\sin(2t)$. Then we have $\dot{y}_1 = -2A\sin(2t) + 2B\cos(2t)$ and $\ddot{y}_1 = -4A\cos(2t) - 4B\sin(2t)$. Hence we have

$$\begin{aligned} \ddot{y}_1 + 2\dot{y}_1 + 5y_1 &= (-4A + 4B + 5A)\cos(2t) + (-4B - 4A + 5B)\sin(2t) \\ &= (A + 4B)\cos(2t) + (B - 4A)\sin(2t). \end{aligned} \tag{0.25}$$

Hence, to solve the equation $\ddot{y}_1 + 2\dot{y}_1 + 5y_1 = \sin(2t)$ we precisely need $B - 4A = 1, A + 4B = 0$. It follows that $B = 1/17, A = -4/17$.

Next we set $y_2 = At + B$. Then $\dot{y}_2 = A, \ddot{y}_2 = 0$. Hence $\ddot{y}_2 + 2\dot{y}_2 + 5y_2 = 2A + 5At + 5B = (2A + 5B) + 5At$. Setting it equal to t we deduce $2A + 5B = 0, 5A = 1$. Hence $A = 1/5, B = -2/25$.

Putting the above together we obtain the general solution of (0.23):

$$x = e^{-t}(a_1\cos(2t) + a_2\sin(2t)) - \frac{4}{17}\cos(2t) + \frac{1}{17}\sin(2t) + \frac{1}{5}t - \frac{2}{25}. \tag{0.26}$$

b) First we have

$$x(0) = a_1 - \frac{4}{17} - \frac{2}{25}. \tag{0.27}$$

Next there holds

$$\begin{aligned} \dot{x} &= -e^{-t}(a_1\cos(2t) + a_2\sin(2t)) + e^{-t}(-2a_1\sin(2t) + 2a_2\cos(2t)) \\ &\quad + \frac{8}{17}\sin(2t) + \frac{2}{17}\cos(2t) + \frac{1}{5}. \end{aligned} \tag{0.28}$$

We deduce

$$\dot{x}(0) = -a_1 + 2a_2 + \frac{2}{17} + \frac{1}{5}. \tag{0.29}$$

We set $x(0) = -4$ and $\dot{x}(0) = 3$. This leads to

$$\begin{aligned} a_1 - \frac{4}{17} - \frac{2}{25} &= -4, \\ -a_1 + 2a_2 + \frac{2}{17} + \frac{1}{5} &= 3. \end{aligned} \tag{0.30}$$

Hence we obtain

$$\begin{aligned} a_1 &= -4 + \frac{4}{17} + \frac{2}{25}, \\ a_2 &= \frac{1}{2}(3 - \frac{2}{17} - \frac{1}{5} - 4 + \frac{4}{17} + \frac{2}{25}) \\ &= \frac{1}{2}(-1 + \frac{2}{17} - \frac{3}{25}). \end{aligned} \tag{0.31}$$

c) We set $y = \dot{x}$. Then we deduce

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -2y - 5x + \sin(2t) + t. \end{cases}$$

7. Consider the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}. \tag{0.32}$$

Do the following:
a) (5 points) Find the general solution of the following system of differential equations

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{0.33}$$

b) (1 point) Identify whether the fixed point $\vec{\mathbf{0}}$ is asymptotically stable, neutrally stable, or unstable.
c)(2 points) Identify which of the following types the fixed point $\vec{\mathbf{0}}$ is: *node sink, node source, saddle point, spiral sink, spiral source, degenerate node, or star node.* Hint: sinks are attracting, and sources are repelling.
d) (2 points) Draw a schematic picture of typical trajectories, which indicates the type of the fixed point $\vec{\mathbf{0}}$ clearly.

Solution a) We have $p_A(\lambda) = (\lambda - 3)^2 - 1$. Hence we obtain two eigenvalues $\lambda_1 = 4, \lambda_2 = 2$. Next we find eigenvectors. First consider $A - 4I$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}. \tag{0.34}$$

We infer $a - b = 0$ for eigenvectors $\vec{\mathbf{v}}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$. Choosing $a = 1$ we obtain $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Second we consider $A - 2I$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \tag{0.35}$$

We infer $a+b = 0$ for eigenvectors $\vec{\mathbf{v}}_2 = \begin{pmatrix} a \\ b \end{pmatrix}$. Choosing $a = 1$ we obtain $\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We conclude that the general solution of the system is given by

$$\mathbf{x} = a_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{0.36}$$

b) It is unstable.
c) It is a node source.

8. Consider the matrix

$$A = \begin{pmatrix} 2 & 7 \\ 2 & -3 \end{pmatrix} \tag{0.37}$$

Do the same as in Problem 7 for this matrix, i.e. do a), b), c) and d) in Problem 7 for this matrix.

Solution a) We have $p_A(\lambda) = (\lambda - 2)(\lambda + 3) - 14 = \lambda^2 + \lambda - 20 = (\lambda + 5)(\lambda - 4)$. So we obtain two eigenvalues $\lambda_1 = 4, \lambda_2 = -5$. Next we find eigenvectors. First consider $A - 4I$:

$$\begin{pmatrix} -2 & 7 \\ 2 & -7 \end{pmatrix} \to \begin{pmatrix} 2 & -7 \\ 0 & 0 \end{pmatrix}. \tag{0.38}$$

We infer $2a - 7b = 0$ for eigenvectors $\vec{\mathbf{v}}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$. Choosing $a = 1$ we obtain $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 2/7 \end{pmatrix}$. Second we consider $A + 5I$:

$$\begin{pmatrix} 7 & 7 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \tag{0.39}$$

We infer $a + b = 0$ for eigenvectors $\vec{\mathbf{v}}_2 = \begin{pmatrix} a \\ b \end{pmatrix}$. Choosing $a = 1$ we obtain $\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We conclude that the general solution of the system is given by

$$\mathbf{x} = a_1 e^{4t} \begin{pmatrix} 1 \\ 2/7 \end{pmatrix} + a_2 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{0.40}$$

b) It is unstable.
c) It is a saddle point.

9. Consider the matrix

$$A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}. \tag{0.41}$$

Do the same as in Problem 7 for this matrix, i.e. do a), b), c) and d) in Problem 7 for this matrix.

Solution a) We have $p_A(\lambda) = (\lambda+2)^2+1$. Hence we obtain two eigenvalues $\lambda_1 = -2+i$ and $\lambda_2 = -2-i$. Next we find eigenvectors. Consider the matrix $A+(2+i)I$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \to \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}. \tag{0.42}$$

We infer $a+bi=0$ for eigenvectors $\vec{\mathbf{v}}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$. Choosing $b=1$ we obtain

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{0.43}$$

We obtain two linearly independent real solutions of the system

$$\begin{aligned} \mathbf{x}_1 &= e^{-2t}((\cos t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - (\sin t)\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \\ \mathbf{x}_2 &= e^{-2t}((\sin t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + (\cos t)\begin{pmatrix} -1 \\ 0 \end{pmatrix}. \end{aligned} \tag{0.44}$$

The general solution is given by $\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$.
b) It is asymptotically stable.
c) It is a spiral sink.

10. Consider the matrix

$$A = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \tag{0.45}$$

Do the same as in Problem 7 for this matrix, i.e. do a), b), c) and d) in Problem 7 for this matrix.

Solution a) We have $p_A(\lambda) = (\lambda - 1)^2$. So we obtain a single eigenvalue $\lambda = 1$. To find an eigenvector consider the matrix $A - I$

$$A - I = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \tag{0.46}$$

We obtain an equation $-4b = 0$ and hence $b = 0$ for eigenvectors $\vec{\mathbf{v}} = \begin{pmatrix} a \\ b \end{pmatrix}$. Choosing $a = 1$ we deduce $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Next we find a generalized eigenvector:

$$(A - I)\vec{\mathbf{u}} = \vec{\mathbf{v}}. \tag{0.47}$$

This is pecisely

$$\begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.48}$$

for $\vec{\mathbf{u}} = \begin{pmatrix} a \\ b \end{pmatrix}$. It follows that $-4b = 1$ and hence $b = -1/4$. Choosing $a = 0$ we deduce $\vec{\mathbf{u}} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$. Hence the general solution of the system is given by

$$\mathbf{x} = a_1 t e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 e^t \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}. \tag{0.49}$$

b) It is unstable.
c) It is a degenerate node.