Midterm Exam
Math 240B
Prof. R. Ye
Winter 2011

Your Name:
Score:

Each problem is worth 20 points.

1. Problem 11-7 on page 286 of the textbook.
2. 1) Problem 12-3 on page 319 of the textbook. 2) Problem 12-4 on the same page.
3. Problem 12-6 on page 320 of the textbook.
4. Problem 12-17 on page 323 of the textbook.
5. Recall the following definition of the Lie derivative of covariant tensor fields. Let X be a smooth vector field on a smooth manifold M and σ a smooth covariant tensor field on M. Consider the one-parameter family of local diffeomorphisms $\Phi(p,t)$ generated by X, i.e.

$$\frac{d\Phi}{dt}(p,t) = X(\Phi(p,t)), \Phi(p,0) = p.$$

Set $\Phi_t = \Phi(\cdot, t)$. Then

$$(L_X\sigma)(p) = \frac{d}{dt}(\Phi_t)^*_p \sigma_{\Phi(p,t)}|_{t=0}.$$

Prove the following.
1) $L_X\sigma$ is smooth.
2) There hold

$$L_X(\sigma_1 \otimes \sigma_2) = L_X\sigma_1 \otimes \sigma_2 + \sigma_1 \otimes L_X\sigma_2$$

for smooth covariant tensor fields σ_1 and σ_2, and

$$L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta$$

for smooth differential forms α and β.
3) There holds

$$L_{fX}\alpha = df \wedge i_X\alpha + fL_X\alpha$$

for smooth functions f, where the *interior product* i_X is defined as follows. Let k denote the degree of α. Then

$$i_X\alpha(X_1, ..., X_{k-1}) = \alpha(X, X_1, ..., X_{k-1})$$

if $k \geq 0$, and $i_X\alpha = 0$ if $k = 0$. (In particular, $i_X\alpha$ is a $(k-1)$-form.)