On the l-function and the Reduced Volume of Perelman

Rugang Ye
Department of Mathematics
University of California,Santa Barbara

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1 Introduction

In [P], Perelman introduced, among other things, two important tools for analyzing the Ricci flow: the reduced distance, i.e. the l-function, and the reduced volume. The l-function is defined in terms of a natural curve energy along the Ricci flow, which is analogous to the classical curve energy associated with geodesics, but involves the evolving metric and the scalar curvature (as a potential term). The reduced volume is a certain integral involving the l-function. Of fundamental importance are the monotonicity properties of the l-function and the reduced volume. These monotonicities can be used, as demonstrated by Perelman, to classify and analyse blow-up limits of the Ricci flow, and to obtain various estimates for the Ricci flow, such as non-collapsing estimates and curvature estimates. A notable example is Prop. 11.2 in [P], which identifies blow-down limits of the κ-solutions (κ-non-collapsed ancient solutions of the Ricci flow with bounded curvature and nonnegative curvature operator) to be gradient shrinking solitons.

The main purpose of this paper is to present a number of analytic and geometric properties of the l-function and the reduced volume, including the monotonicity, the upper bound and the rigidity of the reduced volume. In Perelman's paper, a general assumption concerning the l-function and the reduced volume is uniformly bounded sectional curvature. The results presented in [P] under this assumption are sufficient for the application to the geometrization of 3-manifolds. Because of the fundamental role of the l-function and the reduced volume for analyzing Ricci flow in general, it is very desirable to allow weaker geometric conditions. Our main focus is to deal with the situation in which only a lower bound for the Ricci curvature is assumed. (Naturally, we follow many arguments in [P].) On the other hand, we hope that our treatment can provide assistance for understanding Perelman's theory, even when one is only interested in the case of bounded sectional curvature.

In this paper we also present a detailed proof of Proposition 11.2 in [P] as mentioned above, which is based on the analytic and geometric properties of the l-function and the reduced volume. The basic ideas of this proof are due to Perelman.

Communications with Perelman were of great help for understanding his ideas for the proof of [Proposition 11.2, P]. We also benefited much from conversations with Guofang Wei and Vitali Kapovich.

2 Basic Properties of the l-Function I

Consider a solution $(M, g = g(\tau))$ of the backward Ricci flow

$$\frac{\partial g}{\partial \tau} = 2Ric \tag{2.1}$$

on a manifold M over an interval $[0, T)$. We assume that $(M, g(\tau))$ is complete for each $\tau \in [0, T)$.

Notations We shall denote the distance between two points q_1, q_2 with respect to the metric $g(\tau)$ by $d(q_1, q_2, \tau)$ or $d_{g(\tau)}(q_1, q_2)$. The geodesic ball of center q and radius r with respect to the metric $g(\tau)$ will be denoted by $B_r(q, \tau)$. The volume form of $g(\tau)$ will be denoted by dq or $dq|_\tau$. The scalar curvature $R_{g(\tau)}$ of $g(\tau)$ at a point q will be written as $R(q, \tau)$. Similar notations are also used for other curvature quantities.

A basic and simple lemma is this.

Lemma 2.1 *If $Ric \geq -cg$ for a nonnegative constant c on the time interval $[0, \tau]$, then*

$$e^{-2cs} g(0) \leq g(s) \leq e^{2c(\tau - s)} g(\tau) \tag{2.2}$$

for $s \in [0, \tau]$. If $Ric \leq Cg$ for a nonnegative constant C on $[0, \tau]$, then

$$e^{2C(s-\tau)} g(\tau) \leq g(s) \leq e^{2Cs} g(0) \tag{2.3}$$

for $s \in [0, \tau]$.

We consider Perelman's $\mathcal{L}$-energy for piecewise C^1 curves $\gamma : [a,b] \to M, 0 \leq a < b < T$:

$$\mathcal{L}_{a,b}(\gamma) = \int_a^b \sqrt{s}(R(\gamma(s), s) + |\dot{\gamma}|^2) ds, \tag{2.4}$$

where $|\cdot| = |\cdot|_{g(s)}$. For a given τ we abbreviate $\mathcal{L}_{0,\tau}$ to $\mathcal{L}$. The $\mathcal{L}_{a,b}$-geodesic (or $\mathcal{L}$-geodesic) equation is:

$$\nabla_{\frac{d}{ds}} \dot{\gamma} - \frac{1}{2}\nabla R + \frac{1}{2s}\dot{\gamma} + 2Ric(\dot{\gamma}, \cdot) = 0, \tag{2.5}$$

where $R = R_{g(s)}, Ric = Ric_{g(s)}$, and ∇ is the Levi-Civita connection of $g(s)$. This is the Euler-Lagrange equation of the $\mathcal{L}$-energy. Its (smooth) solutions are called $\mathcal{L}_{a,b}$-geodesics or $\mathcal{L}$-geodesics.

To better understand the properties of $\mathcal{L}_{a,b}$-geodesics, it is helpful to introduce a convenient reparametrization. We set $t = \sqrt{s}$ and $\gamma' = d\gamma/dt = 2t\dot{\gamma}$. Then

$$\mathcal{L}_{a,b}(\gamma) = \int_{\sqrt{a}}^{\sqrt{b}} (\frac{1}{2}|\gamma'|^2 + 2Rt^2) dt \tag{2.6}$$

and the $\mathcal{L}_{a,b}$-geodesic equation becomes

$$\nabla_{\frac{d}{dt}} \gamma' - 2t^2 \nabla R + 4tRic(\gamma', \cdot) = 0. \tag{2.7}$$

Next we choose a reference point $p \in M$ and define $L(q, \tau) = L_g(q, \tau)$ to be the infimum of $\mathcal{L}(\gamma)$ for $\gamma : [0, \tau] \to M$ with $\gamma(0) = p$ and $\gamma(\tau) = q$. (We write $L_g(q, \tau)$ if we need to indicate the dependence on the solution g.)

Definition 1 We define the *reduced distance* to be

$$l(q, \tau) = l_g(q, \tau) = \frac{L(q, \tau)}{2\sqrt{\tau}}. \tag{2.8}$$

Lemma 2.2 *Assume that $Ric \geq -cg$ on $[0, \bar{\tau}]$ for a nonnegative constant c. Then*

$$e^{2c\tau} \frac{d^2(p, q, \bar{\tau})}{4\tau} + \frac{nc}{3}\tau \geq l(q, \tau) \geq e^{-2c\tau} \frac{d^2(p, q, 0)}{4\tau} - \frac{nc}{3}\tau \tag{2.9}$$

for each $\tau \in [0, \bar{\tau}]$. If we assume instead $Ric \leq Cg$ on $[0, \bar{\tau}]$ for a nonnegative constant C, then

$$e^{-2C\tau} \frac{d^2(p, q, \bar{\tau})}{4\tau} - \frac{nC}{3}\tau \leq l(q, \tau) \leq e^{2C\tau} \frac{d^2(p, q, 0)}{4\tau} + \frac{nC}{3}\tau \tag{2.10}$$

for each $\tau \in [0, \bar{\tau}]$.

Proof. We first assume a lower bound on the Ricci curvature. By (2.2) and (2.6) we have for an arbitrary γ from p to q

$$\mathcal{L}(\gamma) \geq \frac{e^{-2c\tau}}{2}\int_0^{\sqrt{\tau}} |\gamma'|^2_{g(0)} dt - \frac{2nc}{3}\tau^{\frac{3}{2}} \geq e^{-2c\tau}\frac{d^2(p,q,0)}{2\sqrt{\tau}} - \frac{2nc}{3}\tau^{\frac{3}{2}}. \tag{2.11}$$

This leads to the second half of (2.9).

To deduce the first half of (2.9) we apply (2.2) and choose γ to be a minimal geodesic from p to q with respect to the metric $g(\bar{\tau})$. The case of an upper bound for the Ricci curvature is similar, in which we use (2.3) instead of (2.2). ■

Next we consider Perelman's $\mathcal{L}$-*exponential map*.

Definition 2 The $\mathcal{L}$-*exponential map* $exp_p^{\mathcal{L},\tau} : T_pM \to M$ at time $\tau \in [0,T)$ is defined as follows. For $v \in T_pM$, let γ_v denote the $\mathcal{L}$-geodesic such that $\gamma_v(0) = p, \lim_{s\to 0}\sqrt{s}\dot{\gamma}(s) = v$ (equivalently, $\gamma_v'(0) = \frac{v}{2}$). If γ_v exists on $[0,\tau]$, we set $exp_p^{\mathcal{L},\tau}(v) = \gamma_v(\tau)$. Let $\mathcal{U}(\tau)$ denote the maximal domain of $exp_p^{\mathcal{L},\tau}$. By (2.7) and basic ODE, $\mathcal{U}(\tau)$ is an open set and $exp_p^{\mathcal{L},\tau}$ is a smooth map from $\mathcal{U}(\tau)$ into M.

We also have the following extension of the concept of $\mathcal{L}$-exponential map.

Definition 3 For a given reference point $\bar{p}$ and $0 < \varepsilon < \tau$ the $\mathcal{L}_{\varepsilon,\tau}$-exponential map $exp_{\bar{p}}^{\mathcal{L}_{\varepsilon,\tau}}$ is defined as follows. For $v \in T_{\bar{p}}M$, let $\gamma_{v,\varepsilon}$ denote the $\mathcal{L}$-geodesic such that $\gamma_{v,\varepsilon}(\varepsilon) = \bar{p}, \sqrt{\varepsilon}\dot{\gamma}(\varepsilon) = v$ (equivalently, $\gamma_{v,\varepsilon}'(0) = \frac{v}{2}$). If $\gamma_{\varepsilon,v}$ exists on $[0,\tau]$, we set $exp_{\bar{p}}^{\mathcal{L}_{\varepsilon,\tau}}(v) = \gamma_{\varepsilon,v}(\tau)$.

Proposition 2.3 *Assume that the sectional curvature is bounded on* $[0,\bar{\tau}]$ *for* $\bar{\tau} \in (0,T)$*. Then* $\mathcal{U}(\tau) = T_pM$ *for each* $\tau \in (0,\bar{\tau})$*. A similar statement holds true for* $exp_{\bar{p}}^{\mathcal{L}_{\varepsilon,\tau}}$.

Proof. By the local interior estimates in [S], the sectional curvature bound on $[0,\bar{\tau}]$ implies an upper bound on $|\nabla R|$ on $[0,\tau]$ for each $\tau \in (0,\bar{\tau})$. Fix $\tau \in (0,\bar{\tau})$ and let K denote an upper bound for $|Ric|$ and $|\nabla R|$ on $M \times [0,\tau]$.

Consider an $\mathcal{L}$-geodesic γ with initial time 0, defined on its maximal interval. We derive from (2.7)

$$\frac{d}{dt}|\gamma'|^2 \;=\; \frac{\partial g}{\partial s}\frac{\partial s}{\partial t} + 2\gamma' \cdot \nabla_{\frac{d}{dt}}\gamma' = 4t^2\nabla R \cdot \gamma' - 4tRic(\gamma',\gamma'). \tag{2.12}$$

Consequently, we obtain for $t \leq \sqrt{\tau}$ (as long as γ is defined)

$$|\frac{d}{dt}|\gamma'|^2| \leq 2Kt \cdot (2t|\gamma'|) + 4Kt|\gamma'|^2 \leq 2Kt^3 + 6Kt|\gamma'|^2 \tag{2.13}$$

and hence

$$|\gamma'|^2 \le e^{6Kt^2}(|\gamma'(0)|^2 + \int_0^t 2Ku^3 e^{-6Ku^2} du) \tag{2.14}$$

for $t \le \sqrt{\tau}$. This and (2.2) give rise to a uniform upper bound for the length of $\gamma|_{[0,\tau']}$ for $\tau' \le \tau$ measured in $g(0)$. By the completeness of $g(0)$ and basic ODE we conclude that γ is defined on $[0, \tau]$. ■

Proposition 2.4 *We have $\cup_\tau \mathcal{U}(\tau) = T_pM$. In other words, the direct limit of $\mathcal{U}(\tau)$ as $\tau \to 0$ is T_pM. Indeed, for each $r > 0$, there is $\tau > 0$ such that $B_r(0) \subset \mathcal{U}(\tau)$, where the norm on T_pM is induced from $g(0)_p$. A similar statement holds true for $exp_{\bar{p}}^{\mathcal{L}_{\varepsilon,\tau}}$.*

Proof. Fix $0 < \tau^* < \min\{T, 1\}$. Let $r > 0$ be given. Let K be an upper bound for $|Ric|$ and ∇R on $B_{2r}(p, 0) \times [0, \tau^*]$. Consider $v \in B_r(0)$. Applying (2.14) and Lemma 2.1 to $\gamma = \gamma_v$ (parameterized in t) we deduce

$$|\gamma'(t)|^2_{g(0)} \le e^{8Kt^2}(r^2 + At^2), \tag{2.15}$$

where $A = \int_0^{\sqrt{\tau^*}} 2Kue^{-6Ku^2} du$, as long as t is in the maximal existence interval of γ, $t \le \sqrt{\tau^*}$ and $\gamma(t') \in B_{2r}(p, 0)$ for all $t' \in [0, t]$. For such t which also satisfies $t < \frac{r^2}{2\sqrt{A}}$ and $t < \sqrt{\frac{1}{8K} \ln \frac{5}{4}}$ we then have

$$|\gamma'(t')|_{g(0)} \le \frac{3}{2} r \tag{2.16}$$

for all $t' \in [0, t]$, whence $\int_0^t |\gamma'|_{g(0)} dt \le \frac{3}{2} r$. Consequently, $\gamma([0, t]) \subset B_{\frac{3}{2}r}(p, 0)$. Since $\gamma(0) \in B_{\frac{r}{2}}(p, 0)$, a continuity argument shows that $\gamma([0, t]) \subset B_{\frac{3}{2}r}(p, 0)$ for all t in the maximal existence interval of γ such that $0 \le t \le t^*$, where $t^* = \min\{\sqrt{\tau^*}, \frac{r^2}{2\sqrt{A}}, \sqrt{\frac{1}{8K} \ln \frac{9}{5}}\}$. This implies in turn that $[0, t^*]$ is contained in the maximal existence interval of γ. It follows that $v \in \mathcal{U}(\tau)$ for each $\tau \in [0, t^{*2}]$. ■

Proposition 2.5 *For each sufficiently small $\tau > 0$, $exp_p^{\mathcal{L},\tau}$ is a diffeomorphism from a neighborhood of 0 in T_pM onto a neighborhood of p in M. If $\nabla^2 R(p, \tau) \ge 0$ for each τ, then this holds for each τ. A similar statement holds for $exp_{\bar{p}}^{\mathcal{L}_{\varepsilon,\tau}}$.*

Proof. First note that $0 \in \mathcal{U}(\tau)$. Indeed, the $\mathcal{L}$-geodesic γ_0 is the constant curve $\gamma_0 \equiv p$, hence it is defined for all τ. To establish the desired difeomorphism property, it suffices to show that the differential of $exp_p^{\mathcal{L},\tau}$ at 0 is has zero kernel. For this purpose, consider a nonzero $v \in T_pM$ and $exp_p^{\mathcal{L},\tau}(xv) = \gamma_{xv}(\tau)$. Obviously, $dexp_p^{\mathcal{L},\tau}|_0(v) = Y_v(\sqrt{\tau})$, where Y_v (parameterized in t) is the $\mathcal{L}$-Jacobi field along γ_0 associated with the family of $exp_p^{\mathcal{L},\tau}$-geodesics γ_{xv} (with parameter x). Thus $Y_v(0)=0, \nabla_{\frac{d}{dt}} Y_v(0)=v$. By [(7.7), P], the $\mathcal{L}$-Jacobi equation along an $\mathcal{L}$-geodesic γ (parameterized in s) is

$$\nabla_{\frac{d}{ds}}\nabla_{\frac{d}{ds}} Y+\frac{1}{2s}\nabla_{\frac{d}{ds}} Y+Rm(\dot\gamma,Y)\dot\gamma+2\nabla_Y Ric(\dot\gamma,\cdot)-\nabla_{\dot\gamma}Ric(Y,\cdot)-\frac{1}{2}\nabla^2 R(Y,\cdot)=0. \tag{2.17}$$

For $\gamma=\gamma_0$ this becomes, when parameterized in t,

$$\frac{d^2Y}{dt^2}-2t^2\nabla^2 R(p,t^2)(Y,\cdot)=0. \tag{2.18}$$

It is easy to see that for small τ, $Y(0)=0$ and $Y(\sqrt{\tau})=0$ imply that $Y\equiv 0$. The same holds for each τ if $\nabla^2 R(p,\tau)\ge 0$ for each τ. Applying this to Y_v we arrive at the desired conclusions. ■

Proposition 2.6 *If $Ric \ge -cg$ on $[0,\tau]$ for a nonnegative constant c, then there exists a minimal $\mathcal{L}_{0,\tau}$-geodesic from p to q for each q. Consequently, $exp_p^{\mathcal{L},\tau}$ is onto. The same conclusions hold if $Ric \le Cg$ on $[0,\tau]$ for a nonnegative constant C.*

Proof. For a given $q\in M$ we minimize the $\mathcal{L}$-energy in the reparametrized form (2.6) among Sobolev curves which connect p to q. By the estimate (2.9) we can find a minimizer γ. By the standard elliptic regularity, it is a smooth $\mathcal{L}$-geodesic connecting p to q. Set $v=\gamma'(0)/2$. Then $exp_p^{\mathcal{L},\tau}(v)=q$. ■

Definition 4 1) We define the *injectivity domain* $\Omega(\tau)$ at time τ to be

$$\Omega(\tau)=\{q\in M:\ \text{there is a unique minimal } \mathcal{L}-\text{geodesic } \gamma:[0,\tau]\to M$$

$$\text{with } \gamma(0)=p, \gamma(\tau)=q; q \text{ is not conjugate to } p \text{ along } \gamma\}.$$

Here, "conjugate" means the same as in ordinary Riemannian geometry of geodesics, i.e. there is a nontrivial $\mathcal{L}$-Jacobi field J along γ with $J(0)=0, J(\tau)=0$.

The *cut-locus* $C(\tau)$ is defined to be $M-\Omega(\tau)$.

The corresponding concepts, the $\mathcal{L}_{\varepsilon,\tau}$ injectivity domain $\Omega(\varepsilon,\tau)$ and cut-locus $C(\varepsilon,\tau)$ associated with $\mathcal{L}_{\varepsilon,\tau}$-geodesics, are defined in a similar way.

2) The *tangential injectivity domain* $\Omega^{T_p}(\tau)$ at time τ is defined to be

$$\Omega^{T_p}(\tau)=\{v\in\mathcal{U}(\tau):\gamma_v|_{[0,\tau]} \text{ is a unique minimal } \mathcal{L}-\text{geodesic s.t. } \tau \text{ is not a conjugate}$$

$$\text{time.}\}$$

It is easy to see that $\Omega(\tau)=exp_p^{\mathcal{L},\tau}(\Omega^{T_p}(\tau))$.

The tangential $\mathcal{L}_{\varepsilon,\tau}$ injectivity domain $\Omega^{T_{\bar p}}(\varepsilon,\tau)$ is defined in a similar way.

Lemma 2.7 *$exp_p^{\mathcal{L},\tau}$ is a smooth diffeomorphism from $\Omega^{T_p}(\tau)$ onto $\Omega(\tau)$, depending smoothly on the parameter τ. $L(q,\tau)$ is a smooth function on $\cup_\tau \Omega(\tau)\times\{\tau\}$.*

Similarly, $exp_{\bar p}^{\mathcal{L}_{\varepsilon,\tau}}$ is a smooth diffeomorphism from $\Omega^{T_{\bar p}}(,\varepsilon,\tau)$ onto $\Omega(\varepsilon,\tau)$, depending smoothly on ε and τ, and $L_{\varepsilon,\tau}(q)$ is smooth on $\cup_{\varepsilon,\tau}\{(\varepsilon,\tau)\}\times\Omega(\varepsilon,\tau)$, where $L_{\varepsilon,\tau}(q)$ is defined to be the infimum of $\mathcal{L}_{\varepsilon,\tau}(\gamma)$ for $\gamma:[\varepsilon,\tau]\to M$ such that $\gamma(\varepsilon)=\bar p, \gamma(\tau)=q$.

Proof. This is similar to the theory of ordinary geodesics in Riemannian geometry. ■

Lemma 2.8 *Let γ be a minimal $\mathcal{L}_{0,\tau}$-geodesic. Then $\gamma|_{[0,\tau']}$ is the unique minimal $\mathcal{L}_{0,\tau'}$-geodesic from p to $\gamma(\tau')$ for any $\tau'\in(0,\tau)$. Moreover, τ' is not a conjugate time. Thus $\gamma(\tau')\in\Omega(\tau')$. We also have $\gamma(\tau)\in\Omega(\varepsilon,\tau)$ for any $\varepsilon\in(0,\tau)$, where the reference point $\bar p$ for $\Omega(\varepsilon,\tau)$ is chosen to be $\gamma(\varepsilon)$.*

As a consequence, we have $\Omega^{T_p}(\tau_2)\subset\Omega^{T_p}(\tau_1)$ for $\tau_2\geq\tau_1$.

Proof. The arguments in the theory of ordinary geodesics can be applied directly. ■

Note that if $q\in\Omega(\tau)$ and γ is the unique minimal $\mathcal{L}$-geodesic from p to q, then we have

$$\dot\gamma(s)=\nabla l(\gamma(s)), \gamma'(s(t))=\nabla L(\gamma(s(t))) \tag{2.19}$$

for $s\in[0,\tau]$ and $s(t)=t^2$.

Proposition 2.9 *Let $\bar\tau\in(0,T)$. Assume that $Ric\geq -cg$ on $[0,\bar\tau]$ for a nonnegative constant c. Then $L(\cdot,\tau)$ is locally Lipschitz with respect to the metric $g(\tau)$ for each $\tau\in(0,\bar\tau]$. Moreover, for each compact subset E of M, there are positive constants A_1 and A_2 such that $\sqrt{\tau}L\leq A_1$ on $E\times(0,\bar\tau]$ and*

$$|\dot\gamma(s)|^2\leq\frac{A_2}{s}(1+\frac{1}{\tau}) \tag{2.20}$$

for $s\in(0,\tau]$, where $\tau\in(0,\bar\tau]$ and γ denotes an arbitrary minimal $\mathcal{L}_{0,\tau}$-geodesic from p to q for $q\in E$.

Proof. We first derive an upper bound for $\sqrt{\tau}L(q,\tau)$ on $B_\rho(p,\bar\tau)\times(0,\bar\tau]$ for a given $\rho>0$. By smoothness, there is a positive constant C such that $R\leq C$ on $B_\rho(p,\bar\tau)\times[0,\bar\tau]$. For $q\in B_\rho(p,\bar\tau)$ and $\tau\in(0,\bar\tau]$ we choose a minimal geodesic $\gamma:[0,\sqrt{\tau}]\to B_\rho(p,\bar\tau)$ from p to q with respect to $g(\bar\tau)$. By (2.2) and (2.6) we have

$$\mathcal{L}(\gamma)\leq\int_0^{\sqrt{\tau}}(e^{2c(\bar\tau-t)}|\gamma'|^2_{g(\bar\tau)}+2Ct^2)dt\leq e^{2c\bar\tau}\frac{d(p,q,\bar\tau)}{\sqrt{\tau}}+3C\tau^{\frac{3}{2}}.$$

It follows that

$$\sqrt{\tau}L(q,\tau)\leq A(\rho),$$

where $A(\rho) = e^{2c\bar{\tau}}\rho + 3C\bar{\tau}$.

Next consider a given $\rho > 0$. Choose ρ_1 such that $B_\rho(p,0) \subset B_{\rho_1}(p,\bar{\tau})$. We set $\rho^* = \max\{e^{c\bar{\tau}}\sqrt{\frac{4nc}{3}\bar{\tau}^2 + 2A(\rho_1)}, 2\rho\}$. By the smoothness of g, there is an upper bound K for $|Ric|$ and $|\nabla R|$ on $B_{\rho^*}(p,0) \times [0,\bar{\tau}]$.

Now consider $q_1, q_2 \in B_\rho(p,0)$ and $\tau \in (0,\bar{\tau}]$. Let γ_i be a minimal $\mathcal{L}_{0,\tau}$-geodesic from p to $q_i, i = 1,2$. (By Lemma 2.6, they exist.) Let $\gamma_0 : [0,1] \to M$ be a minimal geodesic from q_1 to q_2 with respect to $g(0)$. By the choice of ρ^*, the image of γ is obviously contained in $B_{\rho^*}(p,0)$. We claim that the images of γ_1 and γ_2 are also contained in $B_{\rho^*}(p,0)$. Indeed, we have

$$\int_0^{\tau'} \sqrt{s}(R + |\dot{\gamma_i}|^2)ds \le L(q_i,\tau) - \int_{\tau'}^{\tau} \sqrt{s}Rds \le \frac{A(\rho_1)}{\sqrt{\tau}} + \frac{2nc}{3}\tau^{\frac{3}{2}} \tag{2.21}$$

for $i = 1,2$ and $\tau' \in [0,\tau]$. By (2.8) and (2.9) (applied to τ') we then deduce

$$d(p,\gamma_i(\tau'),0)^2 \le 4e^{2c\tau'}(\frac{1}{2}\sqrt{\tau'}L(q_i,\tau') + \frac{nc}{3}\tau'^2) \le \rho^{*2} \tag{2.22}$$

for $i = 1,2$ and $\tau' \in [0,\tau]$. It follows that the images of γ_1 and γ_2 are contained in $B_{\rho^*}(p,0)$.

Next we estimate $|\dot{\gamma_1}|$ and $|\dot{\gamma_2}|$. It is more convenient to handle γ_1' and γ_2'. By the arguments in the proof of Lemma 2.3 we deduce

$$|\frac{d}{dt}|\gamma_i'|^2| \le 2Kt^3 + 6Kt|\gamma_i'|^2 \tag{2.23}$$

for $i = 1,2$ and $t \in [0,\sqrt{\tau}]$ and hence

$$c_1|\gamma_i{}'(t_1)|^2 - c_2 \le |\gamma_i{}'(t_2)|^2 \le C_1|\gamma_i{}'(t_1)|^2 + C_2 \tag{2.24}$$

for $t_1, t_2 \in [0,\sqrt{\tau}], i = 1,2$ and positive constants c_1, c_2, C_1, C_2 depending on K and $\bar{\tau}$. It follows that

$$4s|\dot{\gamma_i(s)}|^2 = |\gamma_i'|^2 \le 2c_1^{-1}(\frac{L(q_i,\tau)}{\sqrt{\tau}} + \frac{2nc}{3}\tau) + c_2c_1^{-1} \tag{2.25}$$

for $i = 1,2$ and $s \in [0,\tau]$.

To proceed, we set $d = d(q_1,q_2,0)$ and assume that $d < \frac{\tau}{4}$. We define $\hat{\gamma}_1(s) = \gamma_1(s)$ for $s \in [0,\tau - 2d]$, $\hat{\gamma}_1(s) = \gamma_1(\tau - 2d + 2(s - \tau + 2d))$ for $s \in [\tau - 2d, \tau - d]$ and $\hat{\gamma}_1(s) = \gamma_0(\frac{1}{d}(s - \tau + d))$ for $s \in [\tau - d, \tau]$. Then we have

$$\begin{aligned} L(q_2,\tau) \quad \le \quad & \mathcal{L}(\hat{\gamma}_1) \le L(q_1,\tau) - \int_{\tau-2d}^{\tau} \sqrt{s}R(\gamma_1)ds + \int_{\tau-2d}^{\tau-d} \sqrt{s}(R(\gamma_1) + 4|\dot{\gamma}_1|^2)ds \\ & + \int_{\tau-d}^{\tau} \sqrt{s}(R(\gamma_0) + \frac{1}{d^2}|\dot{\gamma_0}|^2)ds, \end{aligned} \tag{2.26}$$

where the arguments for γ_1 and γ_0 in the second and third integrals correspond to the defintion of $\hat{\gamma}_1$. We have

$$-\int_{\tau-2d}^{\tau}\sqrt{s}R(\gamma_1)ds \le \frac{2nc}{3}(\tau^{\frac{3}{2}}-(\tau-2d)^{\frac{3}{2}}), \tag{2.27}$$

$$\int_{\tau-2d}^{\tau-d}\sqrt{s}R(\gamma_1) \le \frac{2}{3}nK((\tau-d)^{\frac{3}{2}}-(\tau-2d)^{\frac{3}{2}}), \tag{2.28}$$

and

$$\int_{\tau-d}^{\tau}\sqrt{s}R(\gamma_0)ds \le \frac{2}{3}nK(\tau^{\frac{3}{2}}-(\tau-d)^{\frac{3}{2}}). \tag{2.29}$$

By (2.3) we have $|\dot{\gamma_0}|^2 \le e^{2K\tau}d^2$, hence

$$\int_{\tau-d}^{\tau}\sqrt{s}\frac{1}{d^2}|\dot{\gamma_0}|^2ds \le \frac{2}{3}e^{2K\tau}(\tau^{\frac{3}{2}}-(\tau-d)^{\frac{3}{2}}). \tag{2.30}$$

On the other hand, we infer from (2.25) that

$$\int_{\tau-2d}^{\tau-d}4\sqrt{s}|\dot{\gamma_1}|^2ds \le \frac{2d}{\tau}c_1^{-1}(\frac{A(\rho_1)}{\tau}+\frac{2nc}{3}\tau+\frac{c_2}{2}). \tag{2.31}$$

We deduce

$$L(q_2,\tau) \le L(q_1,\tau)+I(\tau,d), \tag{2.32}$$

where

$$I(\tau,d) = \frac{2}{3}(2nc+nK+e^{2K\tau})(\tau^{\frac{3}{2}}-(\tau-2d)^{\frac{3}{2}})$$

$$+\frac{2d}{\tau}c_1^{-1}(\frac{A(\rho_1)}{\tau}+\frac{2nc}{3}\tau+\frac{c_2}{2}). \tag{2.33}$$

Similarly, we have

$$L(q_1,\tau) \le L(q_2,\tau)+I(\tau,d). \tag{2.34}$$

The desired Lipschitz continuity follows. The estimate (2.20) follows from (2.25).

Finally, we would like to point out that the local Lipschitz continuity of $L(\cdot,\tau)$ also follows from its local semiconcavity, which is given by Lemma 2.11 below. Note however that the proof of Lemma 2.11 below uses some arguments here. ■

Proposition 2.10 *Assume that the Ricci curvature is bounded from below on $[0,\bar{\tau}]$. Then $L(q,\cdot)$ is locally Lipschitz on $(0,\bar{\tau}]$ for every $q\in M$. Moreover, $\tau^{\frac{3}{2}}|L_\tau|$ is bounded on $E\times(0,\bar{\tau}]$ for each compact subset E of M.*

Proof. This is similar to the proof of Proposition 2.9 above. Fix $\rho > 0$ and let ρ^* and K have the same meanings as in the proof of Proposition 2.9. Consider $q \in B_\rho(p, 0)$ and $\tau_1, \tau_2 \in (0, \tau]$ such that $\tau_1 < \tau_2$ and $\tau_2 < 2\tau_1$. Choose a minimal $\mathcal{L}_{0,\tau_1}$-geodesic γ_1 from p to q and a minimal $\mathcal{L}_{0,\tau_2}$-geodesic γ_2 from p to q. As in the proof of Proposition 2.9, the images of γ_1 and γ_2 are contained in $B_{\rho^*}(p, 0)$. We define $\hat{\gamma}_1(s) = \gamma_1(s), s \in [0, \tau_1]$ and $\hat{\gamma}_1(s) = q, s \in [\tau_1, \tau_2]$. Then

$$L(q, \tau_2) \le \mathcal{L}_{0,\tau_2}(\hat{\gamma}_1) \le L(q, \tau_1) + \int_{\tau_1}^{\tau_2} \sqrt{s} R(q, s) ds \le L(q, \tau_1) + \frac{2}{3} n K (\tau_2^{3/2} - \tau_1^{3/2}). \tag{2.35}$$

Next we set $\tau_3 = 2\tau_1 - \tau_2$, $\hat{\gamma}_2(s) = \gamma_2(s)$ for $s \in [0, \tau_3]$ and $\hat{\gamma}_2(s) = \gamma_2(\tau_3 + 2(s - \tau_3))$ for $s \in [\tau_3, \tau_1]$. Then

$$\begin{aligned} L(q, \tau_1) \quad &\le \quad \mathcal{L}_{0,\tau_1}(\hat{\gamma}_2) \le L(q, \tau_2) - \int_{\tau_3}^{\tau_2} \sqrt{s} R(\gamma_2) ds \\ &\quad + \int_{\tau_3}^{\tau_1} \sqrt{s} (R(\gamma_2) + 4|\dot{\gamma}_2|^2) ds, \end{aligned} \tag{2.36}$$

where the argument of γ_2 in the last integral on the right hand side is $\tau_3 + 2(s - \tau_3)$. Applying (2.20) we then obtain

$$L(q, \tau_1) \le L(q, \tau_2) + \frac{2n(c + K)}{3} (\tau_1^{\frac{3}{2}} - \tau_3^{\frac{3}{2}}) + 8A_2(1 + \frac{1}{\tau})(\tau_1^{\frac{1}{2}} - \tau_3^{\frac{1}{2}}). \tag{2.37}$$

Clearly, (2.35) and (2.37) imply the desired Lipschitz continuity and derivative bound. ■

Proposition 2.11 *Assume that the Ricci curvature is bounded from below on* $[0, \bar{\tau}]$. *Then* $l(\cdot, \tau)$ *is locally semi-concave for each* $\tau \in (0, \bar{\tau}]$, *i.e. for every point* $q \in M$ *there is a smooth function* ϕ *on a neighborhood* U_q *of* q *such that* $l(\cdot, \tau) + \phi$ *is concave in the sense that the composition of* $l(\cdot, \tau) + \phi$ *with every geodesic in* U_q *is a concave function.*

Proof. By [(7.9), P] we have for each $\tau \in (0, T), q \in \Omega(\tau)$ and $v \in T_q M$

$$Hess_L(v, v) \le \frac{1}{\sqrt{\tau}} |v|^2 - 2\sqrt{\tau} Ric(v, v) - \int_0^\tau \sqrt{s} H(X, Y) ds, \tag{2.38}$$

where $X = \dot{\gamma}$ with γ denoting the unique minimal $\mathcal{L}$-geodesic from p to q, Y is a suitable extension of v along γ such that $|Y(s)|^2 = \frac{s}{\tau}|v|^2$, and

$$\begin{aligned} H(X, Y) = -\nabla_Y \nabla_Y R - 2 < Rm(Y, X)Y, X > -4(\nabla_X Ric(Y, Y) - \nabla_Y Ric(Y, X)) \\ -2Ric_\tau(Y, Y) + 2|Ric(Y, \cdot)|^2 - \frac{1}{s} Ric(Y, Y). \end{aligned} \tag{2.39}$$

To estimate $H(X,Y)$ we fix $\rho > 0$ and assume $q \in B_\rho(p,0) \cap \Omega(\tau)$ and $\tau \in (0,\bar{\tau}]$. Let ρ^* be given in the proof of Proposition 2.9. As in the proof of Proposition 2.9, the smoothness of g implies an upper bound C for $|\nabla^2 R|, |\nabla Ric|, |Ric_\tau|, |Rm|$ and $|Ric|$ on $B_{\rho^*}(p,0) \times [0,\bar{\tau}]$. By the proof of Proposition 2.9, γ is contained in $B_{\rho^*}(p,0)$. Hence we have $|H(X,Y)| \leq \frac{s}{\tau}(C(3+2C+\frac{1}{s})+8C|X|+2C|X|^2)|v|^2$. Applying (2.20) we then deduce $|H(X,Y)| \leq \frac{C_1}{\tau}(s+1+\frac{1}{\tau}|v|^2$ for a positive constant C_1. It follows that

$$Hess_L(v,v) \leq C_2|v|^2 \tag{2.40}$$

for a positive constant $C_2 = C_2(\bar{\tau})$. (Note that if the curvature operator is nonnegative, then $H(X,Y)$ can be estimated as in [7.2, P], namely Hamilton's Harnack inequality implies $H(X,Y) \geq -R(\frac{1}{s}+\frac{1}{\tau_0 - s})|Y|^2$, where $\tau < \tau_0 < T$. One can e.g. choose $\tau_0 = \frac{1}{2}(\tau + T)$ if T is finite.)

We claim that (2.40) holds true for all $q \in B_\rho(p,0)$ in the sense of barriers, provided that C_2 is chosen large enough. This means that for each point $q \in B_\rho(p,0)$ and each $\varepsilon > 0$ we can find a smooth function f on a neighborhood of q (called an *ε-barrier* at q) such that $f \geq L(\cdot,\tau), f(q) = L(q,\tau)$ and $Hess_f(q)(v,v) \leq (C_2+\varepsilon)|v|^2$. Consider $q \in B_\rho(p,0)$. (We can assume that $q \in C_\tau$.) Choose a minimal $\mathcal{L}$-geodesic γ from p to q. For a given $\varepsilon > 0$ we define

$$f = L(\gamma(\varepsilon),\tau) + L_{\varepsilon,\tau}(q), \tag{2.41}$$

where $L_{\varepsilon,\tau}$ is defined in Lemma 2.7 with the reference point $\bar{p} = \gamma(\varepsilon)$. By Lemma 2.7, $L_{\varepsilon,\tau}$ is smooth at q. We can estimate its Hessian at q in the same fashion as above. Indeed, all the relevant lemmas can easily be extended to the situation of $L_{\varepsilon,\tau}$. Then one infers readily that f is an ε-barrier at q.

For each $q \in M$ we choose a suitable smooth function on a neighborhood of q (for example $\phi = -C'd(q,\cdot,\tau)^2$ for a suitable C') and deduce that

$$Hess_{L+\phi} \leq 0 \tag{2.42}$$

on a neighborhood of q in the sense of barriers. The maximum principle then implies that $L+\phi$ is concave in this neighborhood (see e.g. [Y]). ■

Lemma 2.12 *Assume that the Ricci curvature is bounded from below on $[0,\tau]$ for $\tau \in (0,T)$. Then the cut-locus $C(\tau)$ is a closed set of measure zero. Consequently, $|\nabla l|$ and l_τ are measurable on $M \times [0,T)$, provided that the Ricci curvature is bounded from below on $[0,\tau]$ for each $\tau \in [0,T)$. The same conclusions hold if we assume upper bounds for the Ricci curvature.*

Proof. Set $B(\tau) = \{q \in M : \exists$ more than one minimal $\mathcal{L}_{0,\tau} -$ geodesics from p to $q\}$ and $D(s) = \{q \in M : \exists$ a unique minimal $\mathcal{L}_{0,\tau}-$geodesic γ from p to q, q is conjugate to p along $\gamma\}$. By Lemma 2.6, we have $C(\tau) = B(\tau) \cup D(\tau)$. As in the theory of ordinary geodesics, $D(\tau)$ is contained in the set of critical values of $exp_p^{\mathcal{L},\tau}$. By Sards' theorem, it has zero measure. On the other hand, $L(\cdot,\tau)$ is obviously non-differentiable at any point of $B(\tau)$. Since $L(\cdot,\tau)$ is almost everywhere differentiable by Proposition 2.9, $B(\tau)$ has zero measure. (The idea of using the Lipschitz continuity of L was suggested by G. Wei.) It follows that $C(\tau)$ has zero measure.

Another argument was suggested by Perelman. By Proposition 2.11 and Aleksandrov's theorem (see [Y]), $L(\cdot,\tau)$ is twice differentiable almost everywhere. This immediately implies that $B(\tau)$ has measure zero. On the other hand, one can show that at a point in $D(\tau)$, $L(\cdot,\tau)$ cannot be twice differentiable. Hence $D(\tau)$ also has measure zero. (Of course, Aleksandrov's theorem yields more information than the mere statement that $C(\tau)$ has measure zero.)

The arguments in the theory of ordinary geodesics apply to show that $C(\tau)$ is closed.

Next we assume that the Ricci curvature is bounded below on $[0,\tau]$ for each τ. Obviously, $\cup_\tau(C(\tau)\times\{\tau\})$ is a closed set of measure zero. Since l is smooth outside of $\cup_\tau(C(\tau)\times\{\tau\})$, it follows that $|\nabla l|$ and l_τ are measurable on $M\times[0,T)$.

In the case of upper bounds for the Ricci curvature, Proposition 2.9 is not applicable. But one can follow the argument outlined in [KL] in this case. (It can also be used in the case of lower bounds for the Ricci curvature.) More precisely, we have as above $C(\tau)=B(\tau)\cup D(\tau)$. By Sards' theorem, we only need to show that $B(\tau)^*$ has zero measure, where $B(\tau)^*$ is the intersection of $B(\tau)$ with the set of regular values of $exp_p^{\mathcal{L},\tau}$. Consider $q\in B(\tau)^*$. Then there are $v_1,v_2\in T_pM$ such that $v_1\neq v_2, exp_p^{\mathcal{L},\tau}(v_1)=exp_p^{\mathcal{L},\tau}(v_2)=q$, and $L(v_1,\tau)=L(v_2,\tau)$, where $L(v,\tau)=\mathcal{L}(\gamma_v)$. Since v_1 and v_2 are non-critical for $exp_p^{\mathcal{L},\tau}$, there are disjoint neighborhoods U_1 of v_1 and U_2 of v_2 such that $F_1=exp_p^{\mathcal{L},\tau}|_{U_1}$ and $F_2=exp_p^{\mathcal{L},\tau}|_{U_2}$ are diffeomorphisms onto their common image U, which is a neighborhood of q.

To proceed, we define $L_*(v,w)=L(v,\tau)-L(w,\tau)$, and set $S=\{(v,w)\in U_1\times U_2: F_1(v)=F_2(w)\}$. Obviously, S is a codimension 1 submanifold of $U_1\times U_2$. We claim that 0 is a regular value of $L_*|_S$. Indeed, consider a curve $(v(t),w(t))$ in S which represents a tangent vector $(v'(0),w'(0))$ of S at a given point $(v(0),w(0))$. Since $exp_p^{\mathcal{L},\tau}(v(t))=exp_p^{\mathcal{L},\tau}(w(t))$, we have

$$d(exp_p^{\mathcal{L},\tau})_{v(0)}(v'(0))=d(exp_p^{\mathcal{L},\tau})_{w(0)}(w'(0)). \tag{2.43}$$

On the other hand, by the first variation formula [(7.1), P] for the $\mathcal{L}$ energy, we have

$$\frac{dL^*(v(t),w(t))}{dt}(0)=2\sqrt{\tau}(<\dot\gamma_{v(0)}(0),Y_1>-<\dot\gamma_{w(0)}(0),Y_2>), \tag{2.44}$$

where $Y_1=d(exp_p^{\mathcal{L},\tau})_{v(0)}(v'(0))$ and $Y_2=d(exp_p^{\mathcal{L},\tau})_{w(0)}(w'(0))$. Since $v(0)\neq w(0)$, we have $\dot\gamma_{v(0)}(\tau)\neq\dot\gamma_{w(0)}(\tau)$. It follows that $dL^*_{((v(0),w(0))}((v'(0),w'(0)))\neq 0$. By the implicit function theorem, $L_*|_S^{-1}(0)$ is an $(n-2)$-dimensional submanifold. Consequently, $S^*=\pi_1\circ F(L^*|_S^{-1}(0))$ is an $(n-2)$-dimensional submanifold, where $F=(F_1,F_2)$ and π_1 denotes the projection from $U\times U$ to the first factor. We call S^* a *local container* for $B(\tau)^*$.

It is easy to see that $B(\tau)^*$ is contained in a countable union of local containers. Hence it has zero measure. ∎

Lemma 2.13 *Assume that the curvature operator is nonnegative for each τ. Then we have*

$$R \leq \frac{Cl}{\tau} \tag{2.45}$$

everywhere,

$$|\nabla l|^2 \leq \frac{Cl}{\tau} \tag{2.46}$$

almost everywhere in M for every $\tau > 0$, and

$$|l_\tau| \leq \frac{Cl}{\tau} \tag{2.47}$$

almost everywhere in $(0, T)$ for every q, where C depends only on n. Integrating (2.47) yields

$$(\frac{\tau_1}{\tau_2})^C \leq \frac{l(q,\tau_2)}{l(q,\tau_1)} \leq (\frac{\tau_2}{\tau_1})^C \tag{2.48}$$

for all $q \in M$ and $\tau_1, \tau_2 \in (0, T)$ with $\tau_2 > \tau_1$.

Proof. By [(7.16), P] we have

$$|\nabla l|^2 + R \leq C\frac{l}{\tau} \tag{2.49}$$

on $\cup_\tau(\Omega(\tau) \times \{\tau\})$. The first and second estimates follow from this, Proposition 2.9, and Lemma 2.12. To derive (2.47) we observe that it is invariant under the rescaling $g(\tau) \to \frac{1}{a}g(a\tau)$, hence it suffices to prove it for $\tau = 1$ at points q such that $l(q, \cdot)$ is differentiable at q. By Hamilton's Harnack inequality ([(11.1), P]), $R_\tau \leq 0$ and hence $R(q, s) \leq R(q, 1)$ in (2.35) with $\tau_1 = 1$. By (2.45) we then infer

$$L(q,\tau_2) \leq L(q,1) + CL(q,1)({\tau_2}^{\frac{3}{2}} - 1). \tag{2.50}$$

On the other hand, we can apply (2.45) and (2.46) in (2.36) to deduce

$$L(q,1) \leq L(q,\tau_2) + CL(q,\tau_2)(\sqrt{\tau_2} - \sqrt{\tau_3}). \tag{2.51}$$

Obviously, (2.50) and (2.51) imply $|l_\tau(q,1)| \leq Cl(q,1)$.

Alternatively, we have by [(7.5), P] and [(7.6), P]

$$L_\tau = \sqrt{\tau}R - \frac{|\nabla L|^2}{4\sqrt{\tau}} \tag{2.52}$$

on $\cup_\tau \Omega(\tau)$. Applying Lemma 2.12 and (2.46) we also obtain (2.47). ∎

Lemma 2.14 *Assume that the Ricci curvature is bounded on* $[0, \bar{\tau}]$. *Then there is a positive constant* $C = C(\tau^*)$ *for every* $\tau^* \in (0, \bar{\tau})$ *with the following properties. For each* $\tau \in (0, \tau^*]$ *we have*

$$|\nabla l|^2 \leq \frac{C}{\tau}(l + \tau + 1) \tag{2.53}$$

almost everywhere in M. *For each* $q \in M$ *we have*

$$|l_\tau| \leq \frac{C}{\tau}(l + \tau + 1) \tag{2.54}$$

almost everywhere in $(0, \tau^*]$.

Proof. Consider $\tau^* \in (0, \bar{\tau}), \tau \in (0, \tau^*]$ and $q \in M$. By the arguments in the proof of Proposition 2.9 we deduce for a minimal $\mathcal{L}_{0,\tau}$-geodesic γ from p to q

$$|\dot{\gamma}|^2 \leq \frac{C}{s}(l(q, \tau) + \tau + 1) \tag{2.55}$$

for a positive constant $C = C(\tau^*)$. Taking $s = \tau$ in (2.55) and applying Lemma 2.12 we then arrive at (2.53).

The estimate (2.54) follows from (2.55) and the arguments in the proof of Proposition 2.10. ■

Lemma 2.15 *Assume that the Ricci curvature is bounded from below on* $[0, \bar{\tau}]$. *Then there holds for every* $\tau \in (0, \bar{\tau}]$

$$\int_M l \Delta \phi dq \leq \int_{*M} \phi \Delta l dq \tag{2.56}$$

for nonnegative smooth functions ϕ *with compact support, where the integral* $\int_{*M}$ *means* $\liminf_{\epsilon \to 0} \int_{M - U_\epsilon}$, *with* $U_\epsilon = U_\epsilon(C_\tau)$ *denoting the* ϵ*-neighborhood of* C_τ $(\epsilon > 0)$. *Consequently, we have*

$$-\int_M \nabla l \cdot \nabla \phi dq \leq \int_{*M} \phi \Delta l dq \tag{2.57}$$

for nonnegative Lipschitz functions ϕ *with compact support.*

Proof. Consider $q_0 \in M$. By Proposition 2.11, there is a neighborhood U of q_0 and a smooth function ψ on U such that $l + \psi$ is concave. We can assume that $l + \psi$ is actually strictly concave, i.e. it is the sum of a concave function and a smooth concave function with negative Hessian. By [GW] or [Y], there exists a sequence of smooth concave functions f_k with negative Hessian on a neighborhood $\hat{U} \subset U$ of q_0 such that: 1) f_k converge uniformly to $l + \phi$ on $\hat{U}$, and 2) the derivatives of f_k converge uniformly to the derivatives of $l + \phi$ on $\hat{U} - U_\epsilon(C(\tau))$ for each $\epsilon > 0$.

Let ϕ be a nonnegative smooth function with compact support contained in $\hat{U}$. Setting $U_\epsilon = U_\epsilon(C(\tau))$ we then have

$$\int_M f_k \Delta\phi dq = \int_M \Delta f_k \phi dq = \int_{M-U_\epsilon} \Delta f_k \phi dq + \int_{U_\epsilon} \Delta f_k \phi dq \le \int_{M-U_\epsilon} \Delta f_k \phi dq. \quad (2.58)$$

Taking limit we deduce

$$\int_M (l+\psi)\Delta\phi dq \le \int_{M-U_\epsilon} \Delta(l+\psi)\phi dq. \quad (2.59)$$

It follows that

$$\int_M (l+\psi)\Delta\phi dq \le \liminf_{\epsilon\to 0} \int_{M-U_\epsilon} \Delta(l+\psi)\phi dq. \quad (2.60)$$

Since $C(\tau)$ is closed and has zero measure by Lemma 2.12 there holds

$$\lim_{\epsilon\to 0} \int_{M-U_\epsilon} \Delta\psi\phi dq = \int_M \Delta\psi\phi dq = \int_M \psi\Delta\phi dq. \quad (2.61)$$

Hence we conclude that

$$\int_M l\Delta\phi dq \le \liminf_{\epsilon\to 0} \int_{M-U_\epsilon} \phi\Delta l dq. \quad (2.62)$$

Since q_0 is arbitrary, we deduce by using a partition of unity that (2.56) holds true for all nonnegative smooth functions ϕ with compact support. ■

Lemma 2.16 *We have on* $\cup_\tau \Omega(\tau) \times \{\tau\}$

$$l_\tau - \Delta l + |\nabla l|^2 - R + \frac{n}{2\tau} \ge 0 \quad (2.63)$$

and

$$2\Delta l - |\nabla l|^2 + R + \frac{l-n}{\tau} \le 0. \quad (2.64)$$

Moreover, (2.63) becomes an equality at a point if and only if

$$\Delta l = -R + \frac{n}{2\tau} - \frac{1}{2\tau^{\frac{3}{2}}} K \quad (2.65)$$

holds true at that point, and (2.64) becomes an equality at a point if and only if (2.65) holds true at that point, where K *is defined on page 16 in [P].*

Proof. The inequality (2.63) is [(7.13), P], while the inequality (2.64) is [(7.14), P]. The claims about the equality cases follow from the arguments in [P] for [(7.13), P] and [(7.14), P]. ■

Theorem 2.17 *Assume that the Ricci curvature is bounded from below on* $[0, \tau]$ *for each* $\tau \in (0, T)$. *Then the equations*

$$l_\tau - \Delta l + |\nabla l|^2 - R + \frac{n}{2\tau} \geq 0 \tag{2.66}$$

and

$$2\Delta l - |\nabla l|^2 + R + \frac{l-n}{\tau} \leq 0 \tag{2.67}$$

hold true on $M \times (0, T)$, *when* Δl *is interpreted in the weak sense, i.e.*

$$\int_{\tau_1}^{\tau_2} \int_M \{\nabla l \cdot \nabla \phi + (l_\tau + |\nabla l|^2 - R + \frac{n}{2\tau})\phi\} dq d\tau \geq 0 \tag{2.68}$$

for nonnegative Lipschitz functions ϕ *on* $M \times [\tau_1, \tau_2]$ *with* $0 < \tau_1 < \tau_2 < T$ *such that the support of* $\phi(\cdot, \tau)$ *is compact for each* $\tau \in [\tau_1, \tau_2]$, *and*

$$\int_M \{-2\nabla l \cdot \nabla \phi + \phi(-|\nabla l|^2 + R + \frac{l-n}{\tau})\} dq \leq 0 \tag{2.69}$$

for nonnegative Lipschitz functions ϕ *on* M *with compact support and each* $\tau \in (0, T)$.

Proof. We first consider (2.66). Let ϕ be a nonnegative Lipschitz function on $M \times [\tau_1, \tau_2]$ with compact support. By Proposition 2.9, Proposition 2.10, Lemma 2.12 and Lemma 2.15 we have

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_M \{\nabla l \cdot \nabla \phi + (l_\tau + |\nabla l|^2 - R + \frac{n}{2\tau})\phi\} dq d\tau \ \geq \ & \int_{\tau^1}^{\tau_2} \int_M^* (l_\tau - \Delta l + |\nabla l|^2 - R \\ & + \frac{n}{2\tau})\phi dq d\tau, \end{aligned} \tag{2.70}$$

where the integral $\int_M^*$ means the limsup of the integral on $M - U_\epsilon$ as $\epsilon \to 0$. By (2.63) the right hand side in (2.70) is nonnegative.

The inequality (2.67) follows from the same argument, using the inequality (2.64) instead of (2.63). ■

3 Basic Properties of the l-Function II

Lemma 3.1 *Assume that the Ricci curvature is bounded from below on* $[0, \tau]$. *Then the minimum of* $l(\cdot, \tau)$ *does not exceed* $\frac{n}{2}$.

Proof. By the argument in the proof of Proposition 2.11, one readily shows that the differential inequality [(7.15), P]:

$$\bar{L}_\tau + \Delta \bar{L} \leq 2n \tag{3.1}$$

holds true in the sense of barriers, where $\bar{L}(q,\tau) = 2\sqrt{\tau}L(q,\tau)$. More precisely, for each $q \in M$, each $\tau \in (0,T)$ and each $\varepsilon > 0$, there is a smooth function u (an ε-barrier at (q,τ)) on a neighborhood of (q,τ) in $M \times [\tau, T)$ such that $u \geq \bar{L}, u(q,\tau) = \bar{L}(q,\tau)$ and $u_\tau(q,\tau) + \Delta u(q,\tau) \leq 2n + \varepsilon$. (We use the forward interval $[\tau, T)$ here because the left hand side of (3.1) is the backward heat operator.) By Lemma 2.2 the minimum of $l(\cdot,\tau)$ and hence of $\bar{L}(\cdot,\tau)$ is achieved for every τ. Consequently, the maximum principle implies that the minimum of $\bar{L}(\cdot,\tau) - 2n\tau$ is nonincreasing. The desired bound for the minimum of l follows.

The details of the said maximum principle are as follows. Set $v = \bar{L} - 2n\tau$. Then v satisfies $v_\tau + \Delta v \leq 0$ in the sense of barriers. Let $h(\tau) = \min v(\cdot,\tau)$. Consider τ and a minimum point q for $v(\cdot,\tau)$. For $\varepsilon > 0$ let u_ε be an ε-barrier of v at (q,τ). Then we have for $\tau' > \tau$ sufficiently close to τ

$$\frac{h(\tau') - h(\tau)}{\tau' - \tau} \leq \frac{v(q,\tau') - v(q,\tau)}{\tau' - \tau} \leq \frac{u(q,\tau') - u(q,\tau)}{\tau' - \tau}. \tag{3.2}$$

Taking limit we obtain

$$\frac{d^+h}{d\tau} \leq \frac{\partial u}{\partial \tau}(q,\tau) \leq -\Delta u(q,\tau) + \varepsilon, \tag{3.3}$$

where $\frac{d^+h}{d\tau} = \limsup_{\tau' \to 0^+} \frac{h(\tau') - h(\tau)}{\tau' - \tau}$. Obviously, q is a minimum point for $u(\cdot,\tau)$, whence $\Delta u(q,\tau) \geq 0$. Letting $\varepsilon \to 0$ we then arrive at

$$\frac{d^+h}{d\tau} \leq 0. \tag{3.4}$$

Consequently, h is nonincreasing, cf. [H]. ■

Next we present the following lemma of Perelman on comparison of l with the distance, which strengthens Lemma 2.2. The basic idea of the lower bound for l in this lemma was communicated to us by Perelman. We present the estimates in a scaling invariant form.

Lemma 3.2 *Assume that the curvature operator is nonnegative for each τ. Fix a point $x \in M$. Then we have*

$$-l(x,\tau) - 1 + C_1 \frac{d^2(x,q,\tau)}{\tau} \leq l(q,\tau) \leq l(x,\tau) + C_2 \frac{d^2(x,q,\tau)}{\tau} \tag{3.5}$$

for all $q \in M$, where C_1 and C_2 are positive constants depending only on n.

Proof. Note that l and the quantity $d^2(x,q,\tau)/\tau$ are both invariant under the rescaling $g(\tau) \to a^{-1}g(a\tau)$. Hence it suffices to prove (3.5) for the case $\tau = 1$.

Since $\Omega(1)$ is dense in M, it suffices to consider the case $x, q \in \Omega(1)$. Let γ_x, γ_q be the minimal $\mathcal{L}_{0,1}$-geodesics from p to x, q respectively. Then

$$\begin{aligned} d(x,q,1) &= \int_0^1 \frac{d}{ds} d(\gamma_x(s), \gamma_q(s), s) ds \\ &= \int_0^1 \left[\frac{\partial d}{\partial s}(\gamma_x(s), \gamma_q(s), s) + \nabla_I d \cdot \gamma_x'(s) + \nabla_{II} d \cdot \gamma_q'(s) \right] ds, \end{aligned} \tag{3.6}$$

where ∇_I refers to the gradient with respect to the first argument, and ∇_{II} that with respect to the second argument.

By (2.19) we have $\gamma_x'(s) = \nabla l(\gamma_x(s), s)$ and $\gamma_q'(s) = \nabla l(\gamma_q(s), s)$. We also have

$$l(\gamma_q(s), s) = \frac{1}{2\sqrt{s}} \mathcal{L}_{0,s}(\gamma_q|_{[0,s]}) \leq \frac{1}{2\sqrt{s}} \mathcal{L}_{0,1}(\gamma_q) = \frac{1}{\sqrt{s}} l(q,1). \tag{3.7}$$

Hence we can apply (2.46) to deduce

$$|\gamma_q'(s)| \leq \sqrt{C} s^{-3/4} \sqrt{l(q,1)}, |\gamma_x'(s)| \leq \sqrt{C} s^{-3/4} \sqrt{l(x,1)}. \tag{3.8}$$

Next we estimate $\frac{\partial}{\partial s} d(\gamma_x(s), \gamma_q(s), s)$. It follows from (2.46) that

$$|\nabla l^{\frac{1}{2}}|^2 \leq \frac{C}{4\tau}. \tag{3.9}$$

Set $r_0(s) = s^{3/4}(l(q,1)+1)^{-1/2}$. Then we have for $\bar{q}$ with $d(\bar{q}, \gamma_q(s), s) \leq r_0(s)$

$$l^{\frac{1}{2}}(\bar{q}, s) \leq l^{\frac{1}{2}}(\gamma_q(s), s) + \frac{\sqrt{C}}{2\sqrt{s}} r_0(s) \leq (s^{-1/4} + \frac{\sqrt{C}}{2}) \sqrt{l(q,1)+1}. \tag{3.10}$$

By (2.49) again, we infer

$$R(\bar{q}, s) \leq C s^{-1} (s^{-1/4} + \frac{\sqrt{C}}{2})^2 (l(q,1)+1). \tag{3.11}$$

Similarly, we have

$$R(\bar{q}, s) \leq C s^{-1} (s^{-1/4} + \frac{\sqrt{C}}{2})^2 (l(x,1)+1) \tag{3.12}$$

for $\bar{q}$ with $d(\bar{q}, \gamma_q(s), s) \leq r_0(s)$.

Now we apply [P, (8.3 (b))] to deduce

$$\begin{aligned} |\frac{\partial}{\partial s} d(\gamma_x(s), \gamma_q(s), s)| &\leq 2(n-1) \left(\frac{2}{3} K r_0(s) + r_0(s)^{-1} \right) && (3.13) \\ &\leq C' \left(s^{-3/4} + s^{-1/2} + s^{3/4} \right) \sqrt{l(q,1)+1}. && (3.14) \end{aligned}$$

Combining (3.6), (3.8) and (3.14) finally yields the left hand side of (3.5) for $\tau = 1$. The right hand side of (3.5) follows from (3.9). ■

Theorem 3.3 *Assume either that the curvature operator is nonnegative for each τ or that the Ricci curvature is bounded on $[0, \tau]$ for each τ. Then the inequality (2.68) holds true for nonnegative Lipschitz functions ϕ on $M \times [\tau_1, \tau_2]$ such that $\phi \le \bar{C}e^{-l}$ and $|\nabla\phi| \le \bar{C}e^{-l}$ for a positive constant $\bar{C}$ which depends on ϕ. Similarly, the inequality (2.69) holds true for nonnegative Lipschitz functions ϕ such that $\phi \le \bar{C}e^{-l}$ and $|\nabla\phi| \le \bar{C}e^{-l}$ with C depending on ϕ. In both cases, the involved integrals are absolutely convergent. In particular, we obtain by choosing $\phi = e^{-l}$ in (2.68)*

$$\int_{\tau_1}^{\tau_2} \int_M (l_\tau - R + \frac{n}{2\tau}) e^{-l} \tau^{\frac{n}{2}} dq ds \ge 0. \tag{3.15}$$

Proof. We present the case of (2.68), while the case of (2.69) is similar. We can assume that M is noncompact. We first handle the case of nonnegative curvature operator. Let η_k be a sequence of Lipschitz continuous functions with compact support such that $0 \le \eta_k \le 1, |\nabla\eta_k| \le 1$ and $\eta_k = 1$ on the geodesic ball $B_k(x(\tau), \tau)$ for a minimum point $x(\tau)$ of $l(\cdot, \tau)$ and $\tau \in [\tau_1, \tau_2]$. By (2.68) we have

$$\int_{\tau_1}^{\tau_2} \int_M \{\nabla l \cdot (\nabla\eta_k\phi + \eta_k\nabla\phi) + (l_\tau + |\nabla l|^2 - R + \frac{n}{2\tau})\eta_k\phi\} dq d\tau \ge 0 \tag{3.16}$$

By the volume comparison, the volume of $g(\tau)$ grows no faster than the euclidean rate. Hence we can apply (3.5) and Lemma 3.1 in the case of nonnegative curvature operator to deduce

$$\begin{aligned} |\int_M \nabla l \cdot \nabla\eta_k\phi dq &\le \bar{C}\int_{M-B_k(x(\tau),\tau)} |\nabla l| e^{-l} dq \le \bar{C}\int_{M-B_k(x(\tau),\tau)} \sqrt{\frac{Cl}{\tau}} e^{-l} dq \\ &\le \bar{C}_1 e^{-c_1 k} \end{aligned} \tag{3.17}$$

for suitable positive constants $\bar{C}_1$ and c_1. Using Lemma 3.2, Lemma 3.1 and Lemma 2.13 we also obtain

$$\int_{M-B_k(x(\tau),\tau)} \{|\nabla l \cdot \nabla\phi| + (|l_\tau| + |\nabla l|^2 + R + \frac{n}{2\tau})\phi\} dq d\tau \le \bar{C}_2 e^{-c_2 k} \tag{3.18}$$

for suitable positive constants $\bar{C}_2$ and c_2. Taking limit in (3.16) we then arrive at the desired inequality.

In the case of bounded Ricci curvature we choose $x(\tau) = p$ and apply Lemma 2.2 and Lemma 2.14 instead of Lemma 3.2. ■

4 The Reduced Volume

We continue with the solution $(M, g = g(\tau))$ of the backward Ricci flow on $[0, T)$ as before (assuming that $g(\tau)$ is complete for each $\tau \in [0, T)$).

Definition 5 We define the *reduced volume* $\tilde{V}(\tau)$ to be

$$\tilde{V}(\tau) = \tilde{V}_g(\tau) = \int_M \tau^{-\frac{n}{2}} e^{-l(q,\tau)} dq. \tag{4.1}$$

Our goal is to obtain monotonicity of the reduced volume and its upper bounds. For this purpose, we need as in [P] the following weighted monotonicity of the Jacobian of the $\mathcal{L}$-exponential map given in [P].

Lemma 4.1 *Let $J(\tau)(v) = J_g(\tau)(v)$ denote the Jacobian of the $\mathcal{L}$-exponential map $exp_p^{\mathcal{L},\tau}$ at $v \in \Omega^{T_pM}(\tau)$, where T_pM is equipped with the metric $g(\tau)_p$. Then we have*

$$\frac{d}{d\tau^-}\tau^{-\frac{n}{2}} e^{-l(v,\tau)} J(\tau)(v) \leq 0 \tag{4.2}$$

for each $v \in \Omega^{T_pM}(\tau)$, where $l(v,\tau) = l(\gamma_v(\tau),\tau)$ (γ_v is given in Definition 2). Moreover, if $\tau_1^{-\frac{n}{2}} e^{-l(v,\tau_1)} J(\tau_1)(v) = \tau_2^{-\frac{n}{2}} e^{-l(v,\tau_2)} J(\tau_2)(v)$ for $\tau_1 < \tau_2$ and $v \in \Omega^{T_pM}(\tau_2)$, then the equation

$$Ric - \frac{1}{2\tau} g + \nabla^2 l = 0 \tag{4.3}$$

holds true along γ_v on the interval $[\tau_1, \tau_2]$.

Proof. This follows from the arguments on pages 16 and 17 in [P]. ∎

Lemma 4.2 *Consider $v \in \Omega(\hat{\tau})$ for some $\hat{\tau}$. Then*

$$\lim_{\tau \to 0} \tau^{-\frac{n}{2}} e^{-l(v,\tau)} J(\tau)(v) = e^{-\frac{|v|^2}{4}}. \tag{4.4}$$

Consequently,

$$\tau^{-\frac{n}{2}} e^{-l(v,\tau)} J(\tau)(v) \leq e^{-\frac{|v|^2}{4}} \tag{4.5}$$

for each τ.

Proof. Set $\tilde{J}(\tau)(v) = \tau^{-\frac{n}{2}} e^{-l(v,\tau)} J(\tau)(v)$. The following transformation formula is easy to verify:

$$\tilde{J}_{g_a}(\tau)(av) dv_{g_a(0)} = \tilde{J}(a\tau)(v) dv, \tag{4.6}$$

where $g_a(\tau) = a^{-1} g(a\tau)$ and $a\tau \leq \hat{\tau}$. In particular

$$\tilde{J}_{g_a}(1)(av) dv_{g_a(0)} = J(a)(v) dv. \tag{4.7}$$

Using $exp^{\mathcal{L},\hat{\tau}}$ we pull back $g, 0 \leq \tau \leq \bar{\tau}$ to $\Omega^{T_p}(\bar{\tau})$, and then pull it back by the scaling map $\Phi_a(v') = av', v' \in T_pM$. The resulting metrics will be denoted by g^*. Applying (4.7) to g^* we deduce that

$$\tilde{J}(a)(v) = \tilde{J}_{g_a^*}(1)(v) \tag{4.8}$$

for $0 < a < \hat{\tau}$. Next observe that over $[0, 2]$, g_a^* converge smoothly on compact sets of T_pM to the euclidean steady soliton $g^0(\tau) \equiv g(0)_p$ as $a \to 0$. Moreover, the image of the minimal $\mathcal{L}$-geodesic from the reference point 0 to v remains in a fixed compact set during the convergence, which follows from the arguments in the proof of Proposition 2.4. It follows that $\lim_{a \to 0} \tilde{J}(a)(v) = \tilde{J}_{g^0}(1)(v) = e^{-\frac{|v|^2}{4}}$.

The inequality (4.5) follows from (4.4) and Lemma 4.1. ■

Theorem 4.3 *Assume that the Ricci curvature is bounded from below on $[0, \bar{\tau}]$ for some $\bar{\tau}$. Then $\tilde{V}(\tau) \le (4\pi)^{\frac{n}{2}}$ for each $\tau \in (0, \bar{\tau}]$. The same holds if the Ricci curvature is bounded from above on $[0, \bar{\tau}]$.*

Proof. By Lemma 2.12 and Lemma 4.2 we have

$$\tilde{V}(\tau) = \int_{\Omega^{T_p}(\tau)} \tau^{-\frac{n}{2}} e^{-l(v,\tau)} J(\tau)(v) dv \le \int_{\Omega^{T_p}(\tau)} e^{-\frac{|v|^2}{4}} dv \le \int_{T_pM} e^{-\frac{|v|^2}{4}} dv = (4\pi)^{\frac{n}{2}}. \tag{4.9}$$

Theorem 4.4 *Assume that the Ricci curvature is nonnegative for $s \in [0, \tau]$. Then $\tilde{V}(\tau) < (4\pi)^{\frac{n}{2}}$ unless $(M, g(0))$ is isometric to $\mathbf{R}^n$ and $g(s) = g(0)$ for all $s \in [0, \tau]$, in which case $\tilde{V}(\tau) = (4\pi)^{\frac{n}{2}}$.*

Proof. By (2.1), we have $\frac{\partial}{\partial t} dq = R dq$. This and (2.9) imply that

$$\tilde{V}(\tau) \le \tau^{-\frac{n}{2}} \int_M e^{-\frac{d^2(p,q,0)}{4\tau}} dq \le \tau^{-\frac{n}{2}} \int_M e^{-\frac{d^2(p,q,0)}{4\tau}} dq|_0. \tag{4.10}$$

By volume comparison, we have

$$\int_M e^{-\frac{d^2(p,q,0)}{4\tau}} dq|_0 \le \int_{\mathbf{R}^n} e^{-\frac{|x|^2}{4\tau}} dx = (4\pi\tau)^{\frac{n}{2}}. \tag{4.11}$$

Hence we arrive at the desired inequality. If $\tilde{V}(\tau) = (4\pi)^{\frac{n}{2}}$, then (4.11) must be an equality, and hence $(M, g(0))$ is isometric to $\mathbf{R}^n$. The second inequality in (4.10) must also be an equality. Consequently, $R \equiv 0$ and hence $Ric \equiv 0$ for $s \in [0, \tau]$. It follows that $g(s) = g(0)$ for $s \in [0, \tau]$. ■

Theorem 4.5 *If the Ricci curvature is bounded from below on $[0, \tau]$ for each τ, then $\tilde{V}(\tau)$ is a nondecreasing function. The same holds if the Ricci curvature is bounded from above on $[0, \tau]$ for each τ.*

Proof. By Lemma 2.12 we have

$$\tilde{V}(\tau) = \int_{\Omega(\tau)} \tau^{-\frac{n}{2}} e^{-l(q,\tau)} dq = \int_{\Omega^{T_p}(\tau)} \tau^{-\frac{n}{2}} e^{-l(v,\tau)} J(\tau) dv, \tag{4.12}$$

where dv denotes the euclidean volume form on T_pM determined by $g(\tau)_p$. By Lemma 2.8 we have for $\tau_1 < \tau_2$ the inequality $\tilde{V}(\tau_2) - \tilde{V}(\tau_1) \le \int_{\Omega(\tau_2)} \tau_2^{-\frac{n}{2}} e^{-l(v,\tau_2)} J(\tau_2) dv - \int_{\Omega(\tau_2)} \tau_1^{-\frac{n}{2}} e^{-l(v,\tau_1)} J(\tau_1) dv$. By Lemma 4.1 we then obtain the desired monotonicity. ■

Proposition 4.6 *Assume either that the Ricci curvature is bounded on* $[0, \tau]$ *for each* τ *or that the curvature operator is nonnegative. Then there holds*

$$\tilde{V}(\tau_2) - \tilde{V}(\tau_1) = -\int_{\tau_1}^{\tau_2} \int_M (l_\tau - R + \frac{n}{2\tau}) e^{-l} \tau^{\frac{n}{2}} dq d\tau \tag{4.13}$$

for all $0 < \tau_1 < \tau_2 < T$.

Proof. We first assume nonnegative curvature operator. Consider $\tau_2 > \tau_1$. Applying Proposition 2.10, Lemma 2.13 and Lemma 3.2 we deduce

$$\tilde{V}(\tau_2) - \tilde{V}(\tau_1) = \int_M \int_{\tau_1}^{\tau_2} \frac{\partial}{\partial \tau} (\tau^{-\frac{n}{2}} e^{-l(q,\tau)} dq) =$$

$$-\int_M \int_{\tau_1}^{\tau_2} (l_\tau - R + \frac{n}{2\tau}) e^{-l} \tau^{\frac{n}{2}} dq d\tau. \tag{4.14}$$

Indeed, the last integral is absolutely convergent, see the proof of Lemma 3.3. In particular we can switch the integration order to arrive at (4.13).

The proof of (4.13) in the case of bounded Ricci curvature is similar, cf. the proof of Lemma 3.3. Note that in this case we also have

$$\frac{\partial \tilde{V}}{\partial \tau} = \int_M (l_\tau - R + \frac{n}{2\tau}) e^{-l} \tau^{\frac{n}{2}} \tag{4.15}$$

for every τ. This can be seen by computing the relevant difference quotient and applying the dominated convergence theorem to pass to limit.

We remark that (4.13) and Lemma 3.3 also imply the monotonicity of $\tilde{V}(\tau)$. ■

Definition 6 In this definition, let g be a smooth solution of the backward Ricci flow on $N \times I$ for a smooth manifold N and an interval I. We say that g is a *gradient shrinking soliton* with *time origin* τ_0 and *potential function* f on an open subset O of $N \times I$, where f is a smooth function on O, provided that g satisfies the gradient shrinking soliton equation

$$Ric - \frac{1}{2(\tau - \tau_0)} g + \nabla^2 f = 0 \tag{4.16}$$

in O.

Lemma 4.7 *Let* g *be a gradient shrinking soliton on* $N \times I$ *with time origin* τ_0 *and potential function* f. *Then* g *evolves by the pullback of a family of diffeomorphisms coupled with scaling. More precisely, we have*

$$g = \frac{\tau - \tau_0}{\bar{\tau} - \tau_0} (\phi^{-1})^* g(\bar{\tau}), \tag{4.17}$$

where $\bar{\tau} \in I$ *and* ϕ *is the solution of the equation* $\frac{\partial \phi}{\partial \tau} = \nabla f(\phi)$ *with* $\phi(\bar{\tau}) = id$ *(id denotes the identity map of* N*).*

Proof. We have

$$\frac{\partial \bar{g}}{\partial \tau} = 2Ric_{\bar{g}} = \frac{1}{\tau - \tau_0}\bar{g} - 2\nabla^2_{\bar{g}} f = \frac{1}{\tau - \tau_0}\bar{g} - L_{\nabla_{\bar{g}} f}\bar{g}. \tag{4.18}$$

Hence

$$\frac{\partial}{\partial \tau}\phi^* \bar{g} = \frac{1}{\tau - \tau_0}\phi^* \bar{g}. \tag{4.19}$$

The equation (4.17) follows. ■

Lemma 4.8 *As in Definition 6, let g be a smooth solution of the backward Ricci flow on $N \times I$. Let f be a smooth function on an open subset O of $N \times I$. We set $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$ and $v = [\tau(2\nabla f - |\nabla f|^2 + R) + f - n]$. Then we have*

$$\Box v = -2\tau |Ric + \nabla^2 f - \frac{1}{\tau} g|^2 u + 2\tau u \Delta(u^{-1}\Box u), \tag{4.20}$$

where $\Box u = u_\tau - \nabla u + Ru$. Consequently, if u satisfies the heat equation $\Box u = 0$, or equivalently

$$\frac{\partial f}{\partial \tau} - \Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} = 0, \tag{4.21}$$

then v satisfies the equation

$$\Box v = -2\tau |Ric + \nabla^2 f - \frac{1}{\tau} g|^2. \tag{4.22}$$

In particular, if (4.21) holds, then g is a gradient shrinking soliton on O with time origin 0 and potential function f if and only if $\Box v = 0$ in O.

Proof. This is a reformulation of [Proposition 9.1, P]. The formula (4.20) follows from routine computations. ■

Now we return to our previous g on $M \times [0, T)$.

Proposition 4.9 *Assume either that the Ricci curvature is bounded on $[0, \tau]$ for each $\tau < T$, or that the curvature operator is nonnegative. Assume that $\tilde{V}(\tau_1) = \tilde{V}(\tau_2)$ for some $\tau_1 < \tau_2$. Then l is smooth on $M \times (\tau_1, \tau_2)$ and g is a gradient shrinking soliton on $M \times (\tau_1, \tau_2)$ with time origin and potential function l.*

Proof. Assume $\tilde{V}(\tau_1) = \tilde{V}(\tau_2)$ for some $\tau_1 < \tau_2$. By Proposition 4.6 we have

$$\int_{\tau_1}^{\tau_2} \int_M (l_\tau - R + \frac{n}{2\tau}) e^{-l} \tau^{\frac{n}{2}} dq d\tau = 0. \tag{4.23}$$

We set

$$Q_{\tau_1,\tau_2}(\phi) = \int_{\tau_1}^{\tau_2} \int_M \{\nabla l \cdot \nabla \phi + (l_\tau + |\nabla l|^2 - R + \frac{n}{2\tau})\phi\} dq d\tau \tag{4.24}$$

for admissible ϕ, which are Lipschitz functions ϕ on $M \times [\tau_1, \tau_2]$ such that $|\phi| \leq Ce^{-l}$ and $|\nabla l| \leq Ce^{-l}$ for some C (depending on ϕ). Then the equation (4.23) says that $Q_{\tau_1,\tau_2}(\tau^{-\frac{n}{2}}e^{-l}) = 0$. For an arbitrary nonnegative admissible ϕ with bound factor C we have by Theorem 3.3, $Q_{\tau_1,\tau_2}(C\tau^{-\frac{n}{2}}e^{-l} - \phi) \geq 0$, whence $Q_{\tau_1,\tau_2}(\phi) \leq Q_{\tau_1,\tau_2}(\phi) \leq CQ_{\tau_1,\tau_2}(\tau^{-\frac{n}{2}}e^{-l}) = 0$. By Theorem 3.3 again we deduce that $Q_{\tau_1,\tau_2}(\phi) = 0$. By linearity of Q_{τ_2,τ_2} we then infer that $Q_{\tau_1,\tau_2}(\phi) = 0$ for all admissible ϕ, in particular for all Lipschitz ϕ with compact support. The standard regularity theory for parabolic equations implies that l is smooth on $M \times (\tau_1, \tau_2)$ and satisfies

$$\frac{\partial l}{\partial \tau} - \Delta l + |\nabla l|^2 - R + \frac{n}{2\tau} = 0. \tag{4.25}$$

By Lemma 2.16 we then also have

$$2\Delta l - |\nabla l|^2 + R + \frac{l-n}{\tau} = 0. \tag{4.26}$$

Now we can apply Lemma 4.8 with $f = l$. By (4.25), the equation (4.21) holds true. By (4.26), $v = 0$. Hence we conclude that g is a gradient shrinking soliton with time origin 0 and potential function l. (The implication of Lemma 4.8, i.e. [9.1, P] was first pointed out to us by G. Wei. Note that a similar argument is used in the proof of Theorem 10.1 in [P].) ■

Theorem 4.10 *Assume that the Ricci curvature is bounded on* $[0, \tau]$. *Then* $\tilde{V}(\tau) < (4\pi)^{\frac{n}{2}}$ *unless* $(M, g(0))$ *is isometric to* $\mathbf{R}^n$ *and* $g(s) = g(0)$ *for each* $s \in [0, \tau]$.

Proof. By Theorem 4.3, $\tilde{V}(\tau) \leq (4\pi)^{\frac{n}{2}}$. Assume the equality holds. By Theorem 4.5 an Proposition 4.9, g is a gradient shrinking soliton on $M \times (0, \tau)$ with time origin 0 and potential function l. By Lemma 4.7, $g(\bar{\tau}) = \frac{\tau'}{\bar{\tau}}\phi^* g(\tau')$ for $\bar{\tau}, \tau' \in (0, \tau)$. Since the Ricci curvature is bounded, we can let $\tau' \to 0$ to deduce that $g(\bar{\tau})$ is Ricci flat for each $\bar{\tau} \in (0, \tau)$. The desired conclusion then follows from Theorem 4.4. ■

5 Asymptotic Limits of κ-Solutions

Let $\tilde{g}(t), -\infty < t \leq 0$ be a κ-solution for some $\kappa > 0$ on a manifold M. Recall [P] that this means that $\tilde{g}(t)$ is a solution of the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric \tag{5.1}$$

on $M \times (-\infty, 0]$, such that for each t, the metric $\tilde{g}(t)$ is a complete non-flat metric of bounded curvature and nonnegative curvature operator. Moreover, each $\tilde{g}(t)$ is κ-noncollapsed on all scales. To understand the structures of $\tilde{g}(t)$, we analyse its rescaled asymptotical limits at the time infinity. One needs to use a blow-down rescaling, because the Ricci flow equation and nonnegative curvature imply that $\tilde{g}(t)$ expands as t decreases.

Pick an arbitrary time $t_0 \leq 0$ and set $\tau = t_0 - t$ for $t \leq t_0$. Then $g(\tau) = \tilde{g}(t_0 - \tau), \tau \in [0, \infty)$ is a solution of the backward Ricci flow (2.1). We choose a reference point $p \in M$ and apply the constructions from the previous sections. By Proposition 4.5, $\tilde{V}_\infty = \lim_{\tau \to \infty} \tilde{V}(\tau)$ exists. Let $x = x(\tau)$ be a minimum point for $l(\cdot, \tau)$. By Lemma 3.1, $l(x, \tau) \leq \frac{n}{2}$. We'll use x as the center for pointed convergence. (The reference point p is not suitable for this purpose, because the estimates around p are not good enough after rescaling.)

For $\bar{\tau} > 0$, we set as before $g_{\bar{\tau}}(\tau) = \frac{1}{\bar{\tau}} g(\bar{\tau}\tau)$. The following theorem is Proposition 11.2 in [P].

Theorem 5.1 *Let $\tau_k \to \infty$ be given. Then the pointed flows $(g_{\tau_k}(\tau), M, x(\tau_k))$ for $\tau \in (0, \infty)$ subconverge smoothly to nonflat gradient shrinking solitons $(M_\infty, g_\infty, x_\infty)$ with time origin 0 and limits of the l-function of g_{τ_k} as potential functions. These solitons will be called "asymptotic solitons" of g.*

Proof. **Part 1**

We set $g_k = g_{\tau_k}$ and consider l_{g_k} and $\tilde{V}_{g_k}$ (the l-function and reduced volume associated with g_k). Since the l-function and reduced volume are invariant under rescaling, we have

$$l_{g_k}(x(\tau_k), 1) \leq n/2 \tag{5.2}$$

and $\tilde{V}_{g_k}(\tau) = \tilde{V}(\tau_k \tau)$. It follows that

$$\lim_{k \to \infty} \tilde{V}_{g_k}(\tau) = \tilde{V}_\infty \tag{5.3}$$

for each τ.

By (3.5) we deduce

$$l_{g_k}(q, 1) \leq C(\frac{n}{2} + d^2_{g_k}(x(\tau_k), q, 1)). \tag{5.4}$$

Then it follows from (2.47) that (with a different C)

$$l_{g_k}(q, \tau) \leq C\tau^{\pm C}(\frac{n}{2} + d^2_{g_k}(x(\tau_k), q, 1)), \tag{5.5}$$

where $\pm$ is $+$ if $\tau \geq 1$ and $-$ if $\tau < 1$. Consequently, we obtain from (2.45) and the nonnegativity of curvature the estimate

$$|Rm_{g_k}|(q, \tau) \leq C\tau^{-1 \pm C}(\frac{n}{2} + d^2_{g_k}(x(\tau_k), q, 1)). \tag{5.6}$$

Part 2

By the κ-noncollapsing assumption, we obtain injectivity radius estimates on compact sets for each $g_k(\tau)$ which depend only on $d_{g_k}(x(\tau_k),\cdot,1)$, τ and κ. By Gromov-Cheeger-Hamilton compactness we then obtain pointed smooth subconvergence of $(M, g_k, x(\tau_k))$ to limit flows on the time interval $(0,\infty)$.

Let $(M_\infty, g_\infty(\tau), x_\infty), \tau \in (0,\infty)$ be such a limit whose corresponding converging subsequence will still be denoted by g_k. By Lemma 2.13, the recaling invariance and (5.5), we can assume that l_{g_k} converge locally uniformly to a locally Lipschitz function l_∞ on $M_\infty \times (0,\infty)$.

Part 3

Let $\tau_2 > \tau_1 > 0$. By (4.13) we have

$$\begin{aligned}\tilde{V}_{g_k}(\tau_2) - \tilde{V}_{g_k}(\tau_1) &= -\int_{\tau_1}^{\tau_2}\int_M (\frac{\partial l_{g_k}}{\partial \tau} - R_{g_k} + \frac{n}{2\tau})e^{-l_{g_k}}\tau^{-\frac{n}{2}}dqd\tau \\ &= \int_{\tau_1}^{\tau_2}\int_M R_{g_k}e^{-l_{g_k}}\tau^{-\frac{n}{2}}dqd\tau + \int_M e^{l_{g_k}(q,\tau_2)}dq|_{\tau_2} \\ &\quad - \int_M e^{l_{g_k}(q,\tau_1)}dq|_{\tau_1}, \end{aligned} \tag{5.7}$$

where $dq = dq_{g_k}$. The estimates in the proof of Lemma 3.3 imply that the three integrals on the right hand side of (5.7) converge to the corresponding integrals on M_∞ with respect to l_∞, g_∞ and dq_{g_∞}. By (5.3) we then conclude that

$$\int_{\tau_1}^{\tau_2}\int_{M_\infty} R_{g_\infty}e^{-l_\infty}\tau^{-\frac{n}{2}}dqd\tau + \int_{M_\infty} e^{-l_\infty(q,\tau_2)}dq|_{\tau_2} - \int_{M_\infty} e^{-l_\infty(q,\tau_1)}dq|_{\tau_1} = 0, \tag{5.8}$$

where $dq = dq_{g_\infty}$. This is equivalent to

$$\int_{\tau_1}^{\tau_2}\int_{M_\infty} (\frac{\partial l_\infty}{\partial \tau} - R_{g_\infty} + \frac{n}{2\tau})e^{-l_\infty}\tau^{-\frac{n}{2}}dqd\tau = 0. \tag{5.9}$$

Part 4 By Theorem 2.17 and the local uniform convergence of l_∞, we deduce that l_∞ satisfies the differential inequality (2.66) when Δl_∞ is interpreted in the weak sense, i.e.

$$Q_{\tau_1,\tau_2}(\phi) \geq 0 \tag{5.10}$$

for arbitray $\tau_2 > \tau_1 > 0$ and nonnegative Lipschitz functions ϕ on $M_\infty \times [\tau_1, \tau_2]$ such that the support of $\phi(\cdot,\tau)$ is compact for each $\tau \in [\tau_1,\tau_2]$, where

$$Q_{\tau_1,\tau_2}(\phi) = \int_{\tau_1}^{\tau_2}\int_{M_\infty} \{\nabla_{g_\infty} l_\infty \cdot \nabla_{g_\infty}\phi + (\frac{\partial l_\infty}{\partial \tau} + |\nabla_{g_\infty} l_\infty|^2 - R_{g_\infty} + \frac{n}{2\tau})\phi\}dqd\tau. \tag{5.11}$$

(Of course, the inner product and norm are also measured in g_∞.) Next we deduce from (3.5) that

$$-\frac{n}{2} - 1 + C_1\frac{d^2_{g_\infty}(x_\infty, q, \tau)}{\tau} \leq l_\infty(q,\tau) \leq \frac{n}{2} + C_2\frac{d^2_{g_\infty}(x_\infty, q, \tau)}{\tau}. \tag{5.12}$$

By this and the proof of Theorem 3.3 we infer that (5.10) holds true for nonnegative ϕ such that

$$\phi(q,\tau) \le \bar{C}e^{-l_\infty}(q,\tau) \tag{5.13}$$

for each $q \in M_\infty$ and $\tau \in [\tau_1, \tau_2]$ and some positive constant $\bar{C}$ depending on ϕ. Now we can apply the arguments in the proof of Theorem 4.9 to conclude that l_∞ is smooth and g_∞ is a gradient shrinking soliton with potential function l_∞.

Part 5

Finally, we show that the limit g_∞ or $\bar{g}$ is nonflat. Assume that g_∞ is flat. Then the soliton equation implies

$$\nabla^2_{g_\infty} l_\infty = \frac{1}{2\tau} g_\infty. \tag{5.14}$$

It follows that $l_\infty(\cdot, 1)$ is a strictly convex function with a unique minimum point (by the construction it has a minimum point x_∞). By (5.13), its level sets are compact. Hence M_∞ is diffeomorphic to $\mathbf{R}^n$ and then each $(M_\infty, g_\infty(\tau))$ is isometric to $\mathbf{R}^n$. Next we observe that (5.14) implies that $\Delta_{g_\infty} l_\infty = \frac{n}{2\tau}$, which together with (4.26) leads to

$$|\nabla l_\infty|^2 = \frac{l_\infty}{\tau}. \tag{5.15}$$

For a fixed τ, we identify (M_∞, g_∞) with $\mathbf{R}^n$ via an isometry. Let x denote the coordinates on $\mathbf{R}^n$. Then (5.14) implies that

$$\nabla l_\infty(x,\tau) = \nabla \frac{|x|^2}{4\tau} + v(\tau) \tag{5.16}$$

for a constant vector $v(\tau)$. Hence

$$l_\infty(x,\tau) = \frac{|x|^2}{4\tau} + v(\tau)\cdot x + c(\tau) \tag{5.17}$$

for a constant $c(\tau)$. A simple calculation using (5.15), (5.16) and (5.17) yields $c(\tau) = \tau|v|^2$. It follows that $l_\infty(x,\tau) = \frac{|x+2\tau v|^2}{4\tau}$ and hence $\tilde{V}_{g_\infty} = (4\pi)^{\frac{n}{2}}$. But $\tilde{V}_{g_\infty} \le V_\infty < (4\pi)^{\frac{n}{2}}$ by Theorem 4.4. Thus we arrive at a contradiction. (Actually, $\tilde{V}_{g_\infty} = V_\infty$.) ■

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Department of Mathematics, University of California, Santa Barbara, CA 93106
yer@math.ucsb.edu