MODULAR CLASSES OF REGULAR TWISTED POISSON STRUCTURES ON LIE ALGEBROIDS

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Abstract. We derive a formula for the the modular class of a Lie algebroid with a regular twisted Poisson structure in terms of a canonical Lie algebroid representation of the image of the Poisson map. We use this formula to compute the modular classes of Lie algebras with a twisted triangular r-matrix. The special case of r-matrices associated to Frobenius Lie algebras is also studied.

1. Introduction

Twisted Poisson structures on manifolds, whose definition we recall in Section 2.1, first appeared in the mathematical physics literature. They were introduced in geometry by Klimčik and Strobl [7], and were studied by Ševera and Weinstein [13] who proved that they can be described as Dirac structures in Courant algebroids. Roytenberg then showed in [12] that, more generally, twisted Poisson structures on Lie algebroids appear in a natural way in his general theory of twisting of Lie bialgebroids.

While the modular vector fields of Poisson manifolds were first defined by Koszul in 1985, the theory of the modular classes of Poisson manifolds was developed by Weinstein in his 1997 article [15] and that of the modular classes of Lie algebroids, generalizing those of Poisson manifolds, by Evens, Lu and Weinstein in 1999 [4].

In [9], Kosmann-Schwarzbach and Laurent-Gengoux defined the modular class of a Lie algebroid A with a twisted Poisson structure (π, ψ). (The untwisted case where $\psi = 0$ corresponds to a triangular Lie bialgebroid, see [8].) The modular class, which we shall denote by $\theta(A, \pi, \psi)$, is a class in the first cohomology group $H^1(A^*)$ of the dual Lie algebroid A^*. Furthermore, Kosmann-Schwarzbach and Weinstein [10] showed that this class is, up to a factor of $\frac{1}{2}$, the relative modular class for the morphism $\pi^\sharp \colon A^* \to A$,

$$2\,\theta(A, \pi, \psi) = \mathrm{Mod}(A^*) - (\pi^\sharp)^*(\mathrm{Mod}\, A)\ . \tag{1.1}$$

Here $\mathrm{Mod}\, A \in H^1(A)$ and $\mathrm{Mod}(A^*) \in H^1(A^*)$ denote the modular classes of the Lie algebroids A and A^* in the sense of Evens, Lu and Weinstein [4], and $\pi^\sharp$ is the map $\alpha \in A^* \mapsto i_\alpha \pi \in A$. As was observed in [9], if $\pi^\sharp \colon A^* \to A$ is invertible, the modular class $\theta(A, \pi, \psi)$ vanishes. This can be seen from (1.1), since, in this

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case, $\pi^\sharp\colon A^* \to A$ is an isomorphism of Lie algebroids, and the two terms in (1.1) cancel out.

Let us call a twisted Poisson structure (π, ψ) on a Lie algebroid A *regular* if $\pi^\sharp$ has constant rank, *i.e.*, if the image B of $\pi^\sharp$ is a vector sub-bundle and thus a Lie subalgebroid of A. For such structures, our main result, Theorem 2.5, expresses $\theta(A, \pi, \psi)$ as the image of the characteristic class of the Lie algebroid B with representation on the top exterior power of the kernel of $\pi^\sharp$ defined by the adjoint action. Formula (2.15) in Theorem 2.5 takes into account the cancellations which occur in (1.1), even when π is not an isomorphism, and this formula can be used to compute modular classes. Section 2 contains the necessary definitions, in particular that of the *canonical representation* of the image of $\pi^\sharp$ on its kernel, and two lemmas that lead to the proof of Theorem 2.5 in Section 2.5.

In Section 3 we describe the regular twisted Poisson structures on Lie algebroids by means of a linearization of condition (2.1), extending and unifying two very different results. On the one hand, Ševera and Weinstein observed in [13] that a twisted Poisson manifold (M, π, ψ) admits a foliation by submanifolds equipped with a non-degenerate 2-form ω which is ψ-twisted, *i.e.*, such that its differential is equal to the pull-back of $-\psi$ under the canonical injection, generalizing the symplectic foliation of Poisson manifolds. On the other hand, it was shown by Stolin [14] and by Gerstenhaber and Giaquinto [5] that non-degenerate triangular r-matrices on Lie algebras are in one-to-one correspondence with quasi-Frobenius structures (a property that was used by Hodges and Yakimov to describe the geometry of triangular Poisson Lie groups in terms of symplectic reduction [6]). In Theorem 3.1 and its corollaries, we extend both these results to the case of regular twisted Poisson structures on Lie algebroids.

In Section 4 we deal with the important particular case where the Lie algebroid is a Lie algebra, considered as a Lie algebroid over a point. As a consequence of Theorem 2.5, in Proposition 4.1 and Corollary 4.2 we obtain two simple formulas for the modular class defined by a *twisted triangular r-matrix* on a Lie algebra. The linearization of the twisted classical Yang-Baxter equation, obtained in Proposition 4.3 and Corollary 4.4, as a consequence of Theorem 3.1, is very useful for explicit constructions, as shown in the examples of Section 4.5. The one-to-one correspondence between non-degenerate triangular r-matrices and quasi-Frobenius structures is extended to the twisted structures in Corollary 4.5, and we further study the case of triangular r-matrices associated to *Frobenius Lie algebras*, in which case we obtain a simple characterization of the modular class (Proposition 4.6). In Section 4.5, we illustrate the use of our formulas with several classes of examples. We explicitly compute the modular classes of the Gerstenhaber–Giaquinto generalized Jordanian r-matrix structures on $\mathfrak{sl}_n(\mathbb{R})$ [5], and we construct twisted triangular r-matrix structures with trivial and with nontrivial modular classes.

2. The modular class of a regular twisted Poisson structure

2.1. **Twisted Poisson structures.** Recall that a Lie algebroid with a twisted Poisson structure is a triple, (A, π, ψ), where A is a Lie algebroid over a base manifold M, π and ψ are sections of $\wedge^2 A$ and $\wedge^3 A^*$, respectively, such that ψ

is d_A-closed, where $d_A\colon \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1}A^*)$ is the Lie algebroid cohomology operator of A, satisfying

$$\frac{1}{2}[\pi,\pi]_A = (\wedge^3\pi^\sharp)\psi \ . \tag{2.1}$$

See [11] for an exposition of the general theory of Lie algebroids, see [13] for the twisted Poisson manifolds and [12] [9] for the twisted Poisson structures on Lie algebroids. If $\rho\colon A \to TM$ is the *anchor* of A, then the dual vector bundle A^* is a Lie algebroid with anchor $\rho\circ\pi^\sharp$ and Lie bracket

$$[\alpha,\beta]_{\pi,\psi} = \mathcal{L}^A_{\pi^\sharp\alpha}\beta - \mathcal{L}^A_{\pi^\sharp\beta}\alpha - d_A(\pi(\alpha,\beta)) + \psi(\pi^\sharp\alpha,\pi^\sharp\beta,\,.\,)\ , \tag{2.2}$$

for α and $\beta \in \Gamma(A^*)$, where $\mathcal{L}^A$ is the Lie derivation defined by $\mathcal{L}^A_X = [i_X, d_A]$, for each $X \in \Gamma(A)$. The map $\pi^\sharp\colon A^* \to A$ is a morphism of Lie algebroids.

2.2. **A canonical representation in the regular case.** Let (π,ψ) be a twisted Poisson structure on the Lie algebroid A, and assume that it is regular. Let $B \subset A$ be the image of $\pi^\sharp$. The morphism of Lie algebroids $\pi^\sharp\colon A^* \to A$ and the canonical embedding of Lie algebroids $\iota_B\colon B \hookrightarrow A$ define maps of cohomology spaces,

$$(\pi^\sharp)^*\colon H^\bullet(A) \to H^\bullet(A^*) \quad \text{and} \quad (\iota_B)^*\colon H^\bullet(A) \to H^\bullet(B)\ .$$

Since the image of $\pi^\sharp\colon A^* \to A$ is B, there is an induced map,

$$\pi^\sharp_B\colon A^* \to B\ ,$$

which in turn defines a map of cohomology spaces,

$$(\pi^\sharp_B)^*\colon H^\bullet(B) \to H^\bullet(A^*)$$

satisfying

$$(\pi^\sharp)^* = (\pi^\sharp_B)^* \circ \iota_B^*\ . \tag{2.3}$$

Let $C \subset A^*$ be the kernel of the morphism $\pi^\sharp$. It follows from the definition (2.2) of the Lie bracket on $\Gamma(A^*)$ that $\Gamma(C)$ is an abelian ideal of $\Gamma(A^*)$, and there is an exact sequence of Lie algebroids over the same base,

$$0 \to C \to A^* \to B \to 0\ . \tag{2.4}$$

Because π is skew-symmetric, C is the orthogonal of B with respect to the fiberwise duality pairing, $\langle\,.\,,\,.\,\rangle$ between A and A^*. Therefore the vector bundles A/B and C^* are isomorphic, and the vector bundles A^*/C and B^* are isomorphic.

The Lie algebra $\Gamma(A)$ acts on $\Gamma(A^*)$ by the coadjoint action (relative to the Lie bracket $[\,.\,,\,.\,]_A$), and the restriction of this action to $\Gamma(B)$ leaves $\Gamma(C)$ invariant. This is proved using the skew-symmetry and morphism properties of $\pi^\sharp$. However this is not a representation of B on C unless the anchor of A vanishes.

On the other hand, from (2.4) we see that the adjoint action of $\Gamma(A^*)$ on $\Gamma(C)$ (relative to the Lie bracket $[\,.\,,\,.\,]_{\pi,\psi}$) factors through the submersion $\pi^\sharp_B$: $A^* \to B$, and therefore induces an action of $\Gamma(B)$, which is in fact a representation of B on C, because $\rho\circ\pi^\sharp$ vanishes on C. It is an example of the representation attached to an abelian extension of Lie algebroids ([11, Proposition 3.3.20]). It follows from formula (2.2) that this action is defined by

$$X \cdot \gamma = \mathcal{L}^A_X(\gamma) \ , \tag{2.5}$$

for all $X \in \Gamma(B)$ and $\gamma \in \Gamma(C)$. Explicitly, for all $Y \in \Gamma(A)$,

$$\langle X \cdot \gamma, Y\rangle = \rho(X)\langle \gamma, Y\rangle - \langle \gamma, [X,Y]_A\rangle \ . \tag{2.6}$$

We shall refer to this representation of the Lie algebroid B on the vector bundle C as its *canonical representation.*

When restricted to $\Gamma(B)$, the adjoint action of $\Gamma(A)$ on itself induces an action of $\Gamma(B)$ on $\Gamma(A/B)$, which is defined by

$$X \cdot clY = cl([X,Y]_A) \ , \tag{2.7}$$

where X is a section of B, Y a section of A, and cl denotes the class of a section of A modulo the sections of B. The right-hand side is well defined because B is a Lie subalgebroid of A, and the map $(X, clY) \mapsto X \cdot clY$ is $C^\infty(M)$-linear in X. It is therefore clear that (2.7) defines a representation of B on A/B. It is an example of a Bott representation as defined by Crainic [1, Examples 4]. It is easy to prove that, under the identification of C^* with A/B, this representation is dual to the canonical representation. (We recall the definition of dual representations in Section 2.3 below.)

Remark 2.1. When the anchor of A vanishes, *e.g.*, in the case of Lie algebras, the action of $\Gamma(B)$ on $\Gamma(C)$ which is the restriction of the coadjoint action of $\Gamma(A)$ on $\Gamma(A^*)$ defines a representation of B on C, and equation (2.6) shows that it coincides with the canonical representation defined by (2.5).

2.3. **Characteristic classes of Lie algebroids.** In [4], Evens, Lu and Weinstein defined the characteristic class of a Lie algebroid E with a representation on a line bundle L, over the same base manifold M. If $L \to M$ is trivial, the characteristic class is the class in $H^1(E)$ of the section $\theta_s \in \Gamma(E^*)$ such that, for all $x \in \Gamma(E)$,

$$x \cdot s = \langle \theta_s, x\rangle s, \tag{2.8}$$

where $s \in \Gamma(L)$ is a nowhere-vanishing section. In fact, the section θ_s is a d_E-cocycle, and its class, which we shall denote by $\theta(E, L)$, is independent of the choice of the section s. If L is not trivial, its characteristic class is defined as one-half that of its square.

The *modular class of a Lie algebroid* E over M is the characteristic class of the canonical representation of E on the line bundle $\wedge^{\rm top}E \otimes \wedge^{\rm top}(T^*M)$, defined by

$$x \cdot (\lambda \otimes \mu) = \mathcal{L}^E_x \lambda \otimes \mu + \lambda \otimes \mathcal{L}_{\rho(x)}\mu \ , \tag{2.9}$$

for all $x \in \Gamma(E)$, where λ and μ are nowhere-vanishing sections of $\wedge^{\rm top}E$ and $\wedge^{\rm top}T^*M$, respectively. We denote the modular class of E by $\operatorname{Mod} E$.

We recall that representations of a Lie algebroid E on vector bundles V and V^* are *dual* if

$$\langle x \cdot s, \sigma\rangle = -\langle s, x \cdot \sigma\rangle + \rho(x)\langle s, \sigma\rangle \ ,$$

for all $x \in \Gamma(E)$, $s \in \Gamma(V)$ and $\sigma \in \Gamma(V^*)$.

The proof of the following lemma is simple and will be omitted.

Lemma 2.2. *(i) If $f\colon A_1 \to A_2$ is a morphism of Lie algebroids over the same base and V is a representation of A_2, then*

$$f^*\theta(A_2, V) = \theta(A_1, V^f) \ ,$$

where V^f is the representation of A_1 induced from f.
(ii) Dual representations have opposite characteristic classes.

2.4. **Relations between** $\mathrm{Mod}(A^*)$**,** $\mathrm{Mod}\, A$ **and** $\mathrm{Mod}\, B$**.** The following lemmas contain formulas that will be used in the proof of our main result, Theorem 2.5.

Lemma 2.3. *Let A be a Lie algebroid with a regular twisted Poisson structure (π, ψ); let B be the image of $\pi^\sharp$, and let C be its kernel. Denote by θ_B the characteristic class of the Lie algebroid B with the representation on the line bundle $\wedge^{\mathrm{top}}(C)$ induced by the canonical representation* (2.5)*. Then*

$$\mathrm{Mod}(A^*) = (\pi_B^\sharp)^*(\mathrm{Mod}\, B + \theta_B) \ . \tag{2.10}$$

Proof. We shall assume that $\wedge^{\mathrm{top}}C$, $\wedge^{\mathrm{top}}B$, and $\wedge^{\mathrm{top}}A^* \otimes \wedge^{\mathrm{top}}T^*M$ are trivial bundles over M. If this is not the case, the proof below is easily modified using densities. Let s_1 be a nowhere-vanishing section of $\wedge^{\mathrm{top}}C$. Let $\mathrm{rk}B$ denote the rank of B, and let $s_2 \in \Gamma(\wedge^{\mathrm{rk}B}A^*)$ be such that $s_1 \wedge s_2$ and $\pi^\sharp s_2$ are nowhere-vanishing sections of $\wedge^{\mathrm{top}}A^*$ and $\wedge^{\mathrm{top}}B$, respectively. Finally let $\mu \in \Gamma(\wedge^{\mathrm{top}}T^*M)$ be a volume form on M. We denote by ξ, η and ζ representatives of $\mathrm{Mod}\, A^*$, $\mathrm{Mod}\, B$ and $\theta(A^*, \wedge^{\mathrm{top}}C)$, corresponding to the sections $s_1 \wedge s_2 \otimes \mu$, $\pi^\sharp s_2 \otimes \mu$ and s_1 of $\wedge^{\mathrm{top}}A^* \otimes \wedge^{\mathrm{top}}T^*M$, $\wedge^{\mathrm{top}}B \otimes \wedge^{\mathrm{top}}T^*M$ and $\wedge^{\mathrm{top}}C$, respectively. By definition, for all $\alpha \in \Gamma(A^*)$,

$$< \xi, \alpha > s_1 \wedge s_2 \otimes \mu = [\alpha, s_1 \wedge s_2]_{\pi,\psi} \otimes \mu + s_1 \wedge s_2 \otimes \mathcal{L}_{\rho \circ \pi^\sharp(\alpha)}\mu \ ,$$

and

$$< \zeta, \alpha > s_1 = [\alpha, s_1]_{\pi,\psi} \ .$$

Since

$$[\alpha, s_1 \wedge s_2]_{\pi,\psi} = [\alpha, s_1]_{\pi,\psi} \wedge s_2 + s_1 \wedge [\alpha, s_2]_{\pi,\psi} \ ,$$

we obtain

$$< \xi - \zeta, \alpha > s_2 \otimes \mu = [\alpha, s_2]_{\pi,\psi} \otimes \mu + s_2 \otimes \mathcal{L}_{\rho \circ \pi^\sharp(\alpha)}\mu \ ,$$

and applying $\pi_B^\sharp \otimes \mathrm{Id}$ to both sides of the preceding equality yields

$$< \xi - \zeta, \alpha > \pi_B^\sharp s_2 \otimes \mu = \pi_B^\sharp[\alpha, s_2]_{\pi,\psi} \otimes \mu + \pi_B^\sharp s_2 \otimes \mathcal{L}_{\rho \circ \pi^\sharp(\alpha)}\mu \ .$$

From the definition of η, we obtain

$$< \eta, \pi^\sharp\alpha > \pi^\sharp s_2 \otimes \mu = [\pi^\sharp\alpha, \pi^\sharp s_2]_A \otimes \mu + \pi^\sharp s_2 \otimes \mathcal{L}_{\rho \circ \pi^\sharp(\alpha)}\mu \ .$$

Formula (2.10) follows, using the morphism property of $\pi^\sharp$ and Lemma 2.2 (*i*). □

Lemma 2.4. *Under the assumptions of Lemma 2.3,*

$$\iota_B^*(\mathrm{Mod}\, A) = \mathrm{Mod}\, B - \theta_B \ . \tag{2.11}$$

Proof. Using Lemma 2.2 (*i*) we obtain

$$\iota_B^*(\mathrm{Mod}\, A) = \theta(B, \wedge^{\mathrm{top}} A \otimes \wedge^{\mathrm{top}} T^* M)\ . \tag{2.12}$$

As in the proof of (2.10), one shows that, in terms of the representation of B on $\wedge^{\mathrm{top}}(A/B)$ induced by the representation defined by (2.7),

$$\theta(B, \wedge^{\mathrm{top}} A \otimes \wedge^{\mathrm{top}} T^* M) = \mathrm{Mod}\, B + \theta(B, \wedge^{\mathrm{top}}(A/B))\ . \tag{2.13}$$

Since the representation (2.7) is dual to the canonical representation (2.5), by Lemma 2.2 (*ii*),

$$\theta(B, \wedge^{\mathrm{top}}(A/B)) = -\,\theta_B\ .$$

Combining this result with (2.12) and (2.13) implies (2.11). □

2.5. **A formula for the modular class.** Taking into account (2.3), we see that (2.11) implies

$$(\pi^\sharp)^*(\mathrm{Mod}\, A) = (\pi_B^\sharp)^*(\mathrm{Mod}\, B - \theta_B)\ . \tag{2.14}$$

Formulas (2.10) and (2.14) express the fact that $\mathrm{Mod}(A^*)$ and $(\pi^\sharp)^*(\mathrm{Mod}\, A)$ differ from $(\pi_B^\sharp)^*(\mathrm{Mod}\, B)$ by opposite quantities. Our main result states that the modular class $\theta(A, \pi, \psi)$ of (A, π, ψ), which is the class of a 1-cocycle of the Lie algebroid A^*, is the pull-back of the class of a 1-cocycle of the image of $\pi^\sharp$.

Theorem 2.5. *Let A be a Lie algebroid with a regular twisted Poisson structure (π, ψ), and let $B = \mathrm{Im}\, \pi^\sharp$ and $C = \mathrm{Ker}\, \pi^\sharp$. Then the modular class $\theta(A, \pi, \psi)$ of (A, π, ψ) satisfies*

$$\theta(A, \pi, \psi) = (\pi_B^\sharp)^*(\theta_B)\ , \tag{2.15}$$

where θ_B is the characteristic class of the Lie algebroid B with the representation on the line bundle $\wedge^{\mathrm{top}}(C)$ induced by the canonical representation (2.5).

Proof. The formula follows from Lemma 2.3 and (2.14) together with (1.1). □

In the particular case of a regular Poisson manifold, the fact that the modular class is in the image of $(\pi_B^\sharp)^*$ was proved by Crainic [1, Corollary 9].

Formula (2.15) permits an effective computation of the modular class without computing terms which mutually cancel in $\mathrm{Mod}(A^*)$ and $(\pi^\sharp)^*(\mathrm{Mod}\, A)$. In Section 4, we shall illust rate this with various examples in the particular case of Lie algebras, considered as Lie algebroids over a point.

3. Non-degenerate twisted Poisson structures

Let (A, π, ψ) be a Lie algebroid with a regular twisted Poisson structure. As above, we denote by B the Lie subalgebroid of A which is the image of the Lie algebroid morphism, $\pi^\sharp \colon A^* \to A$. Because π is skew-symmetric, $\pi^\sharp$ defines an isomorphism from B^* to B, where $B^* = A^*/\,\mathrm{Ker}\, \pi^\sharp$ is the dual of B equipped with the Lie bracket inherited from that of A^*. We shall denote this isomorphism by $\pi^\sharp_{(B)}$. Let $\pi_{(B)}$ be the corresponding bivector on B. Thus $\pi_{(B)}$ is nothing but π thought of as an element of $\Gamma(\wedge^2 B) \subset \Gamma(\wedge^2 A)$. Then $(\pi_{(B)}, \iota_B^* \psi)$ is a non-degenerate twisted Poisson structure on B. (Recall that $\iota_B \colon B \hookrightarrow A$ denotes the inclusion.) We denote the inverse of $\pi^\sharp_{(B)}$ by $\omega^\flat_B$, and we define the 2-form ω_B on B by

$$i_X(\omega_B) = -\omega^\flat_B(X) \ , \tag{3.1}$$

for $X \in \Gamma(B)$. With these conventions, $\omega = \omega_B$ and $\pi = \pi_{(B)}$ satisfy

$$\omega(\pi^\sharp\alpha, \pi^\sharp\beta) = \pi(\alpha, \beta) \ , \tag{3.2}$$

for all α and $\beta \in \Gamma(A^*)$. We call the fiberwise non-degenerate 2-cochain ω_B and the non-degenerate bivector $\pi_{(B)}$ *inverses* of one another.

Set $\psi_B = \iota^*_B(\psi)$. Then $\psi_B \in \Gamma(\wedge^3 B^*)$ is a 3-cocycle of B for which one easily shows that (2.1) implies

$$d_B(\omega_B) = -\,\psi_B \ , \tag{3.3}$$

where d_B denotes the Lie algebroid cohomology operator of B.

Conversely, assume that A is a Lie algebroid, B is a Lie subalgebroid, $\omega \in \Gamma(\wedge^2 B^*)$ is a fiberwise non-degenerate 2-cochain of B, and $\psi \in \Gamma(\wedge^3 A^*)$ is a 3-cocycle whose pull-back to B is $-d_B\omega$. Let $\pi \in \Gamma(\wedge^2 B) \subset \Gamma(\wedge^2 A)$ be the inverse of ω. One shows by a direct computation that (π, ψ) is a regular twisted Poisson structure on the Lie algebroid A, for which B is the image of $\pi^\sharp$. This proves the following result, which yields a linearization of the defining condition (2.1) of twisted Poisson structures for regular Lie algebroids.

Theorem 3.1. *There is a one-to-one correspondence between regular twisted Poisson structures on a Lie algebroid A and triples consisting of*

1. *a Lie subalgebroid B of A,*
2. *a fiberwise non-degenerate 2-cochain $\omega \in \Gamma(\wedge^2 B^*)$, and*
3. *a 3-cocycle $\psi \in \Gamma(\wedge^3 A^*)$ whose pull-back to B is $-d_B\omega$,*

The twisted Poisson bivector $\pi \in \Gamma(\wedge^2 B)$ and the non-degenerate 2-cochain ω of B are inverses of one another.

Corollary 3.2. *Assume that A is a Lie algebroid and $\omega \in \Gamma(\wedge^2 A^*)$ is a 2-cochain of A whose restriction $\omega|_B$ to a Lie subalgebroid B of A is fiberwise non-degenerate. If $\pi \in \Gamma(\wedge^2 B)$ denotes the inverse of $\omega|_B$, then $(\pi, -d_A\omega)$ is a regular twisted Poisson structure on the Lie algebroid A.*

Corollary 3.2 shows how one can construct many regular twisted Poisson structures on Lie algebroids.

4. Lie algebras with twisted triangular r-matrices

4.1. **Twisted triangular r-matrices on Lie algebras.** The case of a twisted Poisson structure for a Lie algebroid over a point yields the concept of a twisted triangular r-matrix for a Lie algebra. Given a Lie algebra $\mathfrak{g}$, we say that (r, ψ) is a *twisted triangular r-matrix structure* on $\mathfrak{g}$, or simply a *twisted triangular structure*, if $\pi = r \in \wedge^2\mathfrak{g}$ and ψ is a 3-cocycle of $\mathfrak{g}$, where this data satisfies (2.1), which is equivalent to

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = -(\wedge^3 r^\sharp)\psi \ ,$$

the *twisted classical Yang-Baxter equation*. Here as above, $r^\sharp\colon \mathfrak{g}^* \to \mathfrak{g}$ is the linear map defined by $r^\sharp(\alpha) = i_\alpha r$. A twisted triangular structure (r, ψ) on $\mathfrak{g}$ gives rise to the Lie algebra structure on $\mathfrak{g}^*$ defined by (2.2), which, in this case, reduces to

$$[\alpha, \beta]_{r,\psi} = \mathrm{ad}^*_{r^\sharp\alpha}(\beta) - \mathrm{ad}^*_{r^\sharp\beta}(\alpha) + \psi(r^\sharp\alpha, r^\sharp\beta, \,.\,), \quad \alpha, \beta \in \mathfrak{g}^* \ .$$

The map $r^\sharp$ is a homomorphism of Lie algebras. Its image, denoted by $\mathfrak{p}$, will be called the *carrier* of the twisted triangular structure (r, ψ), in accordance with the terminology for triangular r-matrices [5]. The kernel of $r^\sharp$ is an abelian ideal of $\mathfrak{g}^*$, and it is stable under the coadjoint action of $\mathfrak{p}$. As in the general case of Lie algebroids, we denote by $r^\sharp_{(\mathfrak{p})}$ the isomorphism from $\mathfrak{p}^*$ to $\mathfrak{p}$ defined by r.

For a finite-dimensional representation V of a Lie algebra $\mathfrak{f}$, we will denote by $\chi_{\mathfrak{f},V} \in \mathfrak{f}^*$ its *infinitesimal character*, defined by

$$\chi_{\mathfrak{f},V}(x) = \mathrm{Tr}_V(x), \quad x \in \mathfrak{f} \ .$$

The infinitesimal character of the representation V is a 1-cocycle of $\mathfrak{f}$ whose cohomology class is the characteristic class of $\mathfrak{f}$ with representation on the 1-dimensional vector space, $\wedge^{\mathrm{top}} V$. angoura@gn.refer.org

4.2. **Modular classes of twisted triangular structures.** The modular class of a twisted triangular structure (r, ψ) on a Lie algebra $\mathfrak{g}$ is the class of a 1-cocycle of the Lie algebra $\mathfrak{g}^*$, which can be computed by means of the formulas in the following proposition and its corollary.

Proposition 4.1. *Let $\theta(\mathfrak{g}, r, \psi)$ be the modular class of the Lie algebra $\mathfrak{g}$ with the twisted triangular structure (r, ψ), with carrier $\mathfrak{p}$. Then*

$$\theta(\mathfrak{g}, r, \psi) = -[r^\sharp_{(\mathfrak{p})}(\chi_{\mathfrak{p},\,\mathrm{Ker}\, r^\sharp})] \ , \tag{4.1}$$

where $\chi_{\mathfrak{p},\,\mathrm{Ker}\, r^\sharp}$ is the infinitesimal character of the coadjoint representation of $\mathfrak{p}$ on $\mathrm{Ker}\, r^\sharp$, and $[x]$ denotes the class in the Lie algebra cohomology of $\mathfrak{g}^*$ of a 1-cocycle $x \in \mathfrak{p} \subset \mathfrak{g}$.

Proof. We have observed in Remark 2.1 that the canonical representation of $\mathfrak{p}$ on $\mathrm{Ker}\, r^\sharp$, which was defined by (2.5) in the general case of regular twisted Poisson structures on Lie algebroids, reduces to the coadjoint representation in the case of Lie algebras. Therefore Proposition 4.1 follows from Theorem 2.5, after observing that $r^\sharp_{(\mathfrak{p})}$ is a skew-symmetric map from $\mathfrak{p}^*$ to $\mathfrak{p} \subset \mathfrak{g}$. □

The Lie algebra $\mathfrak{p}$ also acts on $\mathfrak{g}/\mathfrak{p}$ by the action induced from the adjoint action of $\mathfrak{g}$. This representation is dual to the coadjoint representation of $\mathfrak{p}$ on $\mathrm{Ker}\, r^\sharp$. Therefore

Corollary 4.2. *Let $\chi_{\mathfrak{p},\,\mathfrak{g}/\mathfrak{p}}$ be the infinitesimal character of the representation of $\mathfrak{p}$ on $\mathfrak{g}/\mathfrak{p}$ induced from the adjoint action. Then*

$$\theta(\mathfrak{g}, r, \psi) = [r^\sharp_{(\mathfrak{p})}(\chi_{\mathfrak{p},\,\mathfrak{g}/\mathfrak{p}})] \ . \tag{4.2}$$

4.3. **Non-degenerate twisted triangular structures.** Theorem 3.1 and Corollary 3.2, when applied to the case of Lie algebroids with base a point, yield the following statements.

Proposition 4.3. *There is a one-to-one correspondence between twisted triangular r-matrix structures on a Lie algebra $\mathfrak{g}$ and triples consisting of*

(1) *a Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$,*
(2) *a non-degenerate 2-cochain $\mu \in \Gamma(\wedge^2\mathfrak{p}^*)$, and*
(3) *a 3-cocycle $\psi \in \Gamma(\wedge^3\mathfrak{g}^*)$ whose restriction to $\mathfrak{p}$ is $-d_{\mathfrak{p}}\mu$, where $d_{\mathfrak{p}}$ denotes the Lie algebra cohomology operator of $\mathfrak{p}$.*

The twisted r-matrix $r \in \wedge^2\mathfrak{p}$ and the non-degenerate 2-cochain μ of $\mathfrak{p}$ are inverses of one another.

Corollary 4.4. *Assume that $\mathfrak{g}$ is a Lie algebra and $\mu \in \Gamma(\wedge^2\mathfrak{g}^*)$ is a 2-cochain of $\mathfrak{g}$ whose restriction $\mu|_{\mathfrak{p}}$ to a Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is non-degenerate. If $r \in \Gamma(\wedge^2\mathfrak{p})$ denotes the inverse of $\mu|_{\mathfrak{p}}$, then $(r, -d_{\mathfrak{g}}\mu)$ is a twisted triangular r-matrix structure on the Lie algebra $\mathfrak{g}$.*

Proposition 4.3 shows how the twisted classical Yang-Baxter equation can be linearized, and Corollary 4.4 shows how one can construct many twisted triangular r-matrices. In particular, it explains the origin of Example 4.9 below, as well as the construction of Example 5 in [9], see Example 4.8.

When μ is a non-degenerate 2-cocycle of a Lie algebra $\mathfrak{p}$ with values in the trivial representation, the pair $(\mathfrak{p}, \mu)$ is said to be a *quasi-Frobenius Lie algebra*, see [14, 5, 6]. Let us say that $(\mathfrak{p}, \mu, \psi)$ is a *twisted quasi-Frobenius Lie algebra* if μ is a non-degenerate 2-cochain of the Lie algebra $\mathfrak{p}$ and ψ a 3-cocycle of $\mathfrak{p}$ such that $d_{\mathfrak{p}}\mu = -\psi$. We obtain, as a particular case of Proposition 4.3, the following correspondence.

Corollary 4.5. *There is a one-to-one correspondence between non-degenerate twisted triangular structures and twisted quasi-Frobenius structures on Lie algebras. In this correspondence the non-degenerate twisted triangular r-matrix and the non-degenerate 2-cochain are inverses of one another.*

The well-known one-to-one correspondence between non-degenerate triangular r-matrices and quasi-Frobenius structures on Lie algebras [14, 5] appears as the particular case where $\psi = 0$.

4.4. **Frobenius Lie algebras.** A quasi-Frobenius Lie algebra structure on $\mathfrak{p}$ is defined by a non-degenerate 2-cocycle, μ. The special case of Frobenius Lie algebras corresponds to the case where μ is the coboundary in the Lie algebra cohomology of $\mathfrak{p}$ of some $-\xi \in \mathfrak{p}^*$. In other words $(\mathfrak{p}, \xi)$, with $\xi \in \mathfrak{p}^*$, is a *Frobenius Lie algebra* if the skew-symmetric bilinear form μ, defined by

$$\mu(X, Y) = \xi([X, Y]) \ , \tag{4.3}$$

for $X, Y \in \mathfrak{p}$, is non-degenerate. The bilinear form defined by (4.3) is non-degenerate if and only if the linear map

$$X \in \mathfrak{p} \mapsto \mathrm{ad}^*_X(\xi) \in \mathfrak{p}^* \tag{4.4}$$

is an isomorphism from $\mathfrak{p}$ to $\mathfrak{p}^*$. This implies that $(\mathfrak{p}, \xi)$ is a Frobenius Lie algebra if and only if the coadjoint orbit of ξ, under the action of the adjoint group of $\mathfrak{p}$, is dense in $\mathfrak{p}^*$ (see [6]).

We now consider the case where (r, ψ) is a twisted triangular structure on a Lie algebra $\mathfrak{g}$ derived from a Frobenius structure on the carrier subalgebra $\mathfrak{p}$. The map $r^\sharp \colon \mathfrak{g}^* \to \mathfrak{g}$ defines a linear isomorphism $r^\sharp_{(\mathfrak{p})} : \mathfrak{p}^* \to \mathfrak{p}$. Let $\mu \in \wedge^2\mathfrak{p}^*$ be the 2-cochain on $\mathfrak{p}$ inverse of $r_{(\mathfrak{p})} \in \wedge^2\mathfrak{p}$, and assume that μ is the coboundary of a 1-cochain $-\xi \in \mathfrak{p}^*$, $\mu = -d_{\mathfrak{p}}\xi$. In the following proposition we show how to compute the modular class of a Lie algebra $\mathfrak{g}$ with such a triangular r-matrix. The notation is that of Proposition 4.1 and Corollary 4.2.

Proposition 4.6. *Let r be a triangular r-matrix on a Lie algebra $\mathfrak{g}$ derived from a Frobenius structure ξ on the carrier subalgebra $\mathfrak{p}$ of r. Then the modular class of $(\mathfrak{g}, r, 0)$ is the class in the Lie algebra cohomology of $\mathfrak{g}^*$ of the unique 1-cocycle $X \in \mathfrak{p}$ such that*

$$\mathrm{ad}^*_X(\xi) = \chi_{\mathfrak{p},\mathfrak{g}/\mathfrak{p}} \ . \tag{4.5}$$

Proof. For any $\alpha \in \mathfrak{p}^*$, the element $Z = r^\sharp_{(\mathfrak{p})}(\alpha) \in \mathfrak{p}$ satisfies $\xi([Z, Y]) = -\alpha(Y)$, for all $Y \in \mathfrak{p}$, and therefore

$$\mathrm{ad}^*_Z(\xi) = \alpha \ .$$

Thus $X = r^\sharp_{(\mathfrak{p})}(\chi_{\mathfrak{p},\mathfrak{g}/\mathfrak{p}})$ satisfies

$$\mathrm{ad}^*_X(\xi) = \chi_{\mathfrak{p},\mathfrak{g}/\mathfrak{p}} \ .$$

The uniqueness of X satisfying (4.5) follows from the fact that (4.4) is a linear isomorphism. By Corollary 4.2, the class of X is the modular class of $(\mathfrak{g}, r, 0)$. □

Remark 4.7. There are no twisted analogues of the Frobenius Lie algebras because when μ is a coboundary, the coboundary of μ vanishes. When we apply Theorem 3.1 successively to tangent bundles and to Lie algebras we see that

- twisted quasi-Frobenius structures on Lie algebras correspond to twisted symplectic structures on manifolds (non-degenerate 2-forms),
- quasi-Frobenius structures on Lie algebras correspond to symplectic structures on manifolds (non-degenerate closed 2-forms),
- Frobenius structures on Lie algebras correspond to exact symplectic structures on manifolds (non-degenerate exact 2-forms).

4.5. **Examples.** Denote by e_{ij}, $1 \le i, j \le n$, the standard elementary matrices constituting a basis of $\mathfrak{gl}_n(\mathbb{R})$, and by e^*_{ij}, $1 \le i, j \le n$, the dual basis. First we re-examine [9, Example 5]. The published version of this article contained an error which we now correct.

Example 4.8. Denote by $\mathfrak{g}$ the Lie subalgebra of $\mathfrak{gl}_3(\mathbb{R})$ spanned by $\{e_{ij} \mid 1 \le i \le 2, 1 \le j \le 3\}$. It is a codimension-one subalgebra of a maximal parabolic subalgebra of $\mathfrak{gl}_3(\mathbb{R})$, and it is isomorphic to the Lie algebra of the Lie group of affine transformations of $\mathbb{R}^2$. The pair (r, ψ), where

$$r = e_{11} \wedge e_{22} + e_{13} \wedge e_{23} \qquad \text{and} \qquad \psi = -(e^*_{11} + e^*_{22}) \wedge e^*_{13} \wedge e^*_{23} \ ,$$

defines a twisted triangular r-matrix structure on $\mathfrak{g}$, as shown in [9, Example 5]. The carrier of r is the subalgebra $\mathfrak{p} = \mathrm{Span}\{e_{11}, e_{22}, e_{13}, e_{23}\}$. Then $\dim(\mathfrak{g}/\mathfrak{p}) = 2$, and one can easily show that $\chi_{\mathfrak{p},\,\mathfrak{g}/\mathfrak{p}} = 0$. Therefore $\theta(\mathfrak{g}, r, \psi) = 0$.

We observe that the twisted r-matrix discussed in this example can be easily constructed using Proposition 4.3. Define the 2-cochain on $\mathfrak{g}$,

$$\mu = e_{11}^* \wedge e_{22}^* + e_{13}^* \wedge e_{23}^* \ ,$$

and compute the coboundary of $-\mu$,

$$\psi_1 = -d_{\mathfrak{g}}\mu = -(e_{11}^* + e_{22}^*) \wedge e_{13}^* \wedge e_{23}^* - e_{12}^* \wedge e_{21}^* \wedge e_{22}^* + e_{11}^* \wedge e_{21}^* \wedge e_{12}^* \ .$$

Then the restriction of μ to $\mathfrak{p}$ is non-degenerate, and r is the inverse of $\mu_{|\mathfrak{p}}$. Corollary 4.4 implies that (r, ψ_1) is a twisted triangular r-matrix structure on $\mathfrak{g}$. That the same is true for the pair (r, ψ) above follows from the fact that $\psi - \psi_1$ is a 3-cocycle for $\mathfrak{g}$, together with the equality $(\wedge^3 r^\sharp)(\psi - \psi_1) = 0$.

Next we provide an example of a twisted triangular r-matrix with a nontrivial modular class.

Example 4.9. Let $\mathfrak{q}_{n-1}$ be the Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ spanned by $\{e_{ij} \mid 1 \leq i \leq n-1, 1 \leq j \leq n\}$. It is a codimension-one subalgebra of a maximal parabolic subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ obtained by omitting the $(n-1)$-st negative root of $\mathfrak{gl}_n(\mathbb{R})$. Define the 2-cochain on $\mathfrak{gl}_n(\mathbb{R})$,

$$\mu = \sum_{1 \leq i < j \leq n-1} e_{ij}^* \wedge e_{ji}^* + \sum_{i=1}^{n-1} e_{ii}^* \wedge e_{in}^* \ .$$

The $d_{\mathfrak{gl}_n(\mathbb{R})}$-coboundary of $-\mu$ is the following 3-cocycle on $\mathfrak{gl}_n(\mathbb{R})$:

$$\psi = \sum_{i,j,k=1}^{n} \mathrm{sign}\,(i-j) e_{ik}^* \wedge e_{kj}^* \wedge e_{ji}^* + \sum_{1 \leq i,k \leq n-1, i \neq k} e_{ik}^* \wedge e_{ki}^* \wedge e_{in}^* - \sum_{i,k=1}^{n-1} e_{ii}^* \wedge e_{ik}^* \wedge e_{kn}^* \ ,$$

where, for an integer n, $\mathrm{sign}\, n = 1$ if $n > 0$, $\mathrm{sign}\, n = -1$ if $n < 0$ and $\mathrm{sign}\, 0 = 0$. The restriction of μ to $\mathfrak{q}_{n-1}$ is non-degenerate, and we consider its inverse,

$$r = \sum_{1 \leq i < j \leq n-1} e_{ij} \wedge e_{ji} + \sum_{i=1}^{n-1} e_{ii} \wedge e_{in} \in \wedge^2 \mathfrak{q}_{n-1} \subset \wedge^2(\mathfrak{gl}_n(\mathbb{R})) \ .$$

Corollary 4.4 implies that (r, ψ) is a twisted triangular r-matrix structure on $\mathfrak{gl}_n(\mathbb{R})$. The carrier of r is $\mathfrak{q}_{n-1}$. It is easy to compute

$$\chi_{\mathfrak{q}_{n-1}, \mathfrak{gl}_n(\mathbb{R})/\mathfrak{q}_{n-1}} = -\sum_{i=1}^{n-1} e_{ii}^* \ .$$

Since $r^\sharp e_{ii}^* = e_{in}$, we obtain, from Corollary 4.2,

$$\theta(\mathfrak{gl}_n(\mathbb{R}), r, \psi) = -\left[\sum_{i=1}^{n-1} e_{in}\right] \ .$$

Example 4.10. Let $\mathfrak{p}_1$ be the maximal parabolic subalgebra of $\mathfrak{sl}_n(\mathbb{R})$ containing all the upper triangular matrices, obtained by omitting the first negative root. Then $(\mathfrak{p}_1, \xi)$ is a Frobenius Lie algebra with

$$\xi = \sum_{i=1}^{n-1} e^*_{i,i+1} \ .$$

The corresponding triangular r-matrix on $\mathfrak{sl}_n(\mathbb{R})$ is the Gerstenhaber–Giaquinto generalized Jordanian r-matrix,

$$r_{GG} = \sum_{k=1}^{n-1} d_k \wedge e_{k,k+1} + \sum_{i<j} \sum_{m=1}^{j-i-1} e_{i,j-m+1} \wedge e_{j,i+m} \ ,$$

where

$$d_k = \frac{n-k}{n}(e_{11} + e_{22} + \ldots + e_{kk}) - \frac{k}{n}(e_{k+1,k+1} + e_{k+2,k+2} + \ldots + e_{nn}) \ .$$

See [5], where Gerstenhaber and Giaquinto proved that this skew-symmetric solution of the classical Yang-Baxter equation is a boundary point of a subset of the set of solutions of the modified Yang-Baxter equation containing the semi-classical limit of the generalization of the quantum R-matrices introduced by Cremmer and Gervais in [2], and called it the Cremmer–Gervais r-matrix. It is easy to see that

$$\chi_{\mathfrak{p}_1, \mathfrak{sl}_n(\mathbb{R})/\mathfrak{p}_1} = -(n-1)e^*_{11} + e^*_{22} + e^*_{33} + \ldots + e^*_{nn} \ .$$

The unique $X \in \mathfrak{p}_1$ that satisfies $\mathrm{ad}^*_X(\xi) = \chi_{\mathfrak{p}_1, \mathfrak{sl}_n(\mathbb{R})/\mathfrak{p}_1}$ is the matrix $X = -\sum_{k=1}^{n-1}(n-k)e_{k,k+1}$. Therefore, by Proposition 4.6, the modular class of $\mathfrak{sl}_n(\mathbb{R})$ with the Gerstenhaber–Giaquinto generalized Jordanian r-matrix is

$$\theta(\mathfrak{sl}_n(\mathbb{R}), r_{GG}, 0) = -\left[\sum_{k=1}^{n-1}(n-k)e_{k,k+1}\right] \ .$$

This result constitutes a generalization to all n's of [9, Example 4.2] which is the case of $n = 2$.

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