COMPLETENESS OF DETERMINANTAL HAMILTONIAN FLOWS ON THE MATRIX AFFINE POISSON SPACE

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Abstract. The matrix affine Poisson space $(M_{m,n}, \pi_{m,n})$ is the space of complex rectangular matrices equipped with a canonical quadratic Poisson structure which in the square case $m = n$ reduces to the standard Poisson structure on $\mathrm{GL}_n(\mathbb{C})$. We prove that the Hamiltonian flows of all minors are complete. As a corollary we obtain that all Kogan–Zelevinsky integrable systems on $M_{n,n}$ are complete and thus induce (analytic) Hamiltonian actions of $\mathbb{C}^{n(n-1)/2}$ on $(M_{n,n}, \pi_{n,n})$ (as well as on $\mathrm{GL}_n(\mathbb{C})$ and on $\mathrm{SL}_n(\mathbb{C})$).

We define Gelfand–Zeitlin integrable systems on $(M_{n,n}, \pi_{n,n})$ from chains of Poisson projections and prove that their flows are also complete. This is an analog for the quadratic Poisson structure $\pi_{n,n}$ of the recent result of Kostant and Wallach [10] that the flows of the complexified classical Gelfand–Zeitlin integrable systems are complete.

1. Introduction

The matrix affine Poisson space is the complex affine space $M_{m,n}$ consisting of rectangular matrices of size $m \times n$ equipped with the quadratic Poisson structure

$$\pi_{m,n} = \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} (\mathrm{sign}(k-i) + \mathrm{sign}(l-j)) x_{il} x_{kj} \frac{\partial}{\partial x_{ij}} \wedge \frac{\partial}{\partial x_{kl}}, \tag{1.1}$$

where x_{ij} are the standard coordinate functions on $M_{m,n}$. The geometry of this Poisson structure was studied in [7, 3, 8], motivated by its relation to cluster algebras [6] and the ring theory of the algebra of quantum matrices [2].

The square case is closely related to the theory of Poisson Lie groups. Denote $M_n = M_{n,n}$ and $\pi_n = \pi_{n,n}$. The standard Poisson structure on $\mathrm{GL}_n(\mathbb{C})$, defined by Drinfeld, is given by

$$\pi_{\mathrm{GL}_n} = \sum_{i<j} \left(\mathcal{L}_{E_{ij}} \wedge \mathcal{L}_{E_{ji}} - \mathcal{R}_{E_{ij}} \wedge \mathcal{R}_{E_{ji}} \right).$$

Here $\mathcal{L}$ and $\mathcal{R}$ denote the left and right invariant vector fields on $\mathrm{GL}_n(\mathbb{C})$, and E_{ij} denote the standard elementary matrices considered as elements of $\mathfrak{gl}_n(\mathbb{C})$. The canonical embedding

$$(\mathrm{GL}_n(\mathbb{C}), \pi_{\mathrm{GL}_n}) \hookrightarrow (M_n, \pi_n)$$

is Poisson.

A canonical compactification of $(M_{m,n}, \pi_{m,n})$ as a Poisson manifold was constructed in [7, 3]. Define the parabolic subgroup $\mathrm{GL}_{m+n}(\mathbb{C})$

$$P_{n,m} = \{ \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right) \in \mathrm{GL}_{m+n}(\mathbb{C}) \mid A \in M_{n,n}, B \in M_{n,m}, C \in M_{m \times m} \}.$$

Then $\mathrm{GL}_{m+n}(\mathbb{C})/P_{n,m}$ is diffeomorphic to the Grassmannian $\mathrm{Gr}(n, m+n)$ and $M_{m,n}$ is embedded in it as the open Schubert cell by

$$X \in M_{m,n} \mapsto \left(\begin{smallmatrix} I_n & 0 \\ X & I_m \end{smallmatrix}\right).P_{n,m} \tag{1.2}$$

where I_n denotes the identity matrix of size $n \times n$. One can define the canonical Poisson structure on $\mathrm{Gr}(n, m+n)$

$$\pi_{\mathrm{Gr}(n,m+n)} = -\sum_{1 \le i < j \le m+n} \chi(E_{ij}) \wedge \chi(E_{ji})$$

where $\chi\colon \mathfrak{gl}_{m+n}(\mathbb{C}) \to \mathrm{Vect}(\mathrm{Gr}(m, m+n))$ refers to the induced infinitesimal action from the left action of $\mathrm{GL}_{m+n}(\mathbb{C})$ on $\mathrm{Gr}(n, m+n)$. For a smooth manifold M, $\mathrm{Vect}(M)$ stands for the Lie algebra of smooth vector fields on M. The Poisson structure $\pi_{\mathrm{Gr}(n,m+n)}$ can be also defined as the pushforward of $\pi_{\mathrm{GL}_{m+n}}$ under the natural projection $\mathrm{GL}_{m+n}(\mathbb{C}) \to \mathrm{GL}_{m+n}(\mathbb{C})/P_{n,m} \cong \mathrm{Gr}(n, m+n)$. The map (1.2) is not Poisson but a minor correction turns it into a Poisson map. Let us identify the symmetric group S_n with the subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of permutation matrices. Denote the longest element of S_n by $w_\circ^n$. Then by [7, (3.11)] and [3, Proposition 3.3] the map

$$i_{m,n}\colon (M_{m,n}, \pi_{m,n}) \hookrightarrow (\mathrm{GL}_{m+n}(\mathbb{C})/P_{n,m}, \pi_{GL_{m+n}}), \quad i_{m,n}(X) = \left(\begin{smallmatrix} I_n & 0 \\ X w_\circ^n & I_m \end{smallmatrix}\right).P_{n,m}$$

is Poisson, where I_m and I_n are the identity matrices of sizes $m \times m$ and $n \times n$, respectively.

Given two subsets $I \subset \{1, \dots, m\}$ and $J \subset \{1, \dots, n\}$ with the same number of elements, define by $\Delta_{I,J}(X)$ the minor of $X \in M_{m,n}$ (i.e. the determinant of the submatrix of X) corresponding to rows I and J. Our first result is that the Hamiltonian flows of $\Delta_{I,J}$ are complete; that is the corresponding integrable curves are defined over the full $\mathbb{C}$. In addition, we show that the corresponding dynamics is given by

$$x_{ij}(t) = p(t) + \sum_{a=1}^{N} p_k(t) e^{\alpha_a t}, \ t \in \mathbb{C}, \text{ for some polynomials } p(t), p_1(t), \dots, p_N(t),$$

such that $\deg p(t) \le 2|I| - 1$, $\deg p_a(t) \le 2|I| - 2$, $a = 1, \dots, N$, and some $\alpha_1, \dots, \alpha_N \in \mathbb{C}^*$. Here $|I|$ denotes the number of elements of I. The integer N, the polynomials $p(t)$, $p_a(t)$ and the exponents α_a depend, in general, on the initial conditions. This result is proved in Section 2.

Kogan and Zelevinsky [9] constructed completely integrable systems on symplectic leaves of the double Bruhat cells (corresponding to equal elements of the Weyl group) of any complex reductive Lie group, equipped with the standard Poisson structure. Their Hamiltonians are either twisted generalized minors or generalized minors. If we restrict ourselves to $\mathrm{GL}_n(\mathbb{C})$, this construction produces families of $(n-1)(n-2)/2$ functionally independent Hamiltonians on M_n (given by minors) parametrized by reduced expressions of the longest element $w_\circ^n \in S_n$. Our result implies that, in the A_{n-1} case, the flows of all Kogan-Zelevinsky integrable systems are complete, even when extended to M_n. In particular each of these systems provides a Hamiltonian action of $\mathbb{C}^{(n-1)(n-2)/2}$ on (M_n, π_n) (and on $(\mathrm{GL}_n(\mathbb{C}), \pi_{\mathrm{GL}_n})$ and on $(\mathrm{SL}_n(\mathbb{C}), \pi_{\mathrm{GL}_n}|_{\mathrm{SL}_n})$). *The action is analytic but not* *algebraic.* This is carried out in Section 3. One should note here that, although the left and right actions of the standard maximal torus of $\mathrm{GL}_n(\mathbb{C})$ preserve π_{GL_n}, they are not Hamiltonian. In addition, they provide a lot smaller abelian symmetry groups, compared to the constructed ones of dimension $(n-1)(n-2)/2$.

Kostant and Wallach have recently shown that all flows of the complexified Gelfand–Zeitlin integrable systems are complete everywhere on $\mathfrak{gl}_n^*(\mathbb{C})$. (This is not true for the flows of the classical system on $\mathfrak{u}_n$.) This paper grew out of our attempt to understand whether a similar fact is true for a Gelfand–Zeitlin integrable system for the quadratic Poisson structure π_n on M_n. We construct a Gelfand–Zeitlin type integrable system on (M_n, π_n) as follows. Any projection from (M_n, π_n) to (M_k, π_k) sending an $n \times n$ matrix to (any) given $k \times k$ submatrix is Poisson for all $k < n$. We get the chain of Poisson maps

$$(M_n, \pi_n) \to (M_{n-1}, \pi_{n-1}) \to \ldots \to (M_1, \pi_1)$$

by sending a $k \times k$ matrix to its principal $(k-1) \times (k-1)$ minor for all $k = n, n-1, \ldots, 2$. The field of rational functions on any M_k is a Poisson field. We determine its Poisson center $Z(\mathbb{C}(M_k))$ (i.e. its center with respect to the Poisson bracket) using a result of Kogan and Zelevinsky [9], see Proposition 4.2 for details. Pulling back all Poisson centers to $\mathbb{C}(M_n)$ produces the Poisson commutative subfield of $\mathbb{C}(M_n)$ which is the pure transcendental extension of $\mathbb{C}$ by

$$\Delta_{l;k}/\Delta_{l;l-k}, 1 \le k < l \le n, \Delta_l, 1 \le l \le n. \tag{1.3}$$

We call the set of Hamiltonians (1.3) the Gelfand–Zeitlin integrable system for the quadratic Poisson bracket π_n. Those Hamiltonians indeed define an integrable system on any symplectic leaf of the open double Bruhat cell in $\mathrm{GL}_n(\mathbb{C})$. Every such leaf has dimension $n(n-1)/2$.

Finally we prove that the flow of each of the Hamiltonians (1.3) is complete in the following sense (see Theorem 4.5). Denote by D_n the subvariety of M_n which is the union of zero level sets of the denominators of the Hamiltonians in (1.3). Fix h to be one of the Hamiltonians (1.3) and $X_0 \in M_n \backslash D_n$. We prove that

there exists a curve $\gamma\colon \mathbb{C} \to M_n$ whose entry functions are given by

$$\sum_{p=1}^{N} (c_{ij}^p + d_{ij}^p t) e^{\alpha_{ij}^p t} + (b_{ij} + c_{ij} t + d_{ij} t^2)$$

for some integer N and $b_{ij}, c^{ij}, d^{ij}, c_{ij}^p, d_{ij}^p \in \mathbb{C}$, $\alpha_{ij}^p \in \mathbb{C}^*$ depending on X_0 with the following properties:

1. $\gamma^{-1}(D_n)$ is a discrete subset of $\mathbb{C}$ and

2. $\gamma\colon \mathbb{C}\backslash\gamma^{-1}(D_n) \to M_n$ is the integral curve of the Hamiltonian h such that $\gamma(0) = X_0$.

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2. Completeness of determinantal Hamiltonian flows

Let

$$I = \{i_1 < \ldots < i_r\} \subset \{1, \ldots, m\}, \quad J = \{j_1 < \ldots < j_r\} \subset \{1, \ldots, n\}. \tag{2.1}$$

For $k \in \{1, \ldots, n\}$ set

$$I(i_p \to k) = (I \backslash \{i_p\}) \cup \{k\}.$$

Recall the Poisson brackets [7]

$$\{x_{kl}, \Delta_{I,J}\} = \sum_{q=1}^{r} \operatorname{sign}(i_q - k) x_{i_q l} \Delta_{I(i_q \to k),J} + \sum_{q=1}^{r} \operatorname{sign}(j_q - l) x_{k j_q} \Delta_{I,J(j_q \to l)}, \tag{2.2}$$

for all $k = 1, \ldots, m$ and $l = 1, \ldots, n$ (cf. the proof of [7, Lemma 3.2]). Following [7], we say that $\operatorname{sign}(I - k)$ is defined if $k \in I$, $k < i_1$, or $k > i_r$ in which cases it is equal to 0, 1, or -1, respectively. We recall the following lemma for the convenience of the reader.

Lemma 2.1. [7, Lemma 3.2] *If, in the above notation,* $\operatorname{sign}(k - I)$ *and* $\operatorname{sign}(J - l)$ *are defined and*

$$|\operatorname{sign}(I - k) + \operatorname{sign}(J - l)| \leq 1,$$

then

$$\{x_{kl}, \Delta_{I,J}\} = (\operatorname{sign}(I - k) + \operatorname{sign}(J - l)) x_{kl} \Delta_{I,J}.$$

The main result of this section is:

Theorem 2.2. *The Hamiltonian flow of any minor $\Delta_{I,J}$ on $(M_{m,n}, \pi_{m,n})$ is complete over $\mathbb{C}$. In addition,*

each matrix entry evolves quasi-exponentially according to this flow:

$$x_{kl}(t) = p(t) + \sum_{a=1}^{N} p_a(t) e^{\alpha_a t}, \ t \in \mathbb{C}, \textit{ for some polynomials } p(t), p_1(t), \ldots, p_N(t), \tag{2.3}$$

such that $\deg p(t) \leq 2r - 1$, $\deg p_a(t) \leq 2r - 2$, $a = 1, \ldots, N$, and some $\alpha_1, \ldots, \alpha_N \in \mathbb{C}^*$. The integer N, the polynomials $p(t)$, $p_a(t)$, and the exponents α_a depend on the initial condition.

Proof. Assume that I and J are given by (2.1). We will consider several cases for k and l.

Case 1: $(k > i_r, l < j_1)$ or $(k < i_1, l > j_r)$ or $(k \in I, l \in J)$. In this case $\operatorname{sign}(I - k)$ and $\operatorname{sign}(J - l)$ are defined, and $\operatorname{sign}(I - k) + \operatorname{sign}(J - l) = 0$. According to Lemma 2.1

$$x'_{kl}(t) = \{x_{kl}, \Delta_{I,J}\}(X(t)) = 0$$

and $x_{kl}(t) = x_{kl}(0)$ for all $t \in \mathbb{C}$.

Case 2: $(k \notin I, l \in J)$. The case $(k \in I, l \notin J)$ is treated analogously. In this case (2.2) implies

$$x'_{kl}(t) = \sum_{q=1}^{r} \mathrm{sign}(i_q - k) x_{i_q l}(t) \Delta_{I(i_q \to k), J}(X(t)).$$

Taking into account case 1, we obtain that

$$x'_{kl}(t) = c_{kli_1} x_{ki_1}(t) + \cdots + c_{kli_r} x_{ki_r}(t)$$

for some constants $c_{ki_a j_b}$ $(k \notin I,\ a, b = 1, \ldots, r)$ which depend on the initial condition. Thus the vectors

$$\vec{x}_k(t) = (x_{kj_1}(t), \ldots, x_{kj_r}(t))^T, \quad k \notin I$$

satisfy

$$\vec{x}'_k(t) = C_k \vec{x}_k(t)$$

for the constant matrices $C_k = (c_{ki_a j_b})_{a,b=1}^r$. Therefore, in this case, the entries $x_{kl}(t)$ evolve according to (2.3) with $\deg p(t) + \deg p_1(t) + \cdots + \deg p_N(t) \leq r - N$.

Case 3: $(k \notin I, l \notin J)$. Eq. (2.2) and cases 1-2 imply

$$x'_{kl}(t) = p_0(t) + \sum_{a=1}^{N} p_a(t) e^{\alpha_a t}, \quad t \in \mathbb{C},$$

for some polynomials $p_0(t), \ldots, p_N(t)$ such that $\deg p_a(t) \leq 2r - 2$, $a = 0, \ldots, N$ and some $\alpha_1, \ldots, \alpha_N \in \mathbb{C}^*$. The statement of the theorem now follows from this. Note that some of the cases in 1 are included in case 3, but the dynamics in 1 is simpler. □

3. Completeness of the Kogan–Zelevinsky integrable systems on (M_n, π_n) and Hamiltonian group actions

In [9] Kogan and Zelevinsky constructed completely integrable systems on certain symplectic leaves of a complex simple group G equipped with the standard Poisson structure (the leaves in all double Bruhat cells of the type $G^{u,u}$ for some elements u of the corresponding Weyl group). We restrict our attention to GL_n and the leaves in the open double Bruhat cell $\mathrm{GL}_n^{w_\circ^n, w_\circ^n} = B_+ w_\circ^n B_+ \cap B_- w_\circ^n B_-$ (recall that $w_\circ^n$ denotes the longest element of S_n). *Since $B_+ w_\circ^n B_+ \cap B_- w_\circ^n B_-$ is Zariski open in M_n and the corresponding Hamiltonians are regular functions on M_n, the Kogan–Zelevinsky integrable systems for this case provide sets of $n(n-1)/2$ functionally independent commuting Hamiltonians on (M_n, π_n).*

Denote by $s_1, \ldots, s_{n-1}$ the simple reflections in S_n. For two integers $k \leq l$ set

$$[k, l] = \{k, k+1, \ldots, l\}. \tag{3.1}$$

Fix a reduced expression for the longest element of S_n:

$$w_\circ^n = s_{j_1} \ldots s_{j_{n(n-1)/2}}. \tag{3.2}$$

Following [9], for $k = 1, \ldots, n(n-1)/2$ set

$$v_k = s_{j_1} \ldots s_{j_{k-1}}, \quad u_k = (w_\circ^n)^{-1} v_k = s_{j_{n(n-1)/2}} \ldots s_{j_k}$$

and define the Hamiltonians

$$H_k = \Delta_{u_k[1,k],v_k[1,k]} \in \mathbb{C}[M_n], \quad k = 1, \ldots, n(n-1)/2. \tag{3.3}$$

Theorem 3.1. (*Kogan–Zelevinsky,* [9]) *For any reduced expression* (3.2) *of the longest element of S_n, the Hamiltonians $H_1, \ldots, H_{n(n-1)/2}$ Poisson commute with respect to π_n.*

Theorem 2.2 has the following immediate corollary.

Corollary 3.2. *For any reduced expression* (3.2) *of the longest element of S_n the flows of the Kogan–Zelevinsky Hamiltonians* (3.3) *are complete on (M_n, π_n). As a consequence from each such reduced expression one obtains an analytic Hamiltonian action of $\mathbb{C}^{n(n-1)/2}$ on (M_n, π_n).*

As was pointed out in the introduction, the left and right actions of the standard maximal torus of $\mathrm{GL}_n(\mathbb{C})$ are not Hamiltonian and provide a lot smaller abelian symmetry groups of (M_n, π_n).

4. Gelfand-Zeitlin integrable systems on (M_n, π_n)

The field of rational functions $\mathbb{C}(P)$ of an irreducible affine complex Poisson variety (P, π) is naturally a Poisson field. Its center with respect to the Poisson bracket will be denoted by $Z(\mathbb{C}(P))$ and will be called the Poisson center of $\mathbb{C}(P)$.

4.1. **Poisson maps between different** $(M_{m,n}, \pi_{m,n})$**.** For two subsets $I \subset \{1, \ldots, m\}$ and $J \subset \{1, \ldots, n\}$ we have the natural projection and embedding maps:

$$\phi^{m,n}_{I,J} \colon M_{m,n} \to M_{|I|,|J|}, \quad i^{m,n}_{I,J} \colon M_{|I|,|J|} \hookrightarrow M_{m,n}$$

where $|I|$ and $|J|$ denote the number of elements of I and J. Given $X \in M_{m,n}$, $\phi^{m,n}_{I,J}(X)$ denotes the submatrix of X corresponding to rows in I and columns in J. For $X \in M_{|I|,|J|}$, $i^{m,n}_{I,J}(X) \in M_{m,n}$ is the matrix whose I, J'th submatrix is X and all other entries are equal to 0.

One immediately checks that:

Lemma 4.1. *For all subsets $I \subset \{1, \ldots, m\}$, $J \subset \{1, \ldots, n\}$ the maps*

$$\phi^{m,n}_{I,J} \colon M_{m,n} \to M_{|I|,|J|} \ \text{ and } \ i^{m,n}_{I,J} \colon M_{|I|,|J|} \to M_{m,n}$$

are morphisms of affine Poisson spaces.

Note that if one identifies M_n with $\mathfrak{gl}_n^*(\mathbb{C})$, then for the Kirillov–Kostant Poisson structure the morphisms $\phi^{n,n}_{I,J}$ are Poisson only if $I = J$ and $i^{n,n}_{I,J}$ are never Poisson.

4.2. **The Poisson center of** $\mathbb{C}(M_n)$**.** We will construct Poisson commutative subfields of $\mathbb{C}(M_n)$ by pulling back Poisson centers under a chain of projection maps. For this purpose, we will need a description of the Poisson centers $Z(\mathbb{C}(M_n))$.

For $1 \leq k < l \leq n$ define $\Delta_{l;k}, \Delta'_{l;k} \in \mathbb{C}[M_n]$ by

$$\Delta_{l;k}(X) = \Delta_{[1,k],[l-k+1,l]}(X) \quad \text{and} \quad \Delta'_{l;k}(X) = \Delta_{[l-k+1,l],[1,k]}(X), \ \ X \in M_n.$$

In other words $\Delta_{l;k}(X)$ and $\Delta'_{l;k}(X)$ are the upper–right $k \times k$ minor and the lower–left $k \times k$ minors of $\phi^{n,n}_{[1,l],[1,l]}(X)$.

For $1 \le l \le n$, define also $\Delta_l \in \mathbb{C}[M_n]$ by

$$\Delta_l(X) = \Delta_{[1,l],[1,l]}(X) = \det(\phi_{[1,l],[1,l]}(X)), \quad X \in M_n.$$

The following proposition describes the Poisson centers $Z(\mathbb{C}(M_n)$.

Proposition 4.2. *The (Poisson) center of the Poisson field $\mathbb{C}(M_n)$ is*

$$\mathbb{C}(\Delta_{n;1}/\Delta'_{n;n-1}, \dots, \Delta_{n;n-1}/\Delta'_{n;1}, \Delta_n).$$

It will be obtained as an easy consequence of the following theorem of Kogan and Zelevinsky (which is a special case of their result [9, Theorem 2.3] describing the symplectic leaves of the standard Poisson structure on any complex simple Lie group, see [9, Example 2.10]).

Theorem 4.3. (*Kogan–Zelevinsky*) *The symplectic leaves of the Zariski open subset $\mathrm{GL}_n^{w_\circ^m, w_\circ^n}$ of M_n are*

$$\begin{aligned}(4.1)\quad \{X \in \mathrm{GL}_n^{w_\circ^m, w_\circ^n} \mid \Delta_{n;1}(X)/\Delta'_{n;n-1}(X) = \mathrm{const}, \dots,\\ \Delta_{n;n-1}(X)/\Delta'_{n;1}(X) = \mathrm{const}, \Delta_n(X) = \mathrm{const}\}.\end{aligned}$$

It is clear that Theorem 4.3 implies the inclusion

$$Z(M_n) \supseteq \mathbb{C}(\Delta_{n;1}/\Delta'_{n;n-1}, \dots, \Delta_{n;n-1}/\Delta'_{n;1}, \Delta_n).$$

The major point of Proposition 4.2 is the equality of the above fields.

Proof of Proposition 4.2 . Clearly $Z(\mathbb{C}(M_n)) = Z(\mathbb{C}(\mathrm{GL}_n^{w_\circ^n, w_\circ^n}))$. Consider the map

$$\psi \colon \mathrm{GL}^{w_\circ^n, w_\circ^n} \to (\mathbb{C}^*)^{\times n}, \quad \psi = (\Delta_{n;1}/\Delta'_{n;n-1}, \dots, \Delta_{n;n-1}/\Delta'_{n;1}, \Delta_n).$$

Any $f \in Z(\mathbb{C}(\mathrm{GL}^{w_\circ^m, w_\circ^n}))$ has to be constant along generic symplectic leaves of $(\mathrm{GL}^{w_\circ^m, w_\circ^n}, \pi_n)$. Because of the Kogan–Zelevinsky theorem such an f has to be constant along the generic fibers of ψ. Now the theorem follows from the fact that

$$(4.2) \qquad \mathbb{C}(M_n) = \mathbb{C}(\Delta_{n;k}/\Delta'_{n;n-k}, 1 \le k < n, \Delta_n, x_{ij}, 1 < i \le n, 1 \le j \le n)$$

(in particular $\mathbb{C}(M_n)/\psi^*\mathbb{C}((\mathbb{C}^*)^{\times n})$ is a purely transcendental extension). This is proved by checking that

$$\Delta_{n;k}/\Delta'_{n;n-k} = \sum_{d=1}^{k} Q_{kd} x_{1,n+1-d}, \ k = 1, \dots, n-1, \quad \Delta_n = \sum_{d=1}^{n} Q_{nd} x_{1,n+1-d}$$

for some

$$Q_{kd} \in \mathbb{C}(x_{ij}, 1 < i \le n, 1 \le j \le n), 1 \le d \le k \le n, \quad Q_{kk} \neq 0.$$

□

4.3. **A Gelfand–Zeitlin type integrable system.** For $k = 2, \dots, n$ set

$$\phi_k = \phi^{k,k}_{[1,k-1],[1,k-1]} \colon (M_k, \pi_k) \to (M_{k-1}, \pi_{k-1}).$$

We have the chain of Poisson maps

$$(M_n, \pi_n) \xrightarrow{\phi_n} (M_{n-1}, \pi_{n-1}) \xrightarrow{\phi_{n-1}} \dots \xrightarrow{\phi_2} (M_1, \pi_1), \tag{4.3}$$

cf. Lemma 4.1. Following the Gelfand–Zeitlin procedure we induce a Poisson commutative subfield of $\mathbb{C}(M_n)$ by pulling back the Poisson centers of all fields $\mathbb{C}(M_k)$ and considering the subfield of $\mathbb{C}(M_n)$ generated by them. More precisely, set

$$\phi_{n;k} = \phi_n \cdots \phi_{k+1} \colon (M_n, \pi_n) \to (M_k, \pi_k)$$

and consider the Poisson commutative subfield of $\mathbb{C}(M_n)$ generated by $\phi^*_{n;k}(\mathbb{C}(M_k))$ for all $k = 1, \dots n$. Because of Proposition 4.2 this is the field

$$\mathbb{C}(\Delta_{l;k}/\Delta'_{l;l-k}, 1 \leq k < l \leq n, \Delta_l, 1 \leq l \leq n).$$

This proves the Poisson commutativity of the set of rational Hamiltonians

$$\Delta_{l;k}/\Delta'_{l;l-k}, 1 \leq k < l \leq n, \Delta_l, 1 \leq l \leq n \tag{4.4}$$

on (M_n, π_n). We call this set the Gelfand–Zeitlin integrable system on the matrix affine Poisson space (M_n, π_n). Let us note that the commutativity of the Hamiltonians can be also deduced from [7, Proposition 3.4].

The set of Hamiltonians (4.4) induces integrable systems on the leaves of the Zariski open subset $\mathrm{GL}_n^{w_\circ^n, w_\circ^n}$ as follows. Among those Hamiltonians, n are Casimirs for $(\mathrm{GL}_n^{w_\circ^n, w_\circ^n}, \pi_n)$ and restrict to constants, cf. Theorem 4.3. The remaining $n(n-1)/2$ are functionally independent on every symplectic leaf of $\mathrm{GL}_n^{w_\circ^n, w_\circ^n}$ because iterating (4.2) gives

$$\mathbb{C}(M_n) = \mathbb{C}(\Delta_{l;k}/\Delta'_{l;l-k}, 1 \leq k < l \leq n, \Delta_l, 1 \leq l \leq n, x_{ij}, 1 \leq j < i \leq n).$$

Since all leaves of $\mathrm{GL}_n^{w_\circ^n, w_\circ^n}$ have dimension $n(n-1)$ this induces integrable systems on them.

The integrable systems (4.4) are related to the Gelfand–Zeitlin integrable systems defined by Flaschka and Ratiu [4, 5] in the sense that they are produced by similar procedures but live on dual Poisson Lie groups. The one of Flaschka and Ratiu is on the dual Poisson–Lie group U_n^* of U_n (and thus its complexification is a Hamiltonian system on the dual Poisson Lie group of $\mathrm{GL}_n(\mathbb{C})$). The Hamiltonians (4.4) define dynamics on M_n and on its Poisson submanifold $\mathrm{GL}_n(\mathbb{C})$. It was conjectured by Flaschka and Ratiu [4, 5], and proved by Alekseev and Meinrenken [1] that the Gelfand–Zeitlin integrable systems on U_n^* are globally isomorphic to the classical ones. At the same time the systems (4.4) are not isomorphic to Hamiltonian systems for linear Poisson structures since the standard Poisson structure on $\mathrm{GL}_n(\mathbb{C})$ cannot be globally linearized.

Remark 4.4. Instead of the chain of maps (4.3) one can consider any chain of projections $\phi^{k,k}_{I,J}$ where $|I| = |J|$. Clearly there are $n!^2$ different chains of this type. For each of them one can obtain a set of $n(n-1)/2$ functionally independent commuting Hamiltonians on (M_n, π_n). The resulting integrable systems are different since the right and left actions of S_n on M_n (permuting rows and columns) do not preserve the Poisson structure π_n.

4.4. **Completeness of the flows of the Gelfand–Zeitlin integrable system for (M_n, π_n).** Denote the singular locus of the Hamiltonians (4.4)

$$D_n = \{X \in M_n \mid \Delta'_{l;k}(X) = 0 \text{ for some } 1 \le k < l \le n\}.$$

We prove that the flows of the Hamiltonians (4.4) are complete in the following sense.

Theorem 4.5. *Let h be one of the Hamiltonians* (4.4) *and $X_0 \in M_n \backslash D_n$.*
There exists a curve $\gamma\colon \mathbb{C} \to M_n$ whose entry functions are given by

$$\sum_{p=1}^{N} (c_{ij}^p + d_{ij}^p t) e^{\alpha_{ij}^p t} + (b_{ij} + c_{ij} t + d_{ij} t^2) \tag{4.5}$$

for some integer N and $b_{ij}, c^{ij}, d^{ij}, c_{ij}^p, d_{ij}^p \in \mathbb{C}$, $\alpha_{ij}^p \in \mathbb{C}^*$ depending on X_0 with the following properties:
1. $\gamma^{-1}(D_n)$ is a discrete subset of $\mathbb{C}$ and
2. $\gamma\colon \mathbb{C}\backslash\gamma^{-1}(D_n) \to M_n$ is the integral curve of the Hamiltonian h such that $\gamma(0) = X_0$.

Proof. We will prove the statement for the case $h = \Delta_{l;k}/\Delta'_{l;l-k}$ for some $1 \le k < l \le n$. The case of $h = \Delta_l$ is simpler and is left to the reader.

1. Let $i, j \in [1, l]$. Then

$$\{x_{ij}, \Delta_{l;k}/\Delta'_{l;l-k}\} = 0$$

because the map $\phi^{n,n}_{[1,k],[1,k]}$ is Poisson and we define

$$\gamma_{ij}(t) = (X_0)_{ij} = \text{const} \tag{4.6}$$

for those i, j's.

2. Let $i \in [1, k]$, $j \in [l + 1, n]$ or $i \in [l + 1, n]$, $j \in [1, l - k]$. We set $\epsilon = -1$ and 1, respectively. From Lemma 2.1 one gets

$$\{x_{ij}, \Delta_{l;k}/\Delta'_{l;l-k}\} = \epsilon\, x_{ij} \Delta_{l;k}/\Delta'_{l;l-k}.$$

Define

$$\gamma_{ij}(t) = (X_0)_{ij} e^{\epsilon \Delta_{l;k}(X_0)/\Delta'_{l;l-k}(X_0)}. \tag{4.7}$$

For those values of i and j, this is the evolution of the matrix entries of the integral curve of h through X_0 outside the singular locus D_n.

3. Let $i \in [k + 1, l]$, $j \in [l + 1, n]$. Using (2.2) and Lemma 2.1 we get

$$\{x_{ij}, \Delta_{l;k}/\Delta'_{l;l-k}\} = -\Big(\sum_{q=1}^{k} x_{qj} \Delta_{[1,k](q\to i),[l-k+1,l]} + \sum_{q=l-k+1}^{l} x_{iq} \Delta_{[1,k],[l-k+1,l](q\to j)} - \Delta_{l;k}\Big)/\Delta'_{l;l-k}. \tag{4.8}$$

Substituting x_{ab} with $\gamma_{ab}(t)$ from (4.6)–(4.7) in the RHS of (4.8), gives that outside D_n the matrix entries of the integral curve of h through X_0 satisfy

$$x'_{ij}(t) = \sum_{p=1}^{N} c_{ij}^p e^{\alpha_{ij}^p t} + d_{ij} \tag{4.9}$$

and thus

$$x_{ij}(t) = \sum_{p=1}^{N} c_{ij}^p e^{\alpha_{ij}^p t} + (c_{ij} + d_{ij} t) \tag{4.10}$$

for some integer N and $c^{ij}, d^{ij}, c_{ij}^p \in \mathbb{C}$, $\alpha_{ij}^p \in \mathbb{C}^*$. For $i \in [k+1, l]$, $j \in [l+1, n]$ define $\gamma_{ij}(t)$ by the RHS of (4.10).

4. Let $i \in [l+1, n]$, $j \in [l-k+1, l]$. In this case one computes the Poisson bracket $\{x_{ij}, \Delta_{l;k}/\Delta'_{l;l-k}\}$ similarly to (4.8) and proves that it is given by a formula of the type (4.9) where again $c^{ij}, d^{ij}, c_{ij}^p \in \mathbb{C}$, $\alpha_{ij}^p \in \mathbb{C}^*$ depend on X_0. For $i \in [l+1, n]$, $j \in [l-k+1, l]$ we define $\gamma_{ij}(t)$ by the RHS of the latter formula. That formula also gives the evolution of the matrix entries of the integral curve of h through X_0 outside the singular locus D_n.

5. Let $i, j \in [l+1, n]$. Using (2.2) one obtains

$$\begin{aligned}\{x_{ij}, \Delta_{l;k}/\Delta'_{l;l-k}\} = -\Big(\sum_{q=1}^{k} x_{qj} \Delta_{[1,k](q\to i),[l-k+1,l]} + \\ \sum_{q=l-k+1}^{l} x_{iq} \Delta_{[1,k],[l-k+1,l](q\to j)} \Big) / \Delta'_{l;l-k} + \Delta_{l;k} \Big(\sum_{q=1}^{k} x_{qj} \Delta_{[k+1,l](q\to i),[1,l-k]} + \\ \sum_{q=l-k+1}^{l} x_{iq} \Delta_{[k+1,l],[1,l-k](q\to j)} \Big) / (\Delta'_{l;l-k})^2.\end{aligned} \tag{4.11}$$

We substitute x_{ab} with $\gamma_{ab}(t)$ from the formulas in cases 1–4 in the RHS of (4.11), and get that outside D_n the matrix entries of the integral curve of h through X_0 satisfy

$$x'_{ij}(t) = \sum_{p=0}^{N} (c_{ij}^p + d_{ij}^p t) e^{a_{ij}^p t} \tag{4.12}$$

for some $c_{ij}^p, \alpha_{ij}^p \in \mathbb{C}$. We will assume that $\alpha_{ij}^0 = 0$ and set $c_{ij}^0 = d_{ij}^0 = 0$ if polynomial terms are not present. Thus, outside D_n the integral curve of h through X_0 is integrated to

$$x_{ij}(t) = \sum_{p=1}^{N} (c_{ij}^p + d_{ij}^p t) e^{a_{ij}^p t} + (b_{ij} + c_{ij} t + d_{ij} t^2) \tag{4.13}$$

for some $b_{ij}, c_{ij}, d_{ij}, c_{ij}^p, d_{ij}^p \in \mathbb{C}$, $\alpha_{ij}^p \in \mathbb{C}^*$. For $i, j \in [l+1, n]$ we define $\gamma_{ij}(t)$ by the RHS of (4.13).

The cases 1–5 define an analytic curve $\gamma \colon \mathbb{C} \to M_n$ with the property (4.5). If H is any of the Hamiltonians (4.4) and Δ is its denominator, then $\Delta(\gamma(t))$ is an entire function which does not vanish identically since $\Delta(\gamma(0)) = \Delta(X_0) \neq 0$. Therefore

$$\gamma^{-1}(\{X \in M_n \mid \Delta(X) = 0\}) \tag{4.14}$$

is a discrete subset of $\mathbb{C}$. Thus $\gamma^{-1}(D_n)$ is a discrete subset of $\mathbb{C}$ since it is the union of all sets (4.14) where H runs over the Hamiltonians (4.4). Finally, outside $\gamma^{-1}(D_n)$, $\gamma(t)$ is the integral curve of the Hamiltonian h through X_0 because of the way the matrix entries of $\gamma(t)$ were constructed. □

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