INVARIANT PRIME IDEALS IN QUANTIZATIONS OF NILPOTENT LIE ALGEBRAS

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ABSTRACT. De Concini, Kac and Procesi defined a family of subalgebras $\mathcal{U}_+^w$ of a quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$, associated to the elements of the corresponding Weyl group W. They are deformations of the universal enveloping algebras $\mathcal{U}(\mathfrak{n}_+ \cap \mathrm{Ad}_w(\mathfrak{n}_-))$ where $\mathfrak{n}_\pm$ are the nilradicals of a pair of dual Borel subalgebras. Based on results of Gorelik and Joseph and an interpretation of $\mathcal{U}_+^w$ as quantized algebras of functions on Schubert cells, we construct explicitly the H invariant prime ideals of each $\mathcal{U}_+^w$ and show that the corresponding poset is isomorphic to $W^{\leq w}$, where H is the group of group-like elements of $\mathcal{U}_q(\mathfrak{g})$. Moreover, for each H-prime of $\mathcal{U}_+^w$ we construct a generating set in terms of Demazure modules related to fundamental representations.

Using results of Ramanathan and Kempf we prove similar theorems for vanishing ideals of closures of torus orbits of symplectic leaves of related Poisson structures on Schubert cells in flag varieties.

1. Introduction

Let $\mathfrak{g}$ be a be a split semisimple Lie algebra over a field $\mathbb{K}$ of characteristic 0. Fix a pair of opposite Borel subalgebras $\mathfrak{b}_\pm$ with nilradicals $\mathfrak{n}_\pm$. Let $q \in \mathbb{K}$ be transcendental over $\mathbb{Q}$ and $\mathcal{U}_q(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra over $\mathbb{K}$ with standard generators $X_1^\pm, \ldots X_r^\pm$, $K_1^{\pm 1}, \ldots, K_r^{\pm 1}$.

Given an element w of the Weyl group W of $\mathfrak{g}$, one defines the nilpotent subalgebra $\mathfrak{n}_+ \cap \mathrm{Ad}_w(\mathfrak{n}_-)$ of $\mathfrak{g}$, where Ad refers to the adjoint action. The q-analog of $\mathcal{U}(\mathfrak{n}_+ \cap \mathrm{Ad}_w(\mathfrak{n}_-))$ is defined in a less straightforward way. For each reduced expression $w = s_{i_1} \ldots s_{i_n} \in W$ one defines the Lusztig root vectors [23, 5] $X_{i_1}^\pm$, $T_{i_1}(X_{i_2}^\pm)$, $\ldots$, $T_{i_1} \ldots T_{i_{n-1}}(X_{i_n}^\pm)$, where T denotes the action [23, 5] of the braid group of W on $\mathcal{U}_q(\mathfrak{g})$. De Concini, Kac, and Procesi [6] proved that the subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by these root vectors (in the plus and minus cases) do not depend on the choice of a reduced expression of w and studied their representations at roots of unity. Denote the De Concini–Kac–Procesi subalgebras of $\mathcal{U}_q(\mathfrak{g})$ corresponding to $w \in W$ by $\mathcal{U}_\pm^w$.

In this paper we investigate the set of prime ideals of $\mathcal{U}_-^w$ invariant under the conjugation action of the group $H = \langle K_1, \ldots, K_r \rangle$ of group-like elements of $\mathcal{U}_q(\mathfrak{g})$. We identify the (finite) poset of those ideals ordered under inclusions with a Bruhat interval, obtain an explicit description of each ideal using Demazure modules, and construct a small generating set for each ideal. Special examples of $\mathcal{U}_-^w$ are the algebras of quantum matrices $R_q[M_{m,n}]$. Even in this case the generating sets and explicit description of the ideals present new results.

We prove that the algebras $\mathcal{U}_-^w$ are quotients of the Joseph–Gorelik quantum Bruhat cell translates [17, 16]. The latter are quantizations of the algebras of functions on the translated Bruhat cells of the full flag variety associated to $\mathfrak{g}$ with respect to the standard Poisson structure. Along the way we obtain a model for $\mathcal{U}_-^w$ as quantizations of Poisson structures on Schubert cells. Our constructions are similar to the De Concini–Procesi [7] interpretation of $\mathcal{U}_-^w$ as quantum Schubert cells. They constructed an isomorphism between a localization of $H\mathcal{U}_-^w$ and a localization of a quotient of the quantized algebra of functions on a Borel subgroup. We work with a realization of $\mathcal{U}_-^w$ (without H and localization) in terms of Demazure modules. To be more precise, let G be the split simply connected semisimple algebraic group over $\mathbb{K}$ with Lie algebra $\mathfrak{g}$. Denote by $B_\pm$ the Borel subgroups of G corresponding to $\mathfrak{b}_\pm$. It is well known that the coordinate ring of the Schubert cell $B_+w \cdot B_+ \subset G/B_+$ consists of matrix coefficients of Demazure $\mathfrak{b}_+$-modules, cf. §4.6 for details. We construct a quantum version of this coordinate ring as follows. Let P_+ be the set of dominant weights of $\mathfrak{g}$. Denote by $V(\lambda)$ the irreducible $\mathcal{U}_q(\mathfrak{g})$-module with highest weight $\lambda \in P_+$. The Demazure module $V_w(\lambda)$ is the $\mathcal{U}_q(\mathfrak{b}_+)$-module generated by $T_w v_\lambda$ where v_λ is a heighest weight vector of $V(\lambda)$ and T_w refers to the canonical action of the braid group of W on $V(\lambda)$, see [23, 5]. Denote the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $X_1^\pm, \dots, X_r^\pm$ by $\mathcal{U}_\pm$. Identify $U_+^w := U^+ \cap wU^-w^{-1} \cong B_+w \cdot B_+$ and define the quantized coordinate ring $R_q[U_+^w]$ of the Schubert cell $B_+w \cdot B_+$ as the subset of $(\mathcal{U}_+)^*$ consisting of all matrix coefficients $c_\eta^{w,\lambda}(x) := \langle \eta, xT_w v_\lambda \rangle$ for $\eta \in V_w(\lambda)^*$, which is easily seen to be a $\mathbb{K}$-space §3.8. One can make it into a $\mathbb{K}$-algebra by setting

$$c_{\eta_1}^{w,\lambda_1} c_{\eta_2}^{w,\lambda_2} = q^{\langle \lambda_2, \lambda_1 - w^{-1}\mu_1 \rangle} c_\eta^{w,\lambda_1+\lambda_2},$$
$$\text{where } \ \eta = \eta_1 \otimes \eta_2|_{\mathcal{U}_+(T_w v_{\lambda_1} \otimes T_w v_{\lambda_2})} \in V_w(\lambda_1+\lambda_2)^*$$

for $\eta_1 \in V_w(\lambda_1)^*$ of weight μ_1 and $\eta_2 \in V_w(\lambda_2)^*$, see §3.8 for details.

Recall that to each $w \in W$ one associates a quantum R-matrix $\mathcal{R}^w$ which belongs to a certain completion of $\mathcal{U}_+^w \otimes \mathcal{U}_-^w$, see §2.4. Our treatment of $\mathcal{U}_-^w$ rests upon the fact that he map $\psi_w \colon R_q[U_+^w] \to \mathcal{U}_-^w$ given by

$$\psi_w(c_\eta^{w,\lambda}) = (c_\eta^{w,\lambda} \otimes \mathrm{id})(\mathcal{R}^w)$$

is an algebra isomorphism and the fact that $R_q[U_-^w]$ is a quotient of the Joseph–Gorelik quantum Bruhat cell translates (which are quantizations of $wB_- \cdot B_+ \subset G/B_+$). Both facts are proved in Sect. 3.

We then use Gorelik's detailed study [16] of the spectra of the quantum Bruhat cell translates and Joseph's results [18] on generating sets for ideals of $R_q[G]$ to obtain the following Theorem:

Theorem 1.1. *Fix $w \in W$. For each $y \in W^{\leq w}$ define*

$$I_w(y) = \{(c_\eta^{w,\lambda} \otimes \mathrm{id})(\mathcal{R}^w) \mid \lambda \in P_+, \eta \in (V_w(\lambda) \cap \mathcal{U}^- T_y v_\lambda)^\perp\}. \tag{1.1}$$

Then:

(a) $I_w(y)$ is an H-invariant prime ideal of $\mathcal{U}_-^w$ and all H-invariant prime ideals of $\mathcal{U}_-^w$ are of this form.

(b) The correspondence $y \in W^{\leq w} \mapsto I_w(y)$ is an isomorphism from the poset $W^{\leq w}$ to the poset of H invariant prime ideals of $\mathcal{U}_-^w$ ordered under inclusion; that is $I_w(y) \subseteq I_w(y')$ for $y, y' \in W^{\leq w}$ if and only if $y \leq y'$.

(c) $I_w(y)$ is generated as a right ideal by

$$(c^{w,\omega_i}_\eta \otimes \mathrm{id})(\mathcal{R}^w) \quad for \quad \eta \in (V_w(\omega_i) \cap \mathcal{U}^- T_y v_{\omega_i})^\perp, i = 1, \dots, r,$$

where $\omega_1, \dots, \omega_r$ are the fundamental weights of $\mathfrak{g}$.

Assuming only that q is not a root of unity and without restrictions on the characteristic of $\mathbb{K}$ Mériaux and Cauchon [25] obtained a classification of the H-primes of $\mathcal{U}^w_+$ using Cauchon's deletion procedure [4]. Such parametrizations were previously obtained for quantum matrices [22] by Launois. But even for quantum matrices an explicit formula for the ideals $I_w(y)$ of the type (1.1) was unknown.

The poset structure on H-primes was known only for quantum matrices due to Launois [22] under the same restriction that q is transcendental over $\mathbb{Q}$ and $\mathbb{K}$ has characteristic 0.

Generating sets were known only for 3×3 quantum matrices due to Goodearl and Lenagan [13] for arbitrary $\mathbb{K}$, q not a root of unity. Launois [21] proved that the invariant prime ideals in quantum matrices are generated as one sided ideals by quantum minors for the case of q transcendental. In an independent work Goodearl, Launois, and Lenagan [12] determine all quantum minors in a given invariant prime ideal of $R_q[M_{m,n}]$ (for an arbitrary field $\mathbb{K}$, q not a root of unity) and thus construct generating sets for those ideals (in the case when $\mathbb{K}$ has characteristic 0 and q is transcendental over $\mathbb{Q}$). In Sect. 5 we show that the uniform treatment of ideal generators for the prime ideals of all algebras $\mathcal{U}^w_-$ from Theorem 1.1, when specialized to quantum matrices gives explicit generating sets consisting of quantum minors. Our generating sets are smaller than those in [12].

Assume that A is an algebra with a rational action of a torus T by algebra automorphisms. Goodearl and Letzter [13] showed that under some minor conditions Spec$A$ has a natural stratification into strata indexed by T-primes of A, see also Brown–Goodearl [2]. They furthermore proved that each stratum can be identified with the spectrum of a Laurent polynomial ring. Their results apply to iterated skew polynomial rings again under some mild hypotheses which are satisfied for $\mathcal{U}^w_-$ [25]. This in particular provides a stratification of Spec$\mathcal{U}^w_-$ with the property that all strata are tori. Such a stratification can be also directly obtained from Gorelik's results using Theorem 3.7.

In the case when G is a complex simple group, the flag variety G/B_+ has a natural Poisson structure studied by Brown, Goodearl and the author [3, 15]. All Schubert cells $B_+ w \cdot B_+ \subset G/B_+$ are Poisson submanifolds. This induces Poisson structures π_w on $U^w_+ = U^+ \cap wU^- w^{-1} \cong B_+ w \cdot B_+$. All of them are invariant under the action of the maximal torus $T = B_+ \cap B_-$ of G. The torus orbits of symplectic leaves of π_w were described in [3, 15]. In Sect. 5 we show that all results for $\mathcal{U}^w_-$ have Poisson analogs for the vanishing ideals of the torus orbits of leaves of π_w. In particular, one obtains an explicit description of these ideals in terms of Demazure modules, as well as generating sets for the ideals. Another consequence is that we obtain an isomorphism between the poset of H-invariant prime ideals of $\mathcal{U}^w_-$ and the underlying poset of the stratification of (U^w_+, π_w) into T-orbits of symplectic leaves (with inverted order relation). This is the first step towards realizing the orbit method program for the algebras $\mathcal{U}^w_-$ which would amount to constructing a homeomorphism between Spec$\mathcal{U}^w_-$ and the symplectic foliation of (U^w_+, π_w).

To understand the relation between the situation in the Poisson case and Gorelik's construction one is led to consider certain intersections of Schubert cells with respect to three different flags, see Lemma 4.2 and Proposition 4.3. Recently Knutson, Lam and Speyer [20] raised the question of finding intersections of multiple (> 2) Bruhat decompositions with good geometric properties. They proved that each stratum of Lusztig's stratification of Grassmannians can be considered as an intersection of Schubert cells with respect to n cyclically permuted Borel subgroups and used this to obtain a number of results on the geometry of Lusztig's stratification.

We finish the introduction with several notation conventions. For a subgroup $B \subset G$ and $g \in G$ we denote by $g \cdot B$ the coset of g in G/B and by gB the product subset of G. For a subvariety X of Z we denote by $Cl_Z(X)$ the Zariski closure of X in Z. For a subspace L of a vector space V we denote $L^\perp := \{\xi \in V^* \mid \langle \xi, v\rangle = 0, \ \forall v \in L\}$. A submanifold M of a Poisson manifold (X, π) will be called a complete Poisson submanifold if M is a union of symplectic leaves of (X, π).

Acknowledgements. I am grateful to Ken Goodearl for many helpful discussions and to Victor Kac for pointing out the results in [6] at an early stage of this project when I was trying to understand representations of quantized universal enveloping algebras of nilradicals of parabolic subalgebras. The author was partially supported by NSF grant DMS-0701107. I would also like to thank Stéphane Launois and Laurent Rigal for comments on the first draft of the paper.

2. The algebras $\mathcal{U}_q(\mathfrak{g})$, $R_q[G]$, R^w_0, and $\mathcal{U}^w_\pm$

2.1. Let $\mathbb{K}$ be a field of characteristic 0 and $q \in \mathbb{K}$ be transcendental over $\mathbb{Q}$. Let $\mathfrak{g}$ be a split semisimple Lie algebra over $\mathbb{K}$. Denote the rank of $\mathfrak{g}$ by r and its Cartan matrix by (c_{ij}). The quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{K}$-algebra with generators

$$X_i^\pm, K_i^{\pm 1}, \ i = 1, \ldots, r,$$

subject to the relations

$$K_i^{-1}K_i = K_iK_i^{-1} = 1, \ K_iK_j = K_jK_i, \ K_iX_j^\pm K_i^{-1} = q^{\pm c_{ij}} X_j^\pm,$$
$$X_i^+X_j^- - X_j^-X_i^+ = \delta_{i,j}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-c_{ij}} \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_q (X_i^\pm)^k X_j^\pm (X_i^\pm)^{1-c_{ij}-k} = 0, \ i \neq j.$$

Here $q_i = q^{d_i}$ for the standard choice of integers d_i for which the matrix $(d_i c_{ij})$ is symmetric. Recall that $\mathcal{U}_q(\mathfrak{g})$ is a Hopf algebra with comultiplication given by

$$\Delta(K_i) = K_i \otimes K_i,$$
$$\Delta(X_i^+) = X_i^+ \otimes K_i + 1 \otimes X_i^+,$$
$$\Delta(X_i^-) = X_i^- \otimes 1 + K_i \otimes X_i^-,$$

antipode and counit given by

$$S(K_i) = K_i^{-1},\ S(X_i^+) = -X_i^+ K_i^{-1},\ S(X_i^-) = -K_i X_i^-,$$

and

$$\epsilon(K_i) = 1,\ \epsilon(X_i^\pm) = 0.$$

Here

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},\ [n]_q! = [1]_q \dots [n]_q,\ \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}.$$

Denote by $\mathcal{U}_\pm$ the subalgebras of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{X_i^\pm\}_{i=1}^r$. Let H be the group generated by $\{K_i^{\pm 1}\}_{i=1}^r$. Set $\mathcal{U}_q(\mathfrak{b}_\pm) = H\mathcal{U}_\pm$.

2.2. Let P and P_+ be the sets of all integral and dominant integral weights of $\mathfrak{g}$. The sets of simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$ will be denoted by $\{\alpha_i\}_{i=1}^r$, $\{\alpha_i^\vee\}_{i=1}^r$, and $\{\omega_i\}_{i=1}^r$, respectively. The weight spaces of a $\mathcal{U}_q(\mathfrak{g})$-module V are defined by

$$V_\lambda = \{v \in V \mid K_i v = q^{\langle \lambda, \alpha_i^\vee \rangle} v,\ \forall i = 1, \dots, r\}$$

for $\lambda \in P$. A $\mathcal{U}_q(\mathfrak{g})$-module is a weight module if it is the sum of its weight spaces. The irreducible finite dimensional weight $\mathcal{U}_q(\mathfrak{g})$-modules are parametrized by P_+. Denote by $V(\lambda)$ the irreducible module corresponding to $\lambda \in P_+$. For each $\lambda \in P_+$ fix a highest weight vector v_λ of $V(\lambda)$.

The quantized coordinate ring $R_q[G]$ is the Hopf subalgebra of the restricted dual of $\mathcal{U}_q(\mathfrak{g})$ spanned by all matrix entries $c^\lambda_{\xi,v}$, $\lambda \in P_+$, $v \in V(\lambda), \xi \in V(\lambda)^*$. Thus $c^\lambda_{\xi,v}(x) = \langle \xi, xv \rangle$ for $x \in \mathcal{U}_q(\mathfrak{g})$.

We have the canonical left and right actions of $\mathcal{U}_q(\mathfrak{g})$ on $R_q[G]$:

$$x \rightharpoonup c = \sum c_{(2)}(x) c_{(1)},\ c \leftharpoonup x = \sum c_{(1)}(x) c_{(2)},\ x \in \mathcal{U}_q(\mathfrak{g}), c \in R_q[G]. \tag{2.1}$$

The subalgebra of $R_q[G]$ invariant under the left action of $\mathcal{U}_+$ will be denoted by R^+. It is spanned by all matrix entries $c^\lambda_{\xi,v_\lambda}$ where $\lambda \in P_+$, $\xi \in L(\lambda)^*$ and v_λ is the fixed highest weight vector of $V(\lambda)$.

2.3. Denote the Weyl and braid groups of $\mathfrak{g}$ by W and $\mathcal{B}_\mathfrak{g}$, respectively. There is a natural action of $\mathcal{B}_\mathfrak{g}$ on the modules $V(\lambda)$, see [5, 23] for details. The standard generators $T_1, \dots, T_l$ of $\mathcal{B}_\mathfrak{g}$ act by

$$T_i = \sum_{a,b,c \in \mathbb{N}} (-1)^b q_i^{ac-b} (X_i^+)^{(a)} (X_i^-)^{(b)} (X_i^+)^{(c)}$$

where

$$(X_i^\pm)^{(n)} = \frac{(X_i^\pm)^n}{[n]_{q_i}!}.$$

Similarly $\mathcal{B}_\mathfrak{g}$ acts on $\mathcal{U}_q(\mathfrak{g})$. Its generators act by

$$T_i(X_i^+) = -X_i^- K_i, \ T_i(X_i^-) = -K_i^{-1} X_i^+, \ T_i(K_j) = K_j K_i^{-c_{ij}},$$
$$T_i(X_j^+) = \sum_{r=0}^{-c_{ij}} (-q_i)^{-r} (X_i^+)^{(-c_{ij}-r)} X_j^+ (X_i^+)^{(r)}, \ j \neq i,$$
$$T_i(X_j^-) = \sum_{r=0}^{-c_{ij}} (-q_i)^{r} (X_i^-)^{(r)} X_j^- (X_i^-)^{(-c_{ij}-r)}, \ j \neq i.$$

The action of the braid group $\mathcal{B}_\mathfrak{g}$ has the properties

$$T_w V(\lambda)_\mu = V(\lambda)_{w(\mu)}, \ \ T_w(x.v) = (T_w x).(T_w v), \tag{2.2}$$

for all $w \in W$, $x \in \mathcal{U}_q(\mathfrak{g})$, $v \in V(\lambda)$, $\mu \in P$.

2.4. Fix $w \in W$. For a reduced expression

$$w = s_{i_1} \dots s_{i_k} \tag{2.3}$$

define the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \dots, \beta_k = s_{i_1} \dots s_{i_{k-1}} \alpha_{i_k} \tag{2.4}$$

and the root vectors

$$X^\pm_{\beta_1} = X^\pm_{i_1}, X^\pm_{\beta_2} = T_{s_{i_1}} X^\pm_{i_2}, \dots, X^\pm_{\beta_k} = T_{s_{i_1} \dots s_{i_{k-1}}} X^\pm_{i_k}, \tag{2.5}$$

see [23] for details. Following [6], define the subalgebra $\mathcal{U}^w_\pm$ of $\mathcal{U}_\pm$ generated by $X^\pm_{\beta_j}$, $j = 1, \dots, k$.

Theorem 2.1. *(De Concini, Kac, Procesi)* [6, Proposition 2.2] *The definition of the algebra $\mathcal{U}^w_\pm$ does not depend on the choice of a reduced decomposition of w. The algebra $\mathcal{U}^w_\pm$ has the PBW basis*

$$(X^\pm_{\beta_k})^{n_k} \dots (X^\pm_{\beta_1})^{n_1}, \ \ n_1, \dots, n_k \in \mathbb{N}.$$

Set

$$\exp_{q_i} = \sum_{n=0}^{\infty} q_i^{n(n+1)/2} \frac{n^k}{[n]_{q_i}!}.$$

The universal R-matrix associated to w is given by

$$\mathcal{R}^w = \prod_{j=k,\dots,1} \exp_{q_{i_j}} \left((1 - q_{i_j})^{-2} X^+_{\beta_j} \otimes X^-_{\beta_j} \right) \tag{2.6}$$

in terms of the reduced decomposition (2.3) and the root vectors (2.5), cf. [5] for details. In (2.6) the terms are multiplied in the order $j = k, \dots, 1$. The R-matrix $\mathcal{R}^w$ belongs to $\mathcal{U}^w_+ \widehat{\otimes} \mathcal{U}^w_-$, the completion of $\mathcal{U}^w_+ \otimes \mathcal{U}^w_-$ with respect to the descending filtration [23, §4.1.1]. It does not depend on the choice of reduced decomposition of w. For all $\lambda, \mu \in P_+$

$$T_{w, V(\lambda) \otimes V(\mu)} = (\mathcal{R}^w)^{-1} \left(T_{w,V(\lambda)} \otimes T_{w,V(\mu)} \right). \tag{2.7}$$

2.5. Fix again $w \in W$. For each $\lambda \in P_+$ fix $\xi_{w,\lambda} \in (V(\lambda)^*)_{-w\lambda}$ normalized by $\langle \xi_{w,\lambda}, T_w v_\lambda \rangle = 1$.

Following [17], for $\lambda \in P_+$ define

$$c^\lambda_w = c^\lambda_{\xi_{w,\lambda}, v_\lambda},$$

cf. §2.2 for the definition of $v_\lambda \in V(\lambda)_\lambda$.

It is clear from (2.7) that $c^\lambda_w c^\mu_w = c^{\lambda+\mu}_w = c^\mu_w c^\lambda_w$ for all $\lambda, \mu \in P_+$. Denote [17, §9.1.10] the multiplicative commutative subset of R_+:

$$c_w = \{c^\lambda_w \mid \lambda \in P_+\}.$$

Lemma 2.2. *(Joseph)* [17, Lemma 9.1.10] *The set c_w is Ore in R^+.*

Denote the localization

$$R^w = R^+[c_w^{-1}],$$

cf. [17, 16] for details. Recall that the left and right $\mathcal{U}_q(\mathfrak{g})$-actions (2.1) on $R_q[G]$ induce left and right actions of $\mathcal{U}_q(\mathfrak{g})$ on R^w, [17, §4.3.12]. Let R^w_0 be the H invariant subalgebra of R^w with respect to the left action of H. For $\lambda \in P_+$ set $c^{-\lambda}_w = (c^\lambda_w)^{-1}$ in R^w. Note that

$$R^w_0 = \{c^{-\lambda}_w c^\lambda_{\xi, v_\lambda} \mid \lambda \in P_+, \xi \in V(\lambda)^*\} \tag{2.8}$$

since for all $\lambda, \mu \in P_+$ and $\xi \in V(\lambda)^*$, there exists $\xi' \in V(\lambda+\mu)^*$ such that $c^{-\lambda}_w c^\lambda_{\xi, v_\lambda} = c^{-\lambda-\mu}_w c^{\lambda+\mu}_{\xi', v_{\lambda+\mu}}$.

3. The H-spectrum of $\mathcal{U}^w_-$

3.1. We start by recalling several results of Gorelik [16]. For $y \in W$ define the ideals

$$Q(y)^\pm = \mathrm{Span}\{c^\lambda_{\xi, v_\lambda} \mid \xi \in V(\lambda)^*,\ \xi \perp \mathcal{U}_\pm T_y v_\lambda\}$$

of R^+ and

$$Q(y)^\pm_w = \{c^{-\lambda}_w c^\lambda_{\xi, v_\lambda} \mid \xi \in V(\lambda)^*,\ \xi \perp \mathcal{U}_\pm T_y v_\lambda\} \tag{3.1}$$

of R^w_0. In the second case one does not need to take span because of (2.8). The ideals (3.1) are nontrivial if and only if $y \geq w$ in the plus case and $y \leq w$ in the minus case.

Following [16] denote

$$W \overset{\mathrm{w}}{\Diamond} W = \{(y_-, y_+) \in W \times W \mid y_- \leq w \leq y_+\}.$$

Consider the induced poset structure on $W \overset{\mathrm{w}}{\Diamond} W$ from the standard Bruhat order on the first copy of W and the inverse Bruhat order on the second copy of W, i.e. for $(y_-, y_+), (y'_-, y'_+) \in W \times W$

$$(y_-, y_+) \leq (y'_-, y'_+) \text{ if and only if } y_- \leq y'_- \text{ and } y_+ \geq y'_+.$$

Theorem 3.1. *(Gorelik)* [16, Lemma 6.6, Proposition 6.11, Corollary 7.1.2]

(a) For each $(y_-, y_+) \in W \overset{\mathrm{w}}{\Diamond} W$ there exists a unique H invariant prime ideal $Q(y_-, y_+)_w$ of R^w_0 which is minimal among all H invariant prime ideals of R^w_0 containing $Q(y_-)^-_w + Q(y_+)^+_w$.

(b) The ideals in (a) are distinct and exhaust all H invariant prime ideals of R_0^w.

(c) The map $Q(y_-, y_+)_w \mapsto (y_-, y_+)$ is an isomorphism between the posets of H invariant prime ideals of R_0^w ordered under inclusion and $W \overset{w}{\Diamond} W$; that is $Q(y_-, y_+)_w \subseteq Q(y'_-, y'_+)_w$ for $(y_-, y_+), (y'_-, y'_+) \in W \overset{w}{\Diamond} W$ if and only if $y_- \leq y'_- \leq w \leq y'_+ \leq y_+$.

(d) For $y_- \in W^{\leq w}$

$$Q(y_-, w)_w = Q(y_-)^-_w + Q(w)^+_w.$$

3.2. We will construct a surjective homomorphism from R_0^w to $\mathcal{U}_-^w$ similarly to the De Concini–Procesi construction of quantum Schubert cells [7, Sect. 3]. We start with the following simple Lemma. Its proof is included for completeness.

Lemma 3.2. *Assume that H is a Hopf algebra and A is an H-module algebra with a right H action. Let $\epsilon \colon A \to \mathbb{K}$ be an algebra homomorphism, where $\mathbb{K}$ is the ground field. Then the map $\phi \colon A \to H^*$ given by*

$$\phi(a)(h) = \epsilon(a.h)$$

is an algebra homomorphism. If, in addition the action of H is locally finite, then the image of ϕ is contained in the restricted dual H° of H.

Proof. Let $a_1, a_2 \in A$. Using Sweedler's notation

$$\begin{aligned} \phi(a_1 a_2)(h) &= \epsilon((a_1 a_2).h) = \epsilon\big(\sum (a_1.h_{(1)})(a_2.h_{(2)})\big) \\ &= \sum \epsilon(a_1.h_{(1)})\epsilon(a_2.h_{(2)}) = \langle \phi(a_1) \otimes \phi(a_2), \Delta(h)\rangle = (\phi(a_1)\phi(a_2))(h). \end{aligned} \tag{3.2}$$

If the action of H is locally finite, then for all $a \in A$ the annihilator in H of the finite dimensional module $a.H$ is an ideal J of finite codimension. Since $J \subseteq \ker \phi(a)$, $\phi(a) \in H^\circ$. □

3.3.

Lemma 3.3. *The map $\epsilon_w \colon R^+ \to \mathbb{K}$ given by*

$$\epsilon_w(c^\lambda_{\xi, v_\lambda}) = \xi(T_w v_\lambda), \quad \lambda \in P_+, \xi \in V(\lambda)^* \tag{3.3}$$

is an algebra homomorphism. Moreover $\epsilon_w(c^\lambda_w) = 1$ and ϵ_w induces a homomorphism from R^w to $\mathbb{K}$ (which will be denoted by the same letter).

The fact that (3.3) defines a homomorphism is straightforward from (2.7). The equality $\epsilon_w(c^\lambda_w) = 1$ follows from the definition of c^λ_w.

Consider the right action of $\mathcal{U}_q(\mathfrak{b}_+)$ on R_0^w, cf. §2.5 and [17, §4.3.12]. Lemma 3.2 and Lemma 3.3 now imply that the map

$$\phi_w \colon R_0^w \to (\mathcal{U}_q(\mathfrak{b}_+))^*, \quad \langle \phi_w(c), x\rangle = \epsilon_w(c \leftharpoonup x), \ c \in R_0^w, x \in \mathcal{U}_q(\mathfrak{b}_+) \tag{3.4}$$

is an algebra homomorphism.

3.4. Let $w_\circ$ be the longest element of the Weyl group W of $\mathfrak{g}$. Fix a reduced expression for $w \in W$ as in (2.3). There exists a reduced expression of $w_\circ$ starting with the expression (2.3):

$$w_\circ = s_{i_1} \dots s_{i_k} s_{i_{k+1}} \dots s_{i_N}. \tag{3.5}$$

Define the roots

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \dots, \beta_N = s_{i_1} \dots s_{i_{N-1}} \alpha_{i_N} \tag{3.6}$$

of $\mathfrak{g}$ and the root vectors

$$X^\pm_{\beta_1} = X^\pm_{\alpha_{i_1}}, X^\pm_{\beta_2} = T_{s_{i_1}} X^\pm_{\alpha_{i_2}}, \dots, X^\pm_{\beta_N} = T_{s_{i_1} \dots s_{i_{N-1}}} X^\pm_{\alpha_{i_N}}. \tag{3.7}$$

of $\mathcal{U}_q(\mathfrak{g})$. For $j \leq k$, β_j and $X^\pm_{\beta_j}$ are exactly the roots and the root vectors defined by (2.4) and (2.5).

Lemma 3.4. *For all $K \in H$, $j \in \mathbb{N}$, $k < j \leq N$, $Y \in \mathcal{U}_q(\mathfrak{b}_+)$, $c \in R_0^w$*

$$\langle \phi_w(c), YK \rangle = \langle \phi_w(c), Y \rangle$$

and

$$\langle \phi_w(c), Y X^+_{\beta_j} \rangle = 0.$$

Proof. The first equality is simply the definition of R_0^w as an invariant subalgebra of R^w.

Applying [17, §4.3.12] one sees that the second equality follows from

$$\langle \xi, Y X^+_{\beta_j} T_w v_\lambda \rangle = 0, \ \forall j \geq k+1, Y \in \mathcal{U}_+, \xi \in V(\lambda)^*.$$

This in turn is proved by observing that $T_w^{-1}(X^+_{\beta_j}) = T_{s_{i_{k+1}} \dots s_{i_{j-1}}} X^+_{i_j} \in \mathcal{U}_+$ annihilates v_λ. ☐

Corollary 3.5. *For all $n_1, \dots, n_N, n \in \mathbb{N}$*

$$\begin{aligned} &\langle \phi_w(c_w^{-\lambda} c^\lambda_{\xi, v_\lambda}), (X^+_{\beta_1})^{n_1} \dots (X^+_{\beta_N})^{n_N} K^n) \rangle \\ &\quad = \langle \xi, (X^+_{\beta_1})^{n_1} \dots (X^+_{\beta_N})^{n_N} T_w v_\lambda \rangle = \delta_{n_{k+1}, \dots, n_N, 0} \langle \xi, (X^+_{\beta_1})^{n_1} \dots (X^+_{\beta_k})^{n_k} T_w v_\lambda \rangle. \end{aligned} \tag{3.8}$$

3.5. The standard bilinear form $\mathcal{U}_q(\mathfrak{b}_+) \times \mathcal{U}_q(\mathfrak{b}_-) \to \mathbb{K}$ can be used to embed $\mathcal{U}_q(\mathfrak{b}_-)$ in $(\mathcal{U}_q(\mathfrak{b}_+))^*$ (as algebras). We identify $\mathcal{U}_-^w$ with its image in $(\mathcal{U}_q(\mathfrak{b}_+))^*$.

Proposition 3.6. *The image of $\phi_w \colon R_0^w \to (\mathcal{U}_q(\mathfrak{b}_+))^*$ is $\mathcal{U}_-^w$. Its kernel is $Q(w)^+_w$.*

Proof. The inclusion $\mathrm{Im}\phi_w \subseteq \mathcal{U}_-^w$ follows from the fact that

$$\{(X^+_{\beta_1})^{n_1} \dots (X^+_{\beta_N})^{n_N} \mid n_1, \dots, n_n \in \mathbb{N}\}$$

is a PBW basis of $\mathcal{U}_+$ and Corollary 3.5. Assume that ϕ_w is not surjective. Then there exists $X \in \mathcal{U}_w^+$, $X \neq 0$ such that $\xi(XT_w v_\lambda) = 0$ for all $\lambda \in P_+$ and $\xi \in V(\lambda)^*$. Therefore $X_1 = T_w^{-1}(X) \in \mathcal{U}^-$ satisfies $c(X_1) = 0$ for all $c \in R^+$, i.e. $X_1 v_\lambda = 0$ for all $\lambda \in P_+$. It is well known that this implies $X_1 = 0$, see e.g. [17, §4.3.5-4.3.6], which contradicts with $X \neq 0$.

Let $\lambda \in P_+$, $\xi \in V(\lambda)^*$. Using Corollary 3.5 we see that $c_w^{-\lambda} c^\lambda_{\xi, v_\lambda} \in \ker \phi_w$ if and only if

$$\langle \xi, \mathcal{U}_+ T_w v_\lambda \rangle = 0,$$

i.e. $c_w^{-\lambda} c^\lambda_{\xi, v_\lambda} \in Q(w)^+_w$. Thus $\ker \phi_w = Q(w)^+_w$. ☐

3.6. Proposition 3.6 and §3.2 imply the following explicit form of the map ϕ_w, cf. Theorem 3.2 of De Concini and Procesi [7]:

Theorem 3.7. *The $\mathbb{K}$-linear map*

$$\phi_w \colon R_0^w \to \mathcal{U}_-^w, \ \ \phi_w(c_w^{-\lambda} c^\lambda_{\xi, v_\lambda}) = (c^\lambda_{\xi, T_w v_\lambda} \otimes \mathrm{id})(\mathcal{R}^w), \ \lambda \in P_+, \xi \in V(\lambda)^* \tag{3.9}$$

is a surjective homomorhism of algebras. Its kernel is $Q(w)^+_w$.

The explicit form of ϕ_w originally defined by (3.4) follows from the fact that the R-matrix $\mathcal{R}^w$ is by definition equal to a sum of the form $\sum_i Y_i \otimes Z_i$ where $\{Y_i\}$ and $\{Z_i\}$ are dual bases of $\mathcal{U}_+^w$ and $\mathcal{U}_-^w$ with respect the standard bilinear form $\mathcal{U}_q(\mathfrak{b}_+) \times \mathcal{U}_q(\mathfrak{b}_-) \to \mathbb{K}$.

3.7. Corollary 3.5 implies that $\phi_w \colon R_0^w \to \mathcal{U}_-^w$ is H-equivariant with respect to the right action of H on R_0^w (2.1) and the conjugation action of H on $\mathcal{U}_-^w$:

$$K.x = K^{-1} x K, \quad K \in H, x \in \mathcal{U}_-^w. \tag{3.10}$$

The following Theorem describes the poset of H-primes of $\mathcal{U}_-^w$.

Theorem 3.8. *Fix $w \in W$. For each $y \in W^{\leq w}$ define*

$$I_w(y) = \phi_w(Q(y)^-_w) = \{(c^\lambda_{\xi, T_w v_\lambda} \otimes \mathrm{id})(\mathcal{R}^w) \mid \lambda \in P_+, \xi \in V(\lambda)^*, \xi \perp \mathcal{U}^- T_y v_\lambda\}. \tag{3.11}$$

Then:

(a) $I_w(y)$ is an H invariant prime ideal of $\mathcal{U}_-^w$.

(b) All H invariant prime ideals of $\mathcal{U}_-^w$ are of this form.

(c) The correspondence $y \in W^{\leq w} \mapsto I_w(y)$ is an isomorphism from the poset $W^{\leq w}$ to the poset of H invariant prime ideals of $\mathcal{U}_-^w$ ordered under inclusion; that is $I_w(y) \subseteq I_w(y')$ for $y, y' \in W^{\leq w}$ if and only if $y \leq y'$.

Proof. The map ϕ_w establishes a bijection between prime ideals of $\mathcal{U}_-^w$ and prime ideals of R_0^w containing $\ker \phi_w = Q(w)^+_w$ (in order preserving way). The map ϕ_w is also equivariant with respect to the right action of H on R_0^w and the conjugation action of H on $\mathcal{U}_-^w$, cf. 3.5. Thus it provides an isomorphism between the posets of H invariant prime ideals of R_0^w containing $\ker \phi_w = Q(w)^+_w$ and the H-primes of $\mathcal{U}_-^w$. Now Theorem 3.8 follows from Gorelik's Theorem 3.1 because the only H invariant prime ideals of R_0^w that contain $Q(w)^+_w$ are $Q(y_-, w)_w$, $y_- \in W^{\geq w}$ (see Theorem 3.1 (a), (c)). □

Remark 3.9. Gorelik proved [16] that all ideals $Q(y_-, y_+)_w$ are completely prime and as a consequence one gets that all ideals $I_w(y)$ are completely prime. This is true in a greater generality for H-primes of certain iterated skew polynomial rings [14, Proposition 4.2] due to Goodearl and Letzter.

Remark 3.10. There is a natural action of the algebraic torus $\mathbb{K}^r$ on $\mathcal{U}_q(\mathfrak{g})$ by algebra automorphisms constructed by setting

$$(a_1, \ldots, a_r) \cdot X_i^\pm = a_i^{\pm 1} X_i^+, (a_1, \ldots, a_r) \cdot K_i = K_i, \quad i = 1, \ldots r.$$

The subalgebras $\mathcal{U}_\pm^w$ are invariant under it. A subset of $\mathcal{U}_q(\mathfrak{g})$ is invariant under the action of $\mathbb{K}^r$ if and only if it is invariant under the conjugation action of H. From the point of view of Goodearl–Letzter theory of H-primes it is more natural to use the action of $\mathbb{K}^r$ since this group is algebraic. Because the invariance properties under $\mathbb{K}^r$ and H are the same and the latter action is more natural within the Hopf algebra setting, we use the H action.

3.8. An equivalent way to define the algebras $\mathcal{U}_-^w$ and to work with them is by using Demazure modules. This gives an interpretation of $\mathcal{U}_-^w$ as quantized algebras of functions on Schubert cells which is similar to the De Concini–Procesi isomorphism [7, Theorem 3.2]. A notion of quantum Schubert cells in the case of Grassmannians was also defined in [24] using the algebra of quantum matrices.

Recall that the $\mathcal{U}_q(\mathfrak{b}_+)$-modules $V_w(\lambda) = \mathcal{U}_+ T_w v_\lambda = \mathcal{U}_+^w T_w v_\lambda$ are called Demazure modules, cf. [17, §4.4 and 6.3] for details. For $\eta \in V_w(\lambda)^*$ define

$$c_\eta^{w,\lambda} \in (\mathcal{U}_+)^*, \;\; c_\eta^{w,\lambda}(X) = \langle \eta, X T_w v_\lambda \rangle, \;\; X \in \mathcal{U}_+.$$

Set $U_+^w = U_+ \cap w U_- w^{-1}$. Denote by $R_q[U_+^w]$ the subset of $(\mathcal{U}_+)^*$ consisting of

$$c_\eta^{w,\lambda}, \;\; \lambda \in P_+, \eta \in V_w(\lambda)^*.$$

Consider the linear map

$$\varphi_w \colon R_0^w \to (\mathcal{U}_+)^*, \;\; \varphi_w(c_w^{-\lambda} c_\xi^\lambda) = c^{w,\lambda}_{\xi|_{V_w(\lambda)}}, \; \lambda \in P_+, \xi \in V(\lambda)^*. \tag{3.12}$$

(Because of Corollary 3.5 this is nothing but the map $\phi_w \colon R_0^w \to (\mathcal{U}_q(\mathfrak{b}_+))^*$ composed with the the linear projection $(\mathcal{U}_q(\mathfrak{b}_+))^* \to (\mathcal{U}_+)^*$.) The image of φ_w is $R_q[U_+^w]$ and its kernel is $Q(w)_w^+$ because of Corollary 3.5. In particular $R_q[U_+^w]$ is a subspace of $(\mathcal{U}_+)^*$. Since $Q(w)_w^+$ is an ideal of R_0^w one can push forward the algebra structure of R_0^w to an algebra structure on $R_q[U_+^w]$. From now on $R_q[U_+^w]$ will denote the subspace of $(\mathcal{U}_+)^*$ equipped with this algebra structure. Recall [16, §6.5] that for all $\lambda_1, \lambda_2 \in P_+$, $\xi_1 \in (V(\lambda_1)^*)_{\mu_1}$, $\xi_2 \in V(\lambda_2)^*$

$$c_w^{-\lambda_1} c^{\lambda_1}_{\xi_1, v_{\lambda_1}} c_w^{-\lambda_2} c^{\lambda_2}_{\xi_2, v_{\lambda_2}} = q^{\langle \lambda_2, \lambda_1 - w^{-1}\mu_1 \rangle} c_w^{-\lambda_1 - \lambda_2} c^{\lambda_1}_{\xi_1, v_{\lambda_1}} c^{\lambda_2}_{\xi_2, v_{\lambda_2}}, \tag{3.13}$$

see [17, §9.1] for more details on commutation relations in $R_q[G]$. Let $\eta_1 \in (V_w(\lambda_1)^*)_{\mu_1}$ and $\eta_2 \in V_w(\lambda_2)^*$. The induced algebra structure on $R_q[U_+^w]$ is given by:

$$c^{w,\lambda_1}_{\eta_1} c^{w,\lambda_2}_{\eta_2} = q^{\langle \lambda_2, \lambda_1 - w^{-1}\mu_1 \rangle} c^{w, \lambda_1 + \lambda_2}_{\eta}, \quad \text{where } \; \eta = \eta_1 \otimes \eta_2|_{\mathcal{U}_+(T_w v_{\lambda_1} \otimes T_w v_{\lambda_2})} \in V_w(\lambda_1 + \lambda_2)^*. \tag{3.14}$$

Here we use that $\mathcal{U}_+(T_w v_{\lambda_1} \otimes T_w v_{\lambda_2}) \subset \mathcal{U}_+ T_w v_{\lambda_1} \otimes \mathcal{U}_+ T_w v_{\lambda_2}$. Note that it is not a priori obvious from (3.14) that this is a well defined multiplication and that it is associative. The unit in $R_q[U_+^w]$ is equal to $c^{w,\lambda}_{\eta_{w,\lambda}}$ where $\eta_{w,\lambda} = \xi_{w,\lambda}|_{V_w(\lambda)}$, $\lambda \in P_+$, recall §2.5. (The elements $c^{w,\lambda}_{\eta_{w,\lambda}} \in (\mathcal{U}_+)^*$ are all equal to each other.)

Define the linear map

$$\psi_w \colon R_q[U_+^w] \to \mathcal{U}_-^w, \quad \psi_w(c_\eta^{w,\lambda}) = (c_\eta^{w,\lambda} \otimes \mathrm{id})(\mathcal{R}^w), \; \lambda \in P_+, \eta \in V_w(\lambda)^*. \tag{3.15}$$

Corollary 3.5 and Theorem 3.7 imply that $\psi_w \colon R_q[U_+^w] \to \mathcal{U}_-^w$ is an algebra isomorphism.

Recall that $\phi_w \colon R_0^w \to \mathcal{U}_-^w$ is H-equivariant with respect to the right action of H on R_0^w (2.1) and the conjugation action of H on $\mathcal{U}_-^w$ (3.10).

From the definition of φ_w one obtains that $\varphi_w \colon R_0^w \to R_q[U_+^w]$ is H-equivariant with respect to the right action of H on R_0^w and the restriction to $R_q[U_+^w]$ of the following action of H on $\mathcal{U}_+$

$$K.c = K^{-1} \rightharpoonup c \leftharpoonup K, \langle K^{-1} \rightharpoonup c \leftharpoonup K, X \rangle = c(KXK^{-1}) \tag{3.16}$$

$K \in H, c \in (\mathcal{U}_+)^*, X \in \mathcal{U}_+$. Finally, the isomorphism $\psi_w \colon R_q[U_+^w] \to \mathcal{U}_-^w$ is H-equivariant with respect to the action (3.16) and the conjugation action of H on $\mathcal{U}_-^w$ (3.10).

Theorem 3.11. *(1) The homomorphism $\phi_w \colon R_0^w \to \mathcal{U}_-^w$ factors through the surjective homomorphism* (3.12) *$\varphi_w \colon R_0^w \to R_q[U_+^w]$ and the isomorphism* (3.15) *$\psi_w \colon R_q[U_+^w] \to \mathcal{U}_-^w$. Both maps are H-equivariant.*

(2) Under the isomorphism $\psi_w \colon R_q[U_+^w] \to \mathcal{U}_-^w$ the ideals $I_w(y)$, $y \in W^{\leq w}$ of $\mathcal{U}_-^w$ correspond to the H invariant prime ideals

$$J_w(y) = \{c_\eta^{w,\lambda} \mid \lambda \in P_+, \eta \in (V_w(\lambda) \cap \mathcal{U}_- T_y v_\lambda)^\perp\}$$

of $R_q[U_+^w]$.

(3) The ideals $I_w(y)$, $y \in W^{\leq w}$ of $\mathcal{U}_-^w$ are also given by

$$I_w(y) = \{(c_\eta^{w,\lambda} \otimes \mathrm{id})(\mathcal{R}^w) \mid \lambda \in P_+, \eta \in (V_w(\lambda) \cap \mathcal{U}_- T_y v_\lambda)^\perp\}.$$

Part (1) has already been established. Recall that

$$I_w(y) = \phi_w(Q(y)_w^- + Q(w)_w^+).$$

Part (2) is a direct computation of $J_w(y) := \varphi_w(Q(y)_w^-)$. Part (3) follows from the fact that $I_w(y) = \psi_w(J_w(y))$.

3.9. In this subsection, based on results of Joseph [18], we construct generating sets of the H-primes $I_w(y)$ of $\mathcal{U}_-^w$. Denote the subalgebra

$$R^- = \{c^\lambda_{\xi, T_{w_\circ} v_\lambda} \mid \lambda \in P_+, \xi \in V(\lambda)^*\}$$

of $R_q[G]$. Recall §2.2 that $\omega_1, \ldots, \omega_l$ denote the fundamental weights of $\mathfrak{g}$.

Theorem 3.12. *(Joseph)* [18, Théorème 3] *For all $w \in W$*

$$\mathrm{Span}\{c^\lambda_{\xi, v_\lambda} \mid \lambda \in P_+, \xi \in (\mathcal{U}_+ T_w v_\lambda)^\perp\} = \sum_{i=1}^r \{c^\lambda_{\xi, v_{\omega_i}} \mid \xi \in (\mathcal{U}_+ T_w v_{\omega_i})^\perp\} R^+ \tag{3.17}$$

and

$$\begin{aligned} &\mathrm{Span}\{c^\lambda_{\xi, T_{w_\circ} v_\lambda} \mid \lambda \in P_+, \xi \in (\mathcal{U}_- T_w v_\lambda)^\perp\} \\ &\qquad = \sum_{i=1}^r \{c^\lambda_{\xi, T_{w_\circ} v_{\omega_i}} \mid \xi \in (\mathcal{U}_- T_w v_{\omega_i})^\perp\} R^-. \end{aligned} \tag{3.18}$$

The left hand sides of (3.17)–(3.18) are H invariant prime ideals of $R^\pm$ and the right hand sides give efficient generating sets of them as right ideals.

Using (2.7), one sees that the map

$$c^\lambda_{\xi, T_{w_\circ} v_\lambda} \in R^- \mapsto c^\lambda_{\xi, v_\lambda} \in R^+$$

is an isomorphism of algebras. Thus (3.18) implies

$$\mathrm{Span}\{c^{\lambda}_{\xi,v_\lambda} \mid \lambda \in P_+, \xi \in (\mathcal{U}_- T_w v_\lambda)^\perp\} = \sum_{i=1}^{r} \{c^{\lambda}_{\xi,v_{\omega_i}} \mid \xi \in (\mathcal{U}_- T_w v_{\omega_i})^\perp\} R^+.$$

Therefore for all $y \in W^{\leq w}$:

$$\begin{aligned} Q(y)^-_w &= \{c^{\lambda}_{\xi,v_\lambda} c^{-\lambda}_w \mid \lambda \in P_+, \xi \in (\mathcal{U}_- T_y v_\lambda)^\perp\} \\ &\qquad \subset \sum_{i=1}^{r} \{c^{\omega_i}_{\xi,v_{\omega_i}} c^{-\omega_i}_w \mid \xi \in (\mathcal{U}_- T_y v_{\omega_i})^\perp\} R^+[c_w^{-1}] \end{aligned}$$

Using the left action of H on $R^+[c_w^{-1}]$ (2.1) and the fact that $R^+[c_w^{-1}]$ is a semisimple H-module we obtain

$$Q(y)^-_w = \sum_{i=1}^{r} \{c^{\omega_i}_{\xi,v_{\omega_i}} c^{-\omega_i}_w \mid \xi \in (\mathcal{U}_- T_y v_{\omega_i})^\perp\} R^w_0.$$

Using (3.13) we see that the ideals $I_w(y)$ of $\mathcal{U}^w_-$ (and $J_w(y)$ of $R_q[U^w_+]$) are generated as right ideals by the subsets in Theorem 3.8 and Theorem 3.11 corresponding to the fundamental weights $\omega_1, \ldots, \omega_l$.

Theorem 3.13. *For all $w \in W$ and $y \in W^{\leq w}$:*

$$I_w(y) = \sum_{i=1}^{r} \{(c^{\omega_i}_{\xi, T_w v_{\omega_i}} \otimes \mathrm{id})(\mathcal{R}^w) \mid \xi \in (\mathcal{U}^- T_y v_{\omega_i})^\perp\} \mathcal{U}^w_-$$

and

$$J_w(y) = \sum_{i=1}^{r} \{c^{w,\omega_i}_{\eta} \mid \eta \in (V_w(\omega_i) \cap \mathcal{U}_- T_y v_{\omega_i})^\perp\} R_q[U^w_+].$$

For both generating sets one can restrict to root vectors ξ and η.

4. Results for the underlying Poisson structures

In this section we prove results for the underlying Poisson structures for the algebras $\mathcal{U}^w_-$ which are Poisson analogs of Theorems 3.8, 3.11 and 3.13.

4.1. Let G be a simply connected complex semisimple Lie group and $\mathfrak{g} = \mathrm{Lie}\, G$. Fix a pair of opposite Borel subgroups $B_\pm$. Let $T = B_+ \cap B_-$ be the corresponding maximal torus of G, and $U_\pm$ be unipotent radicals of $B_\pm$. Let W be the Weyl group of G. For all $w \in W$ fix representatives $\dot{w}$ in $N(T)/T$. Here $N(T)$ denotes the normalizer of T in G.

Denote by Δ_+ the set of positive roots of $\mathfrak{g}$. Fix root vectors $x^\pm_\alpha \in \mathfrak{g}^{\pm\alpha}$, $\alpha \in \Delta_+$, normalized by

$$\langle x^+_\alpha, x^-_\alpha \rangle = 1$$

where $\langle .,. \rangle$ denotes the Killing form on $\mathfrak{g}$. Define the bivector field

$$\pi = - \sum_{\alpha \in \Delta_+} \chi(x^+_\alpha) \wedge \chi(x^-_\alpha)$$

on the flag variety G/B_+. Here $\chi \colon \mathfrak{g} \to Vect(G/B_+)$ refers to the infinitesimal action of $\mathfrak{g}$ on G/B_+. It is well known that π is a Poisson structure on G/B_+, see e.g. [15] for details. The group T acts on $(G/B_+, \pi)$ by Poisson automorphisms.

For $y_-, y_+ \in W$ define

$$R_{y_-,y_+} = B_- y_- \cdot B_+ \cap B_+ y_+ \cdot B_+ \subset G/B_+.$$

This intersection is nontrivial if and only if $y_- \leq y_+$ in which case it is irreducible [8].

The following Proposition follows from [10, Theorem 4.14] of Evens and Lu, and [15, Theorem 0.4] of Goodearl and the author.

Proposition 4.1. *The T-orbits of symplectic leaves of $(G/B_+, \pi)$ are precisely the intersections R_{y_-,y_+}, for $y_\pm \in W$, $y_- \leq y_+$.*

The closure relation between symplectic leaves is described by the well known fact that:

$$\overline{R_{y_-,y_+}} = \bigsqcup \{ R_{y'_-,y'_+} \mid y'_\pm \in W, y_- \leq y'_- \leq y'_+ \leq y_+ \}.$$

4.2. From now on we fix an element $w \in W$. The Schubert (Bruhat) cell translate $wB_- \cdot B_+ \subset G/B_+$ is an open subset of G/B_+ and is thus a Poisson variety with the restriction of π.

Lemma 4.2. *For $y_\pm \in W$ the intersection $wB_- \cdot B_+ \cap R_{y_-,y_+} \subset G/B_+$ is nonempty if and only if $y_- \leq w \leq y_+$. In the case when it is nontrivial, it is a dense subset of R_{y_-,y_+}.*

Proof. The second statement holds because $wB_- \cdot B_+$ is a Zariski open subset of G/B_+ and R_{y_-,y_+} are all irreducible.

Assume that $wB_- \cdot B_+ \cap R_{y_-,y_+}$ is nonempty. Then

$$\begin{aligned} (4.1) \quad wB_-B_+ \cap B_-y_-B_+ \neq \emptyset \Rightarrow wB_- \cap B_-y_-B_+ \neq \emptyset \Rightarrow \\ B_-wB_- \cap B_-y_-B_+ \neq \emptyset \Rightarrow w \geq y_-. \end{aligned}$$

Analogously $wB_-B_+ \cap B_+y_+B_+ \neq \emptyset$ implies $w \leq y_+$. Therefore, if $wB_- \cdot B_+ \cap R_{y_-,y_+}$ is nonempty, then $y_- \leq w \leq y_+$.

Now assume that that $y_- \leq w \leq y_+$. Analogously we get $wB_-B_+ \cap B_+y_+B_+ \neq \emptyset$ and $wB_-B_+ \cap B_-y_-B_+ \neq \emptyset$. Let $wb_- \in wB_-B_+ \cap B_+y_+B_+$ for some $b_- \in B_-$. Since $wB_-B_+ \cap B_+y_+B_+$ is invariant under the left action of $B_+ \cap wB_-w^{-1}$

$$wB_-B_+ \cap B_+y_+B_+ \supset w(B_- \cap w^{-1}B_+w)b_-. \tag{4.2}$$

Because $wB_-B_+ \cap B_-y_-B_+ \neq \emptyset$, $wB_- \cap B_-y_-B_+ \neq \emptyset$. The latter set is invariant under the left action of $B_- \cap wB_-w^{-1}$ and $B_- = (B_- \cap w^{-1}B_-w)(B_- \cap w^{-1}B_+w)$, thus $wB_- \cap B_-y_-B_+ \neq \emptyset$ implies

$$\big(w(B_- \cap w^{-1}B_+w)b_-\big) \cap B_-y_-B_+ \neq \emptyset. \tag{4.3}$$

Then (4.2) and (4.3) imply

$$B_-y_-B_+ \cap wB_-B_+ \cap B_+y_+B_+ \neq \emptyset.$$

□

Proposition 4.3. *(1) The T-orbits of symplectic leaves of $(wB_- \cdot B_+, \pi)$ are the intersections*

$$S_w(y_-, y_+) = wB_- \cdot B_+ \cap R_{y_-,y_+} \quad \textit{for } (y_1, y_2) \in W \overset{\mathrm{w}}{\Diamond} W.$$

Their Zariski closures are given by

$$\overline{S_w(y_-,y_+)} = \bigsqcup \{S_w(y'_-,y'_+) \mid y'_\pm \in W, y_- \leq y'_- \leq w \leq y'_+ \leq y_+\}.$$

(2) Let $(y_-,y_+) \in W \overset{w}{\Diamond} W$*. For each symplectic leaf* $\mathcal{S}$ *of* R_{y_-,y_+} *the intersection* $\mathcal{S} \cap wB_- \cdot B_+$ *is nontrivial and is a symplectic leaf of* $S_w(y_-,y_+)$*. All symplectic leaves of* $S_w(y_-,y_+)$ *are obtained in this way.*

Proof. Let $\mathcal{S}$ be a symplectic leaf of R_{y_-,y_+} and $(y_-,y_+) \in W \overset{w}{\Diamond} W$. Proposition 4.1 implies that $R_{y_-,y_+} = T \cdot \mathcal{S}$. Since the intersection $wB_- \cdot B_+ \cap R_{y_-,y_+}$ is nonempty there exists $t \in T$ such that $wB_- \cdot B_+ \cap t\mathcal{S} \neq \emptyset$. But $wB_- \cap B_+$ is T-stable, so $wB_- \cdot B_+ \cap \mathcal{S} \neq \emptyset$. The complement of $wB_- \cdot B_+ \cap \mathcal{S}$ in $\mathcal{S}$ has real codimension at least 2. Thus $wB_- \cdot B_+ \cap \mathcal{S}$ is connected and is a symplectic leaf of $wB_- \cdot B_+$. Obviously all symplectic leaves of $wB_- \cdot B_+$ are obtained in this way, which completes the proof of (2). It is clear that

$$S_w(y_-,y_+) = T \cdot (wB_- \cdot B_+ \cap \mathcal{S}).$$

This implies (1). □

4.3. Denote

$$U_+^w = U_+ \cap wU_-w^{-1} \text{ and } \mathfrak{n}_+^w = \operatorname{Lie} U_+^w. \tag{4.4}$$

Identify

$$i_w \colon U_+^w \cong B_+w \cdot B_+ \subset G/B_+, \quad i_w(u) = uw \cdot B_+,\ u \in U_+^w. \tag{4.5}$$

Observe that $B_+w \cdot B_+ = U_+^w w \cdot B_+ = w(w^{-1}U_+^w w) \cdot B_+$ lies inside $wB_- \cdot B_+$. Proposition 4.1 implies that $B_+w \cdot B_+$ is a complete Poisson (locally closed) subset of G/B_+. Denote the Poisson structure

$$\pi_w = i_w^{-1}(\pi|_{B_+w \cdot B_+})$$

on U_+^w.

Consider the conjugation action of T on U_+^w. It preserves π_w since i_w intertwines it with the canonical left action of T on G/B_+.

Corollary 4.4. *The* T*-orbits of symplectic leaves of* (U_+^w, π_w) *are parametrized by* $y \in W^{\leq w}$*:*

$$y \in W^{\leq w} \mapsto S_w(y) := i_w^{-1}(R_{y,w}) = U_+^w \cap B_-yB_+w^{-1}. \tag{4.6}$$

Moreover

$$\overline{S_w(y)} = \bigsqcup \{S_w(y') \mid y' \in W^{\leq w}, y' \geq y\}.$$

In particular, (4.6) *is an isomorphism of posets from* $W^{\leq w}$ *with the inverse Bruhat order to the underlying poset of the stratification of* (U_+^w, π_w) *into* T*-orbits of symplectic leaves.*

The Corollary follows from Proposition 4.1 since i_w is T-equivariant and $B_+w \cdot B_+$ is a complete Poisson (locally closed) subset of $wB_- \cdot B_+$.

4.4. The irreducible finite dimensional representations of G are parametrized by its set of positive dominant weights P_+. Denote by $L(\lambda)$ the corresponding G-module. Let $d^\lambda_{\zeta,u} \in \mathbb{C}[G]$ be the the matrix coefficient corresponding to $\zeta \in L^*(\lambda)$ and $u \in L(\lambda)$. Then

$$\mathbb{C}[G] = \mathrm{Span}\{d^\lambda_{\zeta,u} \mid \lambda \in P_+, u \in L(\lambda), \zeta \in L(\lambda)^*\}.$$

We will denote the root spaces of a G-module M by M_μ. For each $\lambda \in P_+$ fix a highest weight vector u_λ of $L(\lambda)$ and a dual vector $\zeta_\lambda \in L^*(\lambda)_{-\lambda}$ normalized by $\langle \zeta_\lambda, u_\lambda \rangle = 1$. Denote

$$d^\lambda_w = d^\lambda_{\dot{w}\zeta_\lambda, u_\lambda} \text{ and } d_w = \{d^\lambda_w \mid \lambda \in P_+\}.$$

Then for $wB_-B_+ \subset G$

$$\mathbb{C}[wB_-B_+] = \mathbb{C}[G][d_w^{-1}].$$

Identify

$$\mathbb{C}[wB_- \cdot B_+] \cong \mathbb{C}[wB_-B_+]^{B_+}, \tag{4.7}$$

where $(.)^{B_+}$ refers to the ring of invariant functions with respect to the right action of B_+ on G. One verifies that under the isomorphism (4.7)

$$\mathbb{C}[wB_- \cdot B_+] = \{d^\lambda_{\zeta,u_\lambda}/d^\lambda_w \mid \lambda \in P_+, \zeta \in L(\lambda)^*\}. \tag{4.8}$$

Analogously to (2.8) one does not need to take span in (4.8).

4.5. Denote $\mathfrak{n}_\pm = \mathrm{Lie}\, U_\pm$. For $y \in W$, define the ideals

$$\widetilde{Q}(y)^\pm_w = \{d^\lambda_{\zeta,u_\lambda}/d^\lambda_w \mid \lambda \in P_+, \xi \in (\mathcal{U}(\mathfrak{n}_\pm) y v_\lambda)^\perp \subset L(\lambda)^*\}$$

of $\mathbb{C}[wB_- \cdot B_+]$.

Proposition 4.5. *The vanishing ideal of the Zariski closure of $S_w(y,w)$ in $wB_- \cdot B_+$ is*

$$\begin{aligned}\mathcal{V}\left(Cl_{wB_-\cdot B_+}(S_w(y,w))\right) &= \widetilde{Q}(y)^-_w + \widetilde{Q}(w)^+_w \\ &= \{d^\lambda_{\zeta,u_\lambda}/d^\lambda_w \mid \lambda \in P_+, \zeta \in (\mathcal{U}(\mathfrak{n}_-) y v_\lambda \cap \mathcal{U}(\mathfrak{n}_+) w v_\lambda)^\perp \subset L(\lambda)^*\}.\end{aligned}$$

Proof. The ideal $\widetilde{Q}(y)^-_w$ is the vanishing ideal of $Cl_{wB_-\cdot B_+}(wB_- \cdot B_+ \cap B_- y \cdot B_+)$, in particular it is prime. Indeed

$$d^\lambda_{\zeta,u_\lambda}/d^\lambda_w \in \mathcal{V}\left(Cl_{wB_-\cdot B_+}(wB_- \cdot B_+ \cap B_- y \cdot B_+)\right)$$

if and only if $\langle \zeta, (wB_-B_+ \cap B_-yB_+u_\lambda)\rangle$ which is equivalent to $\langle \zeta, B_-yB_+u_\lambda\rangle$ and to $\zeta \in (\overline{\mathcal{U}}_- u_\lambda)^\perp$ since $wB_-B_+ \cap B_-yB_+$ is dense in B_-yB_+. Analogously, in §4.6 we verify that $\widetilde{Q}(w)^+_w$ is the vanishing ideal of $B_+w \cdot B_+$ in $\mathbb{C}[wB_- \cdot B_+]$.

Ramanathan proved [26, Corollary 1.10 and Theorem 3.5] that the scheme theoretic intersection of the opposite Schubert varieties $\overline{B_+w \cdot B_+}$ and $\overline{B_-y \cdot B_+}$ in G/B_+ is reduced. Therefore the same is true for the scheme theoretic intersection of $B_+w \cdot B_+ = wB_- \cdot B_+ \cap \overline{B_+w \cdot B_+}$ and $Cl_{wB_-\cdot B_+}(wB_- \cdot B_+ \cap B_-y \cdot B_+) = wB_- \cdot B_+ \cap \overline{B_-y \cdot B_+}$. This implies the statement of the Proposition. $\square$

4.6. For $\lambda \in P_+$ consider the $\mathcal{U}(\mathfrak{b}_+)$ submodules $L_w(\lambda) = \mathcal{U}(\mathfrak{b}_+)\dot{w}v_\lambda = \mathcal{U}(\mathfrak{n}_+^w)\dot{w}v_\lambda$ of $L(\lambda)$ (cf. (4.4)) called Demazure modules, where $\mathfrak{b}_\pm = \operatorname{Lie} B_\pm$.

Each $\eta \in L_w(\lambda)^*$ gives rise to a regular function $d_\eta^{w,\lambda}$ on U_+^w, $d_\eta^{w,\lambda}(u) = \langle \eta, u\dot{w}\zeta_\lambda\rangle$, $u \in U_+^w$. One has

$$\mathbb{C}[U_+^w] = \{d_\eta^{w,\lambda} \mid \lambda \in P_+, \eta \in L_w(\lambda)^*\}. \tag{4.9}$$

Let us trace back (4.9) to (4.8). The composition of the isomorphism $i_w \colon U_+^w \cong B_+w \cdot B_+$ and the embedding $B_+w \cdot B_+ \hookrightarrow wB_- \cdot B_+$ give rise to the embedding

$$j_w \colon U_+^w \hookrightarrow wB_- \cdot B_+, \quad j_w(u) = uwB_+, u \in U_+^w. \tag{4.10}$$

In terms of (4.8) and (4.9) j_w^* is given by

$$j_w^*(d_{\zeta,u_\lambda}^\lambda / d_w^\lambda) = d^{w,\lambda}_{\zeta|_{L_w(\lambda)}}, \quad \lambda \in P_+, \zeta \in L(\lambda)^*. \tag{4.11}$$

In particular, the kernel of j_w^* is

$$\begin{aligned} \ker j_w^* = \mathcal{V}(B_+w \cdot B_+) = \mathcal{V}(\overline{S_w(1,w)}) \\ = \{d_{\zeta,u_\lambda}^\lambda / d_w^\lambda \mid \lambda \in P_+, \zeta \in (\mathcal{U}(\mathfrak{n}_+)wv_\lambda)^\perp \subset L(\lambda)^*\}, \end{aligned} \tag{4.12}$$

cf. Proposition 4.3 and Proposition 4.5.

Theorem 4.6. *For all $y \in W^{\leq w}$ the vanishing ideal of the Zariski closure of the symplectic leaf $S_w(y)$ in (U_+^w, π_w) is*

$$\mathcal{V}(\overline{S_w(y)}) = \{d_\eta^{w,\lambda} \mid \eta \in (L_w(\lambda) \cap \mathcal{U}(\mathfrak{n}_-)yu_\lambda)^\perp \subset L_w(\lambda)^*\}.$$

Proof. Clearly $j_w(\overline{S_w(y)}) = \overline{S_w(y,w)}$, cf. Proposition 4.3 and Corollary 4.4 (equivalently one can use that j_w is closed). Thus $\mathcal{V}(\overline{S_w(y)}) = j_w^*(\mathcal{V}(\overline{S_w(y,w)}))$ and the Theorem follows from Proposition 4.5 and (4.11). □

We complete this subsection with a proof of the fact that for $y \in W^{\leq w}$ the ideal $\widetilde{Q}(y,w)_w = \widetilde{Q}(y)_w^- + \widetilde{Q}(w)_w^+$ is prime.

4.7. The following Theorem is a Poisson analog of Theorem 3.13.

Theorem 4.7. *For all $y \in W^{\leq w}$ the vanishing ideal of the Zariski closure of the T-orbit of symplectic leaves $S_w(y)$ in (U_+^w, π_w) is generated by d_η^{w,ω_i} where $i = 1, \dots, r$ and $\eta \in (L_w(\omega_i) \cap \mathcal{U}(\mathfrak{n}_-)yu_{\omega_i})^\perp \subset L_w(\omega_i)^*$ is a root vector.*

Proof. Consider the algebra $\mathbb{C}[G]^{U_+}$ of right U^+-invariant functions on G. It is spanned by the matrix coefficients $d_{\zeta,u_\lambda}^\lambda$, $\lambda \in P_+$, $\zeta \in L(\lambda)^*$. Kempf and Ramanathan proved [19, Theorem 3(i)] that Schubert varieties are linearly defined. This implies that for all $y \in W$

$$\operatorname{Span}\{d_{\zeta,u_\lambda}^\lambda \mid \lambda \in P_+, \zeta \in (\mathcal{U}(\mathfrak{n}_-)\dot{y}u_\lambda)^\perp\} = \sum_{i=1}^r \{d_{\zeta,u_{\omega_i}}^\lambda \mid \zeta \in (\mathcal{U}(\mathfrak{n}_-)\dot{y}u_{\omega_i})^\perp\}\mathbb{C}[G]^{U_+}.$$

Then inside $\mathbb{C}[wB_- \cdot B_+]$ one has

$$\begin{aligned} \operatorname{Span}\{d_{\zeta,u_\lambda}^\lambda / d_w^\lambda \mid \lambda \in P_+, \zeta \in (\mathcal{U}(\mathfrak{n}_-)\dot{y}u_\lambda)^\perp\} \\ = \sum_{i=1}^r \{d_{\zeta,u_{\omega_i}}^\lambda / d_w^{\omega_1} \mid \zeta \in (\mathcal{U}(\mathfrak{n}_-)\dot{y}u_{\omega_i})^\perp\}\mathbb{C}[wB_- \cdot B_+], \end{aligned}$$

recall (4.8). Now the Theorem follows from $\mathcal{V}(\overline{S_w(y)}) = j_w^*(\mathcal{V}(\overline{S_w(y,w)}))$ and Proposition 4.5. □

5. Quantum matrices

5.1. Throughout this section we fix two positive integers m and n. Let $G = SL_{m+n}(\mathbb{C})$ and $B_\pm$ be its standard Borel subgroups.

Denote by w°_{m+n} the longest element of S_{m+n}. For each $k \leq m+n$ denote by w°_k and $w^{\circ r}_k$ the longest elements of $S(\{1,\dots,k\}) \subseteq S_{m+n}$ and $S(\{m+n-k+1,\dots,m+n\}) \subseteq S_{m+n}$, respectively.

Denote the Coxeter element $c = (12\dots m+n) \in S_{m+n}$. Then

$$c^m = w^\circ_m w^{\circ r}_n w^\circ_{m+n}. \tag{5.1}$$

In §5.1-5.3 we will apply the results of the previous Section to the case $G = SL_{m+n}(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_{m+n}(\mathbb{C})$ and $w = c^m$. All notation $L(\omega_k)$, $L_w(\omega_k)$, $\mathfrak{n}_+$, U^w_+, π_w will refer to this case.

For two integers $k \leq l$ set $\overline{k,l} = \{k,\dots,l\}$.

5.2. The matrix affine Poisson space is the complex affine space $M_{m,n}$ consisting of rectangular matrices of size $m \times n$ equipped with the quadratic Poisson structure

$$\pi_{m,n} = \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} (\operatorname{sign}(k-i) + \operatorname{sign}(l-j)) x_{il} x_{kj} \frac{\partial}{\partial x_{ij}} \wedge \frac{\partial}{\partial x_{kl}}, \tag{5.2}$$

where x_{ij} are the standard coordinate functions on $M_{m,n}$.

One has, cf. [3, Proposition 3.4], [9, (3.11)], [15, Proposition 1.6]:

Proposition 5.1. *The map $f\colon (M_{m,n}, \pi_{m,n}) \to (U^{c^m}_+, \pi_{c^m})$ given by*

$$f(x) = \begin{pmatrix} I_m & w^\circ_m x \\ 0 & I_n \end{pmatrix}$$

is an isomorphism of Poisson varieties, where $U^{c^m}_+ \subset SL_{m+n}(\mathbb{C})$ is given by (4.4)

Here, for $w \in S_m$ we denote by the same letter the corresponding permutation matrix in $GL_m(\mathbb{C})$.

Define the torus $T := \mathbb{C}^{m+n-1}$ and view it as pairs of diagonal matrices (A,B) of size $m \times m$ and $n \times n$ with $\det(A)\det(B) = 1$. It acts on $M_{m,n}$ by $(A,B)\cdot X = AXB^{-1}$, $X \in M_{m,n}$. The Poisson structure $\pi_{m,n}$ is invariant under the action of T and f intertwines it with the conjugation action of the standard torus of $SL_{m+n}(\mathbb{C})$ on $U^{c^m}_+$, see §4.3. For $y \in S^{\leq c^m}_{m+n}$ denote

$$S(y) = f^{-1}(B_- y B_+ c^{-m}),$$

where $B_\pm$ refer to the standard Borel subgroups of $SL_{m+n}(\mathbb{C})$.

Corollary 5.2. [3, Theorem A] *The T-orbits of symplectic leaves of $(M_{m,n}, \pi_{m,n})$ are $S(y)$, $y \in S^{\leq c^m}_{m+n}$. Their Zariski closures are given by*

$$\overline{S(y)} = \bigsqcup_{y \leq y' \leq c^m} S(y').$$

5.3. Denote by L the vector representation of $SL_{m+n}(\mathbb{C})$ with standard basis $\{u_1, \ldots, u_{m+n}\}$ (such that $E_{ij}u_q = \delta_{jq}u_i$). The fundamental representations of $SL_{m+n}(\mathbb{C})$ are $L(\omega_k) \cong \wedge^k L$, $k = 1, \ldots, m+n-1$. They have bases

$$u_I = u_{i_1} \wedge \ldots \wedge u_{i_k}, \quad I = \{i_1 < \cdots < i_k\} \subset \overline{1, m+n}.$$

For a subset $I \subseteq \overline{1, m+n}$ denote

$$p_1(I) = I \cap \overline{1,m} \ \text{ and } \ p_2(I) = I \cap \overline{m+1, m+n}. \tag{5.3}$$

Consider the partial order on $\{I \subseteq \overline{1,m} \mid |I| = k\}$: for $I = \{i_1 < \cdots < i_k\}$, $J = \{j_1 < \cdots < j_k\} \subseteq \overline{1, m+n}$, $I \leq J$ if $i_l \leq j_l$ for all $l = 1, \ldots, k$.

For $J_1 \subseteq \overline{1,m}$, $J_2 \subseteq \overline{1,n}$, $|J_1| = |J_2|$ denote by $\Delta_{J_1, J_2}(x)$ the corresponding minor of $x \in M_{m,n}$.

Let $I \subset \overline{1, m+n}$. If $k \in \overline{1,n}$, then $I \leq c^m(\overline{1,k}) = \overline{m+1, m+k}$ implies $p_2(I) \subseteq \overline{m+1, m+k}$. If $k \in \overline{n+1, m+n-1}$, then $I \leq c^m(\overline{1,k}) = \overline{1, k-n} \sqcup \overline{m+1, m+n}$ implies $p_1(I) \supseteq \overline{1, k-n}$. For $y \in S_{m+n}^{\leq c^m}$ let $\mathcal{A}(y)$ be the union of the sets of minors

$$\Delta_{w_m^\circ(p_1(I)), (\overline{m+1, m+k} \backslash p_2(I)) - m} \tag{5.4}$$

for $k \in \overline{1,n}$, $I \subseteq \overline{1, m+n}$, $|I| = k$, $I \leq c^m(\overline{1,k})$, $I \not\geq y(\overline{1,k})$ and

$$\Delta_{w_m^\circ(p_1(I) \backslash \overline{1, k-n}), (\overline{m+1, m+n} \backslash p_2(I)) - m} \tag{5.5}$$

for $k \in \overline{n+1, m+n-1}$, $I \subset \overline{1, m+n}$, $|I| = k$, $I \leq c^m(\overline{1,k})$, $I \not\geq y(\overline{1,k})$. In (5.4)–(5.5) $-m$ means subtracting m from each element of the set. Both sets of minors (5.4)–(5.5) can be uniformly described by the less explicit formula $\Delta_{p_1(I) \backslash p_1(c^m(\overline{1,k})), (p_2(c^m(\overline{1,k})) \backslash p_2(I)) - m}$, $k \in \overline{1, m+n-1}$.

Theorem 5.3. *For all $y \in S_{m+n}^{\leq c^m}$ the vanishing ideal of the Zariski closure of the T-orbit of symplectic leaves $S(y)$ in $(M_{m,n}, \pi_{m,n})$ is generated by the minors in $\mathcal{A}(y) \subset \mathbb{C}[M_{m,n}]$.*

Functions cutting the closures $\overline{S(y)}$ were previously obtained by Brown, Goodearl and the author in [3, Theorem 4.2]. Goodearl, Launois and Lenagan [11] independently find all minors that belong to the vanishing ideal of the Zariski closure of any T-orbit of symplectic leaves $S(y)$ in $(M_{m,n}, \pi_{m,n})$.

Proof. Observe that the Demazure module $L_{c^m}(\omega_k)$ is given by

$$L_{c^m}(\omega_k) = \operatorname{Span}\{u_I \mid I \subset \overline{1, m+n}, |I| = k, I \leq c^m(\overline{1,k})\}.$$

Denote by $\{\zeta_I \mid I \subset \overline{1, m+n}, |I| = k\} \subset L(\omega_k)^*$ the dual basis to $\{u_I\}$. We can identify

$$L_{c^m}(\omega_k)^* \cong \operatorname{Span}\{\zeta_I \mid I \subset \overline{1, m+n}, |I| = k, I \leq c^m(\overline{1,k})\} \subset L(\omega_k)^*. \tag{5.6}$$

Then

$$L_{c^m}(\omega_k) \cap \mathcal{U}(\mathfrak{n}_-)\dot{y}u_{\overline{1,k}} = \operatorname{Span}\{u_I \mid I \subseteq \overline{1, m+n}, |I| = k, y(\overline{1,k}) \leq I \leq c^m(\overline{1,k})\},$$

where $\mathfrak{n}_-$ denotes the nilpotent subalgebra of $\mathfrak{sl}_{m+n}$ consisting of lower triangular matrices. Under the identification (5.6) the orthogonal complement $(L_{c^m}(\omega_k) \cap \widetilde{\mathcal{U}}_- \dot{y}u_{\overline{1,k}})^\perp$ in $L_{c^m}(\omega_k)^*$ is

$$(L_{c^m}(\omega_k) \cap \widetilde{\mathcal{U}}_- \dot{y}u_{\overline{1,k}})^\perp = \operatorname{Span}\{\zeta_I \mid I \subseteq \overline{1, m+n}, |I| = k, I \leq c^m(\overline{1,k}), I \not\geq y(\overline{1,k})\}.$$

Theorem 4.7 implies that the vanishing ideal of the Zariski closure of $S(y)$ in $M_{m,n}$ is generated by $d_{\zeta_I}^{c^m,\omega_k}(f(x))$, $k \in \overline{1, m+n-1}$, $I \subset \overline{1, m+n}$, $|I| = k$, $I \leq c^m(\overline{1,k}), I \not\geq y(\overline{1,k})$. It is straighforward to check that

$$d_{\zeta_I}^{c^m,\omega_k}(f(x)) = \begin{cases} \Delta_{w_m^\circ(p_1(I)),(\overline{m+1,m+k}\backslash p_2(I))-m}(x), & \text{if } 1 \leq k \leq n \\ \Delta_{w_m^\circ(p_1(I)\backslash\overline{1,k-n}),(\overline{m+1,m+n}\backslash p_2(I))-m}(x), & \text{if } n+1 \leq k \leq m+n-1 \end{cases}$$

This completes the proof of the Theorem. □

5.4. The algebra of quantum matrices $R_q(M_{m,n})$ is the $\mathbb{K}$-algebra generated by x_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ subject to the relations

$$\begin{aligned} x_{ij}x_{lj} &= qx_{lj}x_{ij}, \quad \text{for } i < l, \\ x_{ij}x_{ik} &= qx_{ik}x_{ij}, \quad \text{for } j < k, \\ x_{ij}x_{lk} &= x_{lk}x_{ij}, \quad \text{for } i < l, j > k, \\ x_{ij}x_{lk} - x_{lk}x_{ij} &= (q - q^{-1})x_{ik}x_{lj}, \quad \text{for } i < l, j < k, \end{aligned}$$

where $\mathbb{K}$ is a field of characteristic 0 and $q \in \mathbb{K}$ is transcendental over $\mathbb{Q}$. For $I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, m\}$ and $J = \{j_1 < \cdots < j_k\} \subset \{1, \ldots, n\}$ one defines the quantum minor $\Delta^q_{I,J} \in R_q(M_{m,n})$ by

$$\begin{aligned} \Delta^q_{I,J} &= \sum_{\sigma \in S_k} (-q)^{l(\sigma)} x_{i_1 j_{\sigma(1)}} \ldots x_{i_k j_{\sigma(k)}} \\ &= \sum_{\sigma \in S_k} (-q)^{-l(\sigma)} x_{i_k j_{\sigma(k)}} \ldots x_{i_1 j_{\sigma(1)}}. \end{aligned} \tag{5.7}$$

The group $\mathbb{Z}^{m+n}$ acts on $R_q[M_{m,n}]$ by algebra automorphisms by setting $(a_1, \ldots, a_m, b_1, \ldots, b_n) \cdot x_{ij} = q^{a_i - b_j} x_{ij}$ on the generators of $R_q[M_{m,n}]$.

5.5. In §5.5–5.7 we apply the results from Sect. 3 to the particular case $\mathfrak{g} = \mathfrak{sl}_{m+n}$, $w = c^m$. In particular $\mathcal{U}_+$, $V(\omega_k)$, $V_w(\omega_k)$ refer to this situation.

Consider the reduced decomposition

$$w^\circ_{m+n} = s_1(s_2 s_1) \ldots (s_{m+n-1} \ldots s_1). \tag{5.8}$$

Denote the corresponding root vectors given by (3.7) by

$$Y_{1,2}; Y_{1,3}, Y_{2,3}; \ldots; Y_{1,m+n}, \ldots, Y_{m+n-1,m+n} \in \mathcal{U}_+^{w^\circ_{m+n}} = \mathcal{U}_+$$

and

$$Y_{2,1}; Y_{3,1}, Y_{3,2}; \ldots; Y_{m+n,1}, \ldots, Y_{m+n,m+n-1} \in \mathcal{U}_-^{w^\circ_{m+n}} = \mathcal{U}_-$$

in the plus and minus cases, respectively. Then by [25, Lemma 2.1.1] $Y_{i,i+1} = X_i^+$, $1 \leq i < m+n-1$ and for $i < j$ Y_{ij} is recursively given by

$$Y_{ij} = Y_{i,j-1}Y_{j-1,j} - q^{-1}Y_{j-1,j}Y_{i,j-1}. \tag{5.9}$$

Analogously one has that $Y_{i+1,i} = X_i^-$, $1 \leq i < m+n-1$ and for $j > i$ Y_{ji} is recursively given by

$$Y_{ji} = Y_{j,j-1}Y_{j-1,i} - qY_{j-1,i}Y_{j,j-1}. \tag{5.10}$$

The expression

$$c^m = (s_m \ldots s_1)(s_{m+1} \ldots s_2) \ldots (s_{m+n-1} \ldots s_n)$$

is reduced since $l(c^m) = mn$. Denote the corresponding root vectors of $\mathcal{U}_+^{c^m}$ by

$$X_{1,m+1}, \dots, X_{m,m+1}; X_{1,m+2}, \dots, X_{m,m+2}; \dots; X_{1,m+n}, \dots, X_{m,m+n}$$

and of $\mathcal{U}_-^{c^m}$ by

$$X_{m+1,1}, \dots, X_{m+1,m}; X_{m+2,1}, \dots, X_{m+2,m}; \dots; X_{m+n,1}, \dots, X_{m+n,m}.$$

Lemma 5.4. *(1) For all $i \in \overline{1,m}$, $j \in \overline{m+1,m+n}$ and $i \in \overline{m+1,m+n}$, $j \in \overline{1,m}$:*

$$X_{ij} = T_{w_m^\circ} Y_{ij}.$$

(2) The map $g \colon R_q[M_{m,n}] \to \mathcal{U}_-^{c^m}$ given by

$$x_{ij} \mapsto (-q)^{j+m-i-1} X_{j+m,i}, \quad i \in \overline{1,m}, j \in \overline{1,n}$$

is an isomorphism of algebras.

Proof. The first part of (1) is [25, Lemma 2.1.3 (3)]. The second part of (1) is similar. Mériaux and Cauchon showed that $x_{ij} \mapsto Y_{i,j+m}$ defines an algebra isomorphism, based on the Alev–Dumas result [1] that the Yamabe root vectors of $\mathcal{U}_q(\mathfrak{sl}_{m+n})$ satisfy the relations for the standard generators of $R_q[M_{m,n}]$. Since $X_i^+ \mapsto X_i^-$ defines an isomorphism from $\mathcal{U}_+$ to $\mathcal{U}_-$ (such that $Y_{ij} \mapsto (-q)^{j-i-1} Y_{ji}$ for $i < j$) and $T_{w_m^\circ}$ is an (algebra) automorphism of $\mathcal{U}_q(\mathfrak{g})$, the map g is a homomorphism. It is an isomorphism because of the PBW basis part of Theorem 2.1. This proves (2). $\square$

5.6. For $y \in S_{m+n}^{\leq c^m}$ let $\mathcal{A}_q(y)$ be the union of the sets of quantum minors

$$\Delta^q_{w_m^\circ(p_1(I)), (\overline{m+1,m+k} \backslash p_2(I))-m} \in R_q[M_{m,n}]$$

for $k \in \overline{1,n}$, $I \subset \overline{1,m+n}$, $|I| = k$, $I \leq c^m(\overline{1,k})$, $I \not\geq y(\overline{1,k})$ and

$$\Delta^q_{w_m^\circ(p_1(I) \backslash \overline{1,k-n}), (\overline{m+1,m+n} \backslash p_2(I))-m} \in R_q[M_{m,n}]$$

for $k \in \overline{n+1,m+n-1}$, $I \subset \overline{1,m+n}$, $|I| = k$, $I \leq c^m(\overline{1,k})$, $I \not\geq y(\overline{1,k})$, cf. (5.3). We refer the reader to (5.4)–(5.5) for a comparison to the Poisson case.

Theorem 5.5. *For all $y \in S_{m+n}^{\leq c^m}$ denote by $I(y)$ the right ideal of $R_q[M_{m,n}]$ generated by $\mathcal{A}_q(y)$.*

Then all ideals $I(y)$ are two sided, prime and $\mathbb{Z}^{m+n}$-invariant. They exhaust all $\mathbb{Z}^{m+n}$-primes of $R_q[M_{m,n}]$. The map $y \in S_{m+n}^{\leq c^m} \mapsto I(y)$ is an isomorphism from the poset $S_{m+n}^{\leq c^m}$ to the poset of $\mathbb{Z}^{m+n}$ invariant prime ideals of $R_q[M_{m,n}]$ ordered under inclusion.

Theorem 5.5 is a corollary of Theorems 3.8, 3.11 and 3.13 for the special case of the algebras $\mathcal{U}_-^{c^m}$, cf. §5.5. Its proof will be given in §5.7.

The parametrization and poset structure of $\mathbb{Z}^{m+n}$-primes is due to Launois [22] who also proved that all of them are generated by quantum minors. Our proof is independent. Generators for the $\mathbb{Z}^{m+n}$-primes of $R_q[M_{m,n}]$ were only known in the case $m = n = 3$ due to Goodearl and Lenagan [13]. Goodearl, Launois, and Lenagan have a recent independent approach constructing ideal generators in the general case [12].

Define the algebra

$$\Lambda_q(\mathbb{K}^{m+n}) = T_{\mathbb{K}}(v_1, \ldots, v_{m+n})/\langle v_i v_j = -q^{-1} v_j v_i, i > j, v_i^2 = 0\rangle$$

where $T_{\mathbb{K}}(.)$ refers to the tensor algebra over $\mathbb{K}$. It has a canonical structure of $\mathcal{U}_q(\mathfrak{sl}_{m+n})$-module algebra for the action:

$$Y_{ij} v_k = \delta_{jk} v_i,$$
$$K_i v_k = q^{a_{ik}} v_k, \quad a_{ki} = 1 \text{ if } k = i, \ a_{ki} = -1 \text{ if } k = i-1, \ a_{ki} = 0 \text{ otherwise.}$$

Moreover $\Lambda_q(\mathbb{K}^{m+n})$ is graded by $\deg v_i = 1$ and its k-graded component is isomorphic to the fundamental representation $V(\omega_k)$ of $\mathcal{U}_q(\mathfrak{sl}_{m+n})$

$$\Lambda_q(\mathbb{K}^{m+n})_k \cong V(\omega_k) \tag{5.11}$$

for $k = 1, \ldots, m+n-1$. We will use the isomorphism (5.11) for the remainder of this Section. Assuming it,

$$v_I := v_{i_1} \ldots v_{i_k}, \quad I = \{i_1 < \cdots < i_k\} \subset \overline{1, m+n}.$$

is a basis of $V(\omega_k)$. Denote the dual basis of $V(\omega_k)^*$ by $\{\xi_I\}$. Since all root spaces of $V(\omega_k)$ are one dimensional (2.2) implies

$$T_w v_I = b v_{w(I)} \tag{5.12}$$

for some nonzero $b \in \mathbb{K}$ (depending on I and w). Recall the partial order on $\{I \subset \overline{1, m+n} \mid |I| = k\}$ from §5.3. Then the Demazure module $V_w(\omega_k)$ is given by

$$V_w(\omega_k) = \mathcal{U}_+ T_w v_{\overline{1,k}} = \operatorname{Span}\{v_I \mid I \subset \overline{1, m+n}, |I| = k, I \leq w(\overline{1,k})\}.$$

Identify the dual space $V_w(\omega_k)^*$ with

$$V_w(\omega_k)^* \cong \operatorname{Span}\{\xi_I \mid I \subset \overline{1, m+n}, |I| = k, I \leq w(\overline{1,k})\} \subset V(\omega_k)^*. \tag{5.13}$$

Under this identification the orthogonal complement $(V_w(\omega_k) \cap \mathcal{U}_- T_y v_{\overline{1,k}})^\perp$ to $V_w(\omega_k) \cap \mathcal{U}_- T_y v_{\overline{1,k}}$ in $V_w(\omega_k)^*$ is given by

$$(V_w(\omega_k) \cap \mathcal{U}_- T_y v_{\overline{1,k}})^\perp = \operatorname{Span}\{\xi_I \mid I \subset \overline{1, m+n}, |I| = k, I \leq c^m(\overline{1,k}), I \not\geq y(\overline{1,k}\}$$

for all $y \in S_{m+n}^{\leq c^m}$.

5.7. We have

$$\mathcal{R}^{c^m} = \left(\exp_q(X_{m,m+n} \otimes X_{m+n,m}) \ldots \exp_q(X_{1,m+n} \otimes X_{m+n,1})\right) \ldots$$
$$\left(\exp_q(X_{m,m+1} \otimes X_{m+1,m}) \ldots \exp_q(X_{1,m+1} \otimes X_{m+1,1})\right),$$

recall (2.6). From (2.2) one obtains that

$$(T^{-1}_{w_m^\circ}(x)).v = T^{-1}_{w_m^\circ} x T_{w_m^\circ} v \tag{5.14}$$

for all $x \in \mathcal{U}_q(\mathfrak{sl}_{m+n})$, $v \in V(\omega_k)$.

Denote

$$c^k_{I,J} = c^{\omega_k}_{\xi_I, v_J}.$$

Taking into account Lemma 5.4, eqs. (5.12), (5.14), and the fact that T_w is an algebra automorphism of $\mathcal{U}_q(\mathfrak{sl}_{m+n})$ for all $w \in W$, we obtain:

$$(5.15)\quad (c^k_{I,J} \otimes g^{-1})(\mathcal{R}^{c^m}) = b\big(c^k_{w^\circ_m(I),w^\circ_m(J)} \otimes \mathrm{id}\big)\big[\big(\exp_q(Y_{m,m+n} \otimes x_{m,n}) \ldots \exp_q(Y_{1,m+n} \otimes x_{1,n})\big) \ldots \big(\exp_q(Y_{m,m+1} \otimes x_{m,1}) \ldots \exp_q(Y_{1,m+1} \otimes x_{1,1})\big)\big]$$

for some nonzero $b \in \mathbb{K}$.

Proof of Theorem 5.5. For $y \in S^{\leq c^m}_{m+n}$ define the right ideals

$$(5.16)\quad \widetilde{I}(y) = \big\{(c^k_{I,c^m(\overline{1,k})} \otimes g^{-1})(\mathcal{R}^{c^m}) \,\big|\, k \in \overline{1,m+n-1}, I \subset \overline{1,m+n}, |I| = k, I \leq c^m(\overline{1,k}), I \not\geq y(\overline{1,k})\big\} R_q[M_{m,n}].$$

Theorems 3.8, 3.11 and 3.13 and Lemma 5.4 (2) imply that all ideals $I(y)$ are two-sided, prime and $\mathbb{Z}^{m+n}$-invariant. Moreover they exhaust all $\mathbb{Z}^{m+n}$-primes of $R_q[M_{m,n}]$ and the map $y \in S^{\leq c^m}_{m+n} \mapsto I(y)$ is an isomorphism from the poset $S^{\leq c^m}_{m+n}$ to the poset of $\mathbb{Z}^{m+n}$ primes of $R_q[M_{m,n}]$ ordered under inclusion.

We claim that for $k \in \overline{1,n}$, $I \subset \overline{1,m+n}$, $|I| = k$, $I \leq c^m(\overline{1,k})$

$$(5.17)\qquad (c^k_{I,c^m(\overline{1,k})} \otimes g^{-1})(\mathcal{R}^{c^m}) = b\Delta^q_{w^\circ_m(p_1(I)),(\overline{m+1,m+k}\backslash p_2(I))-m}$$

and for $k \in \overline{n+1,m+n-1}$, $I \subset \overline{1,m+n}$, $|I| = k$, $I \leq c^m(\overline{1,k})$

$$(5.18)\qquad (c^k_{I,c^m(\overline{1,k})} \otimes g^{-1})(\mathcal{R}^{c^m}) = b\Delta^q_{w^\circ_m(p_1(I)\backslash\overline{1,k-n}),(\overline{m+1,m+n}\backslash p_2(I))-m}$$

for some nonzero $b \in \mathbb{K}$ depending on k and I. This implies that $I(y) = \widetilde{I}(y)$ for all $y \in S^{\leq c^m}_{m+n}$ and the statement of the Theorem.

Eqs. (5.17) and (5.18) are verified in a similar way. We will restrict ourselves to (5.17). Let $i < j \in \overline{1,m+n}$. From (5.9) one checks inductively on $j-i$ that

$$(5.19)\qquad Y_{ij}(u_{I'}u_j) = u_{I'}u_i$$

for all $I' \subset \overline{1,j-1} \sqcup \overline{j+1,m+n}$, $|I'| = k-1$ and

$$(5.20)\qquad Y_{ij}(u_I) = 0$$

for all $I \subset \overline{1,j-1} \sqcup \overline{j+1,m+n}$, $|I| = k$.

Now fix $k \in \overline{1,n}$ and $I \subset \overline{1,m+n}$ such that $|I| = k$, $I \leq c^m(\overline{1,k})$. Compute $w^\circ_m(I) = w^\circ_m(p_1(I)) \sqcup p_2(I)$ and $w^\circ_m c^m(\overline{1,k}) = \overline{m+1,m+k}$. Then

$$w^\circ_m(I) \cap w^\circ_m c^m(\overline{1,k}) = \overline{m+1,m+k} \cap p_2(I)$$

and

$$w^\circ_m(I)\backslash w^\circ_m c^m(\overline{1,k}) = w^\circ_m(p_1(I)), \quad w^\circ_m c^m(\overline{1,k})\backslash w^\circ_m(I) = \overline{m+1,m+k}\backslash p_2(I).$$

Denote

$$w^\circ_m(p_1(I)) = \{i_1 < \ldots < i_l\}, \quad \overline{m+1,m+k}\backslash p_2(I) = \{j_1+m < \ldots < j_l+m\}.$$

Eqs. (5.15) and (5.19)–(5.20) imply

$$(c^k_{I,c^m(\overline{1,k})} \otimes g^{-1})(\mathcal{R}^{c^m}) = b_1 \sum_{\sigma \in S_l} x_{i_{\sigma(l)},j_l} \ldots x_{i_{\sigma(1)},j_1} \langle \xi_{i_1,\ldots,i_l}, Y_{i_{\sigma(l)},j_l+m} \ldots Y_{i_{\sigma(1)},j_1+m} v_{j_l+m} \ldots v_{j_1+m}\rangle$$

for some nonzero $b_1 \in \mathbb{K}$. Using the fact that $v_i v_j = -q^{-1} v_j v_i$ for $i > j$, eq. (5.19) and the fact that for a permutation σ, $l(\sigma)$ is equal to the number of its inversions we obtain

$$\begin{aligned}
(c^k_{I,c^m(\overline{1,k})} \otimes g^{-1})(\mathcal{R}^{c^m}) &= q^{-l(l-1)/2} b_1 \sum_{\sigma \in S_l} x_{i_{\sigma(l)},j_l} \dots x_{i_{\sigma(1)},j_1} \langle \xi_{i_1,\dots,i_l}, v_{i_{\sigma(1)}} \dots v_{i_{\sigma(l)}} \rangle \\
&= q^{-l(l-1)/2} b_1 \sum_{\sigma \in S_l} (-q)^{-l(\sigma)} x_{i_{\sigma(l)},j_l} \dots x_{i_{\sigma(1)},j_1} \\
&= q^{-l(l-1)/2} b_1 \Delta^q_{w_m^\circ(p_1(I)),(\overline{m+1,m+k}\backslash p_2(I))-m}.
\end{aligned}$$

This completes the proof of (5.17) and the Theorem. $\square$

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