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Semiclassical limit of the Schrödinger-Poisson-Landau-Lifshitz-Gilbert system

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Abstract The Schrödinger-Poisson-Landau-Lifshitz-Gilbert (SPLLG) system is an effective microscopic model that describes the coupling between conduction electron spins and the magnetization in ferromagnetic materials. This system has been used in connection to the study of spin transfer and magnetization reversal in ferromagnetic materials. In this paper, we rigorously prove the existence of weak solutions to SPLLG and derive the Vlasov-Poisson-Landau-Lifshitz-Glibert system as the semiclassical limit.

Keywords Schrödinger-Poisson-Landau-Lifshitz-Gilbert · Semiclassical limit · Spin transfer · Magnetization

1 Introduction

This paper is devoted to the study of spin-magnetization coupling in ferromagnetic materials by analyzing the semiclassical limit of the Schrödinger-Poisson-Landau-Lifshitz-Gilbert (SPLLG) system. The spin-magnetization coupling plays a key role in the active control of domain-wall motion [29, 28, 17] and magnetization reversal in magnetic multilayers [4], which are the core techniques used in magnetoresistance random access memories and race-track

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memories [9]. The SPLLG system is used to describe a mechanism known as spin-transfer torque that transfers the spin angular momentum to magnetization dynamics via spin-magnetization coupling, and was introduced in the the seminal works of Slonczweski [34] and Berger [7,8]. The SPLLG system combines two different models, one to describe the conduction electron spin and one to describe the magnetization dynamics, and will be described in details as follows.

Model description. We start from the quantum mixed-state theory where the pure state wave functions $\{\boldsymbol{\psi}_j^\varepsilon\}_{j=1}^\infty$ satisfy the following Schrödinger equation [31],

$$\mathrm{i}\varepsilon\partial_t\boldsymbol{\psi}_j^\varepsilon(\boldsymbol{x},t) = -\frac{\varepsilon^2}{2}\Delta\boldsymbol{\psi}_j^\varepsilon(\boldsymbol{x},t) + V^\varepsilon\boldsymbol{\psi}_j^\varepsilon(\boldsymbol{x},t) - \frac{\varepsilon}{2}\boldsymbol{m}^\varepsilon\cdot\hat{\boldsymbol{\sigma}}\boldsymbol{\psi}_j^\varepsilon(\boldsymbol{x},t). \tag{1.1}$$

Here $0 < \varepsilon \ll 1$ is the renormalized Planck constant in the semiclassical regime, $\boldsymbol{\psi}_j^\varepsilon = (\psi_{j,+}^\varepsilon, \psi_{j,-}^\varepsilon)^T$ stands for the j-th spinor with "±" indicating spin up and down respectively, and the Pauli matrices $\hat{\boldsymbol{\sigma}} = (\sigma_1, \sigma_2, \sigma_3)^T$ are defined as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

The position density ρ^ε, current density $\boldsymbol{j}^\varepsilon$, spin density $\boldsymbol{s}^\varepsilon$, and spin current $J_{\mathrm{s}}^\varepsilon$ are given by

$$\rho^\varepsilon(\boldsymbol{x},t) = \sum_{j=1}^\infty \lambda_j^\varepsilon\,|\boldsymbol{\psi}_j^\varepsilon(\boldsymbol{x},t)|^2, \tag{1.3a}$$

$$\boldsymbol{j}^\varepsilon(\boldsymbol{x},t) = \varepsilon\sum_{j=1}^\infty \lambda_j^\varepsilon\,\mathrm{Im}\big(\boldsymbol{\psi}_j^{\varepsilon\dagger}(\boldsymbol{x},t)\nabla_{\boldsymbol{x}}\boldsymbol{\psi}_j^\varepsilon(\boldsymbol{x},t)\big), \tag{1.3b}$$

$$\boldsymbol{s}^\varepsilon(\boldsymbol{x},t) = \sum_{j=1}^\infty \lambda_j^\varepsilon \mathrm{Tr}_{\mathbb{C}^2}\left(\hat{\boldsymbol{\sigma}}\big(\boldsymbol{\psi}_j^\varepsilon(\boldsymbol{x},t)\boldsymbol{\psi}_j^{\varepsilon\dagger}(\boldsymbol{x},t)\big)\right), \tag{1.3c}$$

$$J_{\mathrm{s}}^\varepsilon(\boldsymbol{x},t) = \varepsilon\sum_{j=1}^\infty \lambda_j^\varepsilon \mathrm{Im}\left(\mathrm{Tr}_{\mathbb{C}^2}\left(\hat{\boldsymbol{\sigma}}\otimes\nabla_{\boldsymbol{x}}\boldsymbol{\psi}_j^\varepsilon(\boldsymbol{x},t)\boldsymbol{\psi}_j^{\varepsilon\dagger}(\boldsymbol{x},t)\right)\right), \tag{1.3d}$$

where the coefficients $\lambda_j^\varepsilon \geq 0$ are the occupation probabilities of the $L^2(\mathbb{R}^3)$-orthonormal initial states $\{\boldsymbol{\varphi}_j^\varepsilon\}_{j=1}^\infty$. Note that $\boldsymbol{\psi}_j^{\varepsilon\dagger}$ is the complex conjugate transpose of $\boldsymbol{\psi}_j^\varepsilon$, $\mathrm{Tr}_{\mathbb{C}^2}$ is the trace operator of a 2×2 complex matrix, and $\otimes$ means a tensor product of two 3-vectors. Therefore $\boldsymbol{s}^\varepsilon$ is a 3-vector and $J_{\mathrm{s}}^\varepsilon$ is a 3×3 matrix.

The potential V^ε in (1.1) is given self-consistently by the Coulomb interaction,

$$V^\varepsilon = -N * \rho^\varepsilon, \tag{1.4}$$

with the kernel function given by

$$N(\boldsymbol{x}) = -\frac{1}{4\pi|\boldsymbol{x}|}, \tag{1.5}$$

and $*$ is the convolution operator in $\boldsymbol{x}$.

We assume that the ferromagnetic material occupies a compact domain Ω with smooth boundary. The magnetization $\boldsymbol{m}^\varepsilon$ satisfies the Landau-Lifshitz-Gilbert equation [24,20],

$$\partial_t \boldsymbol{m}^\varepsilon = -\boldsymbol{m}^\varepsilon \times \boldsymbol{H}^\varepsilon_{\text{eff}} + \alpha \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon, \text{ with } |\boldsymbol{m}^\varepsilon(\boldsymbol{x},t)| = 1, \text{ and } \boldsymbol{x} \in \Omega, \tag{1.6}$$

with Neumann boundary conditions,

$$\partial_\nu \boldsymbol{m}^\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

where α is the dimensionless damping constant, and ν is the outward unit normal vector on $\partial\Omega$. The first term on the right-hand-side of (1.6) describes the precession of magnetization around the local effective field $\boldsymbol{H}^\varepsilon_{\text{eff}}$, and the second term is the Gilbert damping.

In (1.6), the effective field $\boldsymbol{H}^\varepsilon_{\text{eff}}$ is defined as the variational derivative (with respect to $\boldsymbol{m}^\varepsilon$) of the Landau-Lifshitz energy

$$F_{\text{LL}} = \int_\Omega \left(\frac{1}{2}|\nabla \boldsymbol{m}^\varepsilon|^2 + w(\boldsymbol{m}^\varepsilon) - \frac{1}{2}\boldsymbol{H}^\varepsilon_{\text{s}} \cdot \boldsymbol{m}^\varepsilon - \frac{\varepsilon}{2}\boldsymbol{s}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \right) \mathrm{d}\boldsymbol{x}, \tag{1.8}$$

which is given by

$$\boldsymbol{H}^\varepsilon_{\text{eff}} = -\frac{\delta F_{\text{LL}}}{\delta \boldsymbol{m}^\varepsilon} = \Delta \boldsymbol{m}^\varepsilon - w'(\boldsymbol{m}^\varepsilon) + \boldsymbol{H}^\varepsilon_{\text{s}} + \frac{\varepsilon}{2}\boldsymbol{s}^\varepsilon. \tag{1.9}$$

The term $w(\boldsymbol{m}^\varepsilon)$ in (1.8) stands for the anisotropy energy, and we assume that $w \geq 0$ is a polynomial up to degree 4. In particular, this assumption is satisfied for uniaxial anisotropy given by

$$w(\boldsymbol{m}^\varepsilon) = {m_2^\varepsilon}^2 + {m_3^\varepsilon}^2, \tag{1.10}$$

and the cubic anisotropy given by [23]

$$w(\boldsymbol{m}^\varepsilon) = {m_1^\varepsilon}^2 {m_2^\varepsilon}^2 + {m_2^\varepsilon}^2 {m_3^\varepsilon}^2 + {m_3^\varepsilon}^2 {m_1^\varepsilon}^2. \tag{1.11}$$

We use $w'(\boldsymbol{m}^\varepsilon)$ in the variational derivative instead of $\nabla_{\boldsymbol{m}} w(\boldsymbol{m}^\varepsilon)$ for ease of notation. The coupling term $\boldsymbol{s}^\varepsilon \cdot \boldsymbol{m}^\varepsilon$ gives rise to the *spin transfer torque*, which converts the spin angular momentum to magnetization dynamics; and $\boldsymbol{H}^\varepsilon_{\text{s}} = -\nabla u$ is the stray field, where the magnetostatic potential u is given by

$$u = \nabla N * \cdot \boldsymbol{m}^\varepsilon, \tag{1.12}$$

and thus

$$\boldsymbol{H}^\varepsilon_{\text{s}}(\boldsymbol{x}) = -\nabla \left(\nabla N * \cdot \boldsymbol{m}^\varepsilon \right). \tag{1.13}$$

Main result. Our previous work [16] introduced a systematic (but formal) way of deriving mean-field models for spin-magnetization coupling in ferromagnetic materials using the Wigner transform,

$$W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_{\boldsymbol{y}}}\sum_{j=1}^{\infty}\lambda_j^\varepsilon\boldsymbol{\psi}_j^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2},t\right)\boldsymbol{\psi}_j^{\varepsilon\dagger}\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2},t\right)\mathrm{e}^{\mathrm{i}\boldsymbol{v}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,. \tag{1.14}$$

We also numerically implemented the mean-field models in three dimensions, and applied them to predict current-driven domain wall motion [15]. In the current work, we rigorously prove that the SPLLG system has the Vlasov-Poisson-Landau-Lifshitz-Gilbert (VPLLG) system as its semiclassical limit. We describe the main result in the following theorem.

Theorem. Under certain assumptions on the initial conditions to be specified later, there exists a sequence of solutions $(W^\varepsilon,\,\boldsymbol{m}^\varepsilon)$ to the SPLLG system (1.1) - (1.6) and (1.14), such that

$$\begin{aligned}&W^\varepsilon\xrightarrow{\varepsilon\to0}W\text{ in }L^\infty((0,T),L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}))\text{ weak*}\ \ ,\\&\boldsymbol{m}^\varepsilon\xrightarrow{\varepsilon\to0}\boldsymbol{m}\text{ in }L^\infty((0,T),H^1(\Omega))\text{ weak*}\ \ ,\end{aligned}$$

and $(W,\boldsymbol{m})$ is a weak solution to the following VPLLG system,

$$\begin{aligned}&\partial_tW=-\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}W+\nabla_{\boldsymbol{x}}V\cdot\nabla_{\boldsymbol{v}}W+\frac{\mathrm{i}}{2}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m},\,W],\\&\partial_t\boldsymbol{m}=-\boldsymbol{m}\times\boldsymbol{H}_{\text{eff}}+\alpha\boldsymbol{m}\times\partial_t\boldsymbol{m},\end{aligned}$$

where the potential V is given by

$$V=-N*\rho,$$

and the effective magnetic field $\boldsymbol{H}_{\text{eff}}$ is given by

$$\boldsymbol{H}_{\text{eff}}=\varDelta\boldsymbol{m}-w'(\boldsymbol{m})+\boldsymbol{H}_{\text{s}},\quad\boldsymbol{H}_{\text{s}}(\boldsymbol{x})=-\nabla\left(\nabla N*\cdot\boldsymbol{m}\right),$$

with N given in (1.5) and the density $\rho=\int_{\mathbb{R}^3_{\boldsymbol{v}}}W\,\mathrm{d}\boldsymbol{v}$.

Related works. The Wigner transform, first introduced by Wigner in [35], is a powerful tool in studying the semiclassical limit of quantum systems. Under the Wigner transform, the Schrödinger equation becomes a phase-space quantum Liouville equation. Markowich and Neuzert proved that the semiclassical limit of the Schrödinger equation in the presence of an external potential is given by a Liouville equation [27]. There is also a natural connection between semiclassical limits and homogenization analysis, as discussed in [19]. The electron dynamics with spin were considered by the Wigner transform in [3] with a magnetic field given by a fixed, external vector potential. In the spin-less case, the existence and uniqueness of the three-dimensional Schrödinger system with a self-consistent Poisson potential were analyzed in [10, 2, 14], and the semiclassical limit of the Schrödinger-Poisson system to the Vlasov-Poisson system was derived rigorously in [25, 30], and with an additional periodic potential in [26, 5, 6]. The Landau-Lifshitz-Gilbert (LLG) system has also been intensively studied in the literature. Alouges and Soyeur studied the global weak solutions and showed the existence and non-uniqueness in [1]. In [13, 12], local existence and uniqueness of the regular solution was proven in three dimensions, and the global existence of regular solutions was proven in two dimensions for small initial data. The spin-polarized dynamics was studied in [18] by coupling the LLG system with a spin-transport equation, and the existence and non-uniqueness of the weak solutions was discussed in three dimensions. The global existence of weak solutions to several model equations of magnetization reversal by spin-polarized current was also studied in [22]. The existence of a global smooth solution of the spin-polarized transport system was provided in one and two dimensions in [21] and [32], respectively.

Organization of the paper. In Section 2, we prove the existence of weak solutions to the Schödinger-Poisson-Landau-Lifshitz-Gilber (SPLLG) system. We introduce the assumptions, conserved quantities and *a priori* estimates needed for taking the semiclassical limit of SPLLG in Section 3. In Section 4, we rigorously prove the semiclassical limit as the Vlasov-Poisson-Landau-Lifshitz-Gilbert system.

2 Existence of the Weak Solution

Without loss of generality, we consider $\varepsilon = 1$ and the following coupled system consisting of the Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t \boldsymbol{\psi}_j = -\frac{1}{2}\Delta \boldsymbol{\psi}_j + V\boldsymbol{\psi}_j - \frac{1}{2}\boldsymbol{m}\cdot\hat{\boldsymbol{\sigma}}\boldsymbol{\psi}_j, \quad j \in \mathbb{N},\ t > 0, \\ \boldsymbol{\psi}_j(t = 0, \boldsymbol{x}) = \boldsymbol{\varphi}_j(\boldsymbol{x}), \end{cases} \tag{2.1}$$

and the Landau-Lifshitz-Gilbert equation

$$\begin{cases} \partial_t \boldsymbol{m} = -\boldsymbol{m}\times\boldsymbol{H} + \alpha \boldsymbol{m}\times\partial_t \boldsymbol{m}, & (\boldsymbol{x}, t) \in \Omega\times\mathbb{R}^+, \\ \boldsymbol{m}(t = 0, \boldsymbol{x}) = \boldsymbol{m}_0(\boldsymbol{x}), & x \in \Omega, \\ \partial_\nu \boldsymbol{m} = \boldsymbol{0}, & (\boldsymbol{x}, t) \in \partial\Omega\times\mathbb{R}^+, \end{cases} \tag{2.2}$$

where the Poisson potential

$$V = -N * \rho[\boldsymbol{\Psi}], \tag{2.3}$$

the effective field

$$\boldsymbol{H} = \Delta\boldsymbol{m} - w'(\boldsymbol{m}) + \boldsymbol{H}_{\mathrm{s}} + \frac{1}{2}\boldsymbol{s}[\boldsymbol{\Psi}], \tag{2.4}$$

and $\rho[\boldsymbol{\Psi}]$, $\boldsymbol{s}[\boldsymbol{\Psi}]$ and $\boldsymbol{H}_{\mathrm{s}}$ given by

$$\rho[\boldsymbol{\Psi}] = \sum_{j=1}^{\infty} \lambda_j \, |\boldsymbol{\psi}_j|^2, \tag{2.5a}$$

$$\boldsymbol{s}[\boldsymbol{\Psi}] = \sum_{j=1}^{\infty} \lambda_j \mathrm{Tr}_{\mathbb{C}^2} \left(\hat{\boldsymbol{\sigma}} \big(\boldsymbol{\psi}_j \boldsymbol{\psi}_j^{\dagger} \big) \right), \tag{2.5b}$$

$$\boldsymbol{H}_{\mathrm{s}} = -\nabla(\nabla N * \cdot \boldsymbol{m}), \tag{2.5c}$$

respectively. Here $N(x)$ and $w(\boldsymbol{m})$ are given by (1.5) and (1.10) respectively, and we have used the short-hand notation $\boldsymbol{\Psi} = \{\boldsymbol{\psi}_j\}_{j\in\mathbb{N}}$ and introduce $\boldsymbol{\Phi} = \{\boldsymbol{\varphi}_j\}_{j\in\mathbb{N}}$ to be used later. For each $\boldsymbol{\lambda} = \{\lambda_j\}_{j=1}^{\infty}$ and each $r \in \mathbb{R}$, we introduce the following Hilbert norm for $\boldsymbol{\Psi}$ defined on some measurable domain $K \subset \mathbb{R}^3$,

$$\|\boldsymbol{\Psi}\|^2_{\mathcal{H}^r_{\boldsymbol{\lambda}}(K)} := \sum_{j=1}^{\infty} \lambda_j \|\boldsymbol{\psi}_j\|^2_{H^r(K)}, \tag{2.6}$$

then we say $\boldsymbol{\Psi} \in \mathcal{H}^r_{\boldsymbol{\lambda}}(K)$ if $\|\boldsymbol{\Psi}\|_{\mathcal{H}^r_{\boldsymbol{\lambda}}(K)} < \infty$, and we denote $\mathcal{H}^0_{\boldsymbol{\lambda}}(K)$ by $\mathcal{L}^2_{\boldsymbol{\lambda}}(K)$.

We use the following definition of weak solutions:

Definition 1 Let $\boldsymbol{\Phi} \in \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3)$, $\boldsymbol{m}_0 \in H^1(\Omega)$, $|\boldsymbol{m}_0| = 1$ *a.e.* in Ω. We say $(\boldsymbol{\Psi}, \boldsymbol{m})$ is a weak solution to the Schrodinger-Poisson-Landau-Lifshitz system (2.1)-(2.5) if, for all $T > 0$,

- $\boldsymbol{\Psi} \in L^{\infty}([0,\infty), \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3))$, $\boldsymbol{m} \in L^{\infty}([0,\infty), H^1(\Omega)) \cap H^1([0,T] \times \Omega)$, and $|\boldsymbol{m}| = 1$ *a.e.* .
- For all $\chi \in H^1([0,T] \times \Omega)$ and $\eta \in C([0,T], H^1_c(\mathbb{R}^3))$, the following holds

$$\begin{aligned} \mathrm{i}\int_0^T \int_{\mathbb{R}^3} \partial_t \boldsymbol{\psi}_j \eta &= \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \nabla \boldsymbol{\psi}_j \cdot \nabla \eta + \int_0^T \int_{\mathbb{R}^3} V \boldsymbol{\psi}_j \eta \\ &\quad - \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \boldsymbol{m} \cdot \hat{\boldsymbol{\sigma}} \boldsymbol{\psi}_j \eta, \\ \int_0^T \int_{\Omega} \partial_t \boldsymbol{m} \cdot \chi &= \alpha \int_0^T \int_{\Omega} \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \chi - \int_0^T \int_{\Omega} (\boldsymbol{m} \times \boldsymbol{H}) \cdot \chi, \end{aligned} \tag{2.7}$$

  where

$$\begin{aligned} \int_0^T \int_{\Omega} \boldsymbol{m} \times \boldsymbol{H} \cdot \chi &= \int_0^T \int_{\Omega} \boldsymbol{m} \times \left(\boldsymbol{H}_{\mathrm{s}} + \frac{1}{2}\boldsymbol{s} - w'(\boldsymbol{m}) \right) \cdot \chi \\ &\quad - \int_0^T \int_{\Omega} \boldsymbol{m} \times \nabla \boldsymbol{m} \cdot \nabla \chi, \end{aligned}$$

  and V, ρ, $\boldsymbol{s}$, and $\boldsymbol{H}_{\mathrm{s}}$ are given as (2.3)-(2.5).
- $\boldsymbol{\Psi}(\boldsymbol{x}, 0) = \boldsymbol{\Phi}(\boldsymbol{x})$ and $\boldsymbol{m}(\boldsymbol{x}, 0) = \boldsymbol{m}_0(\boldsymbol{x})$ in the trace sense.

We summarize the result in the following existence theorem:

Theorem 1 *Let Ω be a bounded domain with smooth boundary. Given any initial conditions with $\boldsymbol{\Phi} \in \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3)$ and $\boldsymbol{m}_0 \in H^1(\Omega)$, there exists $\boldsymbol{\Psi} \in L^{\infty}([0,\infty), \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3))$ and $\boldsymbol{m} \in L^{\infty}([0,\infty), H^1(\Omega)) \cap H^1([0,T] \times \Omega)$ for all $T > 0$, such that $(\boldsymbol{\Psi}, \boldsymbol{m})$ is a weak solution to* (2.1)-(2.2).

2.1 Bounded domain

We first consider the Schrödinger equation in $K = \{\boldsymbol{x} \in \mathbb{R}^3,\ |x| < R\}$ in the coupled SPLLG system, *i.e.*, for each $\boldsymbol{\lambda} = \{\lambda_j\}_{j=1}^{\infty}$,

$$\begin{cases} \mathrm{i}\partial_t \boldsymbol{\psi}_j = -\frac{1}{2}\Delta\boldsymbol{\psi}_j + V\boldsymbol{\psi}_j - \frac{1}{2}\boldsymbol{m}\cdot\hat{\boldsymbol{\sigma}}\boldsymbol{\psi}_j, & j \in \mathbb{N},\ (\boldsymbol{x},t) \in K \times \mathbb{R}^+, \\ \boldsymbol{\psi}_j(t=0,\boldsymbol{x}) = \boldsymbol{\varphi}_j(\boldsymbol{x}), & x \in K \\ \boldsymbol{\psi}_j(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \partial K \times \mathbb{R}^+, \end{cases} \tag{2.8}$$

with the potential given by the Poisson equation

$$\begin{cases} -\Delta V = \rho[\boldsymbol{\Psi}] & (\boldsymbol{x},t) \in K \times \mathbb{R}^+ \\ V(\boldsymbol{x},t) = 0 & (\boldsymbol{x},t) \in \partial K \times \mathbb{R}^+, \end{cases} \tag{2.9}$$

and magnetization given by the Landau-Lifshitz-Gilbert equation

$$\begin{cases} \partial_t \boldsymbol{m} = -\boldsymbol{m} \times \boldsymbol{H} + \alpha\boldsymbol{m} \times \partial_t \boldsymbol{m}, & (\boldsymbol{x},t) \in \Omega \times \mathbb{R}^+, \\ \boldsymbol{m}(t=0,\boldsymbol{x}) = \boldsymbol{m}_0(\boldsymbol{x}), & x \in \Omega, \\ \partial_\nu \boldsymbol{m} = \boldsymbol{0}, & (\boldsymbol{x},t) \in \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{2.10}$$

We assume the initial condition satisfies $|\boldsymbol{m}_0(\boldsymbol{x})| \equiv 1$ *a.e.* in Ω, and let

$$\boldsymbol{\Psi} \equiv \boldsymbol{0},\ \text{in}\ \left(\mathbb{R}^3 \backslash K\right) \times \mathbb{R}^+, \quad \text{and}\ \ \boldsymbol{m} \equiv \boldsymbol{0},\ \text{in}\ \left(\mathbb{R}^3 \backslash \bar{\Omega}\right) \times \mathbb{R}^+.$$

The main result of this subsection is summarized as the following theorem.

Theorem 2 *Let Ω be a bounded domain with smooth boundary. Given $K \subset \mathbb{R}^3$ as a ball large enough such that $\Omega \subset K$, and given initial condition with $\boldsymbol{\Phi} \in \mathcal{H}^1_{\boldsymbol{\lambda}}(K)$ and $\boldsymbol{m}_0 \in H^1(\Omega)$, then for all $T > 0$, there exists $\boldsymbol{\Psi} \in L^\infty([0,\infty), \mathcal{H}^1_{\boldsymbol{\lambda}}(K))$ and $\boldsymbol{m} \in L^\infty([0,\infty), H^1(\Omega)) \cap H^1([0,T] \times \Omega)$, such that the system* (2.8) - (2.10) *holds weakly.*

To prove this theorem, similar to [1], instead of directly considering (2.10), we first construct weak solutions to a penalized problem, where the constraint $|\boldsymbol{m}| \equiv 1$ is relaxed,

$$\begin{cases} \alpha\partial_t \boldsymbol{m} + \boldsymbol{m} \times \partial_t \boldsymbol{m} = \boldsymbol{H} - k(|\boldsymbol{m}|^2 - 1)\boldsymbol{m}, & (\boldsymbol{x},t) \in \Omega \times \mathbb{R}^+, \\ \boldsymbol{m}(t=0,\boldsymbol{x}) = \boldsymbol{m}_0(\boldsymbol{x}), & x \in \Omega, \\ \partial_\nu \boldsymbol{m} = \boldsymbol{0}, & (\boldsymbol{x},t) \in \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{2.11}$$

with $k > 0$ as a penalization constant. We then apply the Galerkin method to show that the system (2.8) and (2.11) has weak solutions and then let k go to infinity to get weak solutions to the system (2.8) and (2.10).

Galerkin approximation.

Let $\{\theta_n\}_{n\in\mathbb{N}}$ be the normalized eigenfunctions of

$$-\Delta\theta = \mu\theta \quad \text{in } K, \qquad\qquad \theta|_{\partial_K} = 0. \tag{2.12}$$

Let $\{\omega_n\}_{n\in\mathbb{N}}$ be the normalized eigenfunctions of

$$-\Delta\omega = \mu\omega \quad \text{in } \Omega, \qquad\qquad \partial_\nu\omega|_{\partial_\Omega} = 0. \tag{2.13}$$

Note that $\theta_n \in C^\infty(\bar{K})$ and $\omega_n \in C^\infty(\bar{\Omega})$. We define the orthogonal projections Π_N^K and Π_N^Ω as

$$\Pi_N^K(\boldsymbol{u}) = \sum_{n=1}^N (\boldsymbol{u}, \theta_n)_{L^2(K)}\theta_n, \qquad \forall \boldsymbol{u} \in H^1(K), \tag{2.14}$$

$$\Pi_N^\Omega(\boldsymbol{u}) = \sum_{n=1}^N (\boldsymbol{u}, \omega_n)_{L^2(\Omega)}\omega_n, \qquad \forall \boldsymbol{u} \in H^1(\Omega). \tag{2.15}$$

Consider the approximate solutions $\boldsymbol{\Psi}_N = \{\boldsymbol{\psi}_{jN}\}_{j\in\mathbb{N}}$ and $\boldsymbol{m}_N$ in the forms of

$$\boldsymbol{\psi}_{jN}(\boldsymbol{x}, t) = \sum_{n=1}^N \boldsymbol{\alpha}_{jn}(t)\theta_n(\boldsymbol{x}), \qquad \boldsymbol{m}_N(\boldsymbol{x}, t) = \sum_{n=1}^N \boldsymbol{\beta}_n(t)\omega_n(\boldsymbol{x}), \tag{2.16}$$

where $\boldsymbol{\alpha}_{jn}$ and $\boldsymbol{\beta}_n$ are two- and three-dimensional vector-valued functions respectively, and are chosen such that

$$\begin{aligned} \int_K \left(\mathrm{i}\partial_t \boldsymbol{\psi}_{jN} + \frac{1}{2}\Delta\boldsymbol{\psi}_{jN} - V_N\boldsymbol{\psi}_{jN} + \frac{1}{2}\boldsymbol{m}_N \cdot \hat{\boldsymbol{\sigma}}\boldsymbol{\psi}_{jN} \right) \theta_n = 0, \\ \boldsymbol{\psi}_{jN}(\cdot, 0) = \Pi_N^K \boldsymbol{\varphi}_j, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \int_\Omega \left(\alpha\partial_t \boldsymbol{m}_N + \boldsymbol{m}_N \times \partial_t \boldsymbol{m}_N - \boldsymbol{H}_N + k(|\boldsymbol{m}_N|^2 - 1)\boldsymbol{m}_N \right) \omega_n = 0, \\ \boldsymbol{m}_N(\cdot, 0) = \Pi_N^\Omega \boldsymbol{m}_0, \end{aligned} \tag{2.18}$$

for $n = 1, 2, ..., N$, where V_N satisfies $-\Delta V_N = \boldsymbol{\rho}_N$, $V_N|_{\partial K} = 0$, $\boldsymbol{H}_N = \Delta\boldsymbol{m}_N + \boldsymbol{H}_{\mathrm{s}N} + \frac{1}{2}\boldsymbol{s}_N - w'(\boldsymbol{m}_N)$, $\boldsymbol{H}_{\mathrm{s}N} = -\nabla(\nabla N * \cdot \boldsymbol{m}_N)$, $\boldsymbol{\rho}_N = \sum_{j=1}^\infty \lambda_j |\boldsymbol{\psi}_{jN}|^2$, and $\boldsymbol{s}_N = \sum_{j=1}^\infty \lambda_j \mathrm{Tr}_{\mathbb{C}^2}\left(\hat{\boldsymbol{\sigma}}\boldsymbol{\psi}_{jN}\boldsymbol{\psi}_{jN}^\dagger\right)$. The local (in time) existence of solutions to the Cauchy problem (2.17)-(2.18) follows from Picard's theorem.

Lemma 1 *Let* $(\boldsymbol{\Psi}_N, \boldsymbol{m}_N, V_N, \boldsymbol{\rho}_N, \boldsymbol{s}_N, \boldsymbol{H}_{sN})$ *be the solution to* (2.17)-(2.18). *Then the interval of definition of* $(\boldsymbol{\Psi}_N, \boldsymbol{m}_N, V_N, \boldsymbol{\rho}_N, \boldsymbol{s}_N, \boldsymbol{H}_{sN})$ *can be extended to* $[0, \infty)$, *with*

$$\boldsymbol{\Psi}_N \in L^\infty(\mathbb{R}^+, \mathcal{H}^1_{\boldsymbol{\lambda}}(K)), \tag{2.19a}$$

$$\partial_t \boldsymbol{\Psi}_N \in L^\infty(\mathbb{R}^+, \mathcal{H}^{-1}_{\boldsymbol{\lambda}}(K)), \tag{2.19b}$$

$$\boldsymbol{m}_N \in L^\infty(\mathbb{R}^+, H^1(\Omega)), \tag{2.19c}$$

$$\partial_t \boldsymbol{m}_N \in L^2(\mathbb{R}^+, L^2(\Omega)), \tag{2.19d}$$

$$w'(\boldsymbol{m}_N) \in L^\infty(\mathbb{R}^+, L^r(\Omega)), \ \ 1 \le r \le 2, \tag{2.19e}$$

$$\boldsymbol{\rho}_N \in L^\infty(\mathbb{R}^+, L^r(K)), \ \ 1 \le r \le 3, \tag{2.19f}$$

$$\boldsymbol{s}_N \in L^\infty(\mathbb{R}^+, L^r(K)), \ \ 1 \le r \le 3, \tag{2.19g}$$

$$V_N \in L^\infty(\mathbb{R}^+, L^6(K)), \tag{2.19h}$$

$$\nabla V_N \in L^\infty(\mathbb{R}^+, L^2(K)), \tag{2.19i}$$

$$\boldsymbol{H}_{sN} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)), \tag{2.19j}$$

$$|\boldsymbol{m}_N|^2 - 1 \in L^\infty(\mathbb{R}^+, L^2(\Omega)), \tag{2.19k}$$

and the sequences are uniformly bounded in the corresponding spaces.

Proof Multiplying (2.17) by $\alpha^\dagger_{jn}$, summation over n, and separating the real and imaginary parts produce

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_K |\boldsymbol{\psi}_{jN}|^2 = 0, \tag{2.20}$$

therefore

$$\|\boldsymbol{\psi}_{jN}(t, \cdot)\|_{L^2(K)} = \|\Pi^K_N(\boldsymbol{\varphi})\|_{L^2(K)}. \tag{2.21}$$

Multiplying (2.17) by $\frac{\mathrm{d}\alpha^\dagger_{jn}}{\mathrm{d}t}$ and summation over j (with the weight λ_j) and n bring

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_K \sum_{j=1}^\infty \lambda_j |\nabla \boldsymbol{\psi}_{jN}|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_K |\nabla V_N|^2 = \frac{1}{2} \int_K \partial_t \boldsymbol{s}_N \cdot \boldsymbol{m}_N. \tag{2.22}$$

Multiplying (2.18) by $\frac{\mathrm{d}\beta_n}{\mathrm{d}t}$ and summation over n yield

$$\begin{aligned} &\alpha \int_\Omega |\partial_t \boldsymbol{m}_N|^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \boldsymbol{m}_N|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} |\boldsymbol{H}_{sN}|^2 \\ &+ \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w(\boldsymbol{m}_N) + \frac{k}{4}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(|\boldsymbol{m}_N|^2 - 1\right)^2 = \frac{1}{2} \int_\Omega \partial_t \boldsymbol{m}_N \cdot \boldsymbol{s}_N. \end{aligned} \tag{2.23}$$

Adding (2.22) and (2.23) gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_K \sum_{j=1}^{\infty}\lambda_j|\nabla\boldsymbol{\psi}_{jN}|^2+\frac{\mathrm{d}}{\mathrm{d}t}\int_K|\nabla V_N|^2\\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\boldsymbol{m}_N|^2+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\boldsymbol{H}_{\mathrm{s}N}|^2+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega 2w(\boldsymbol{m}_N)\\
&+\frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(|\boldsymbol{m}_N|^2-1\right)^2+2\alpha\int_\Omega|\partial_t\boldsymbol{m}_N|^2=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\boldsymbol{s}_N\cdot\boldsymbol{m}_N.
\end{aligned}\tag{2.24}$$

Thus

$$\begin{aligned}
&\int_K \sum_{j=1}^{\infty}\lambda_j|\nabla\boldsymbol{\psi}_{jN}|^2+\int_K|\nabla V_N|^2\\
&+\int_\Omega|\nabla\boldsymbol{m}_N|^2+\int_{\mathbb{R}^3}|\boldsymbol{H}_{\mathrm{s}N}|^2+\int_\Omega 2w(\boldsymbol{m}_N)\\
&+\frac{k}{2}\int_\Omega\left(|\boldsymbol{m}_N|^2-1\right)^2+2\alpha\int_0^t\int_\Omega|\partial_t\boldsymbol{m}_N|^2=\int_\Omega\boldsymbol{s}_N\cdot\boldsymbol{m}_N+I_N,
\end{aligned}\tag{2.25}$$

where

$$\begin{aligned}
I_N=&\int_K \sum_{j=1}^{\infty}\lambda_j|\nabla\boldsymbol{\psi}_{jN}(\boldsymbol{x},0)|^2+\int_K|\nabla V_N(\boldsymbol{x},0)|^2\\
&+\int_\Omega|\nabla\boldsymbol{m}_N(\boldsymbol{x},0)|^2+\int_{\mathbb{R}^3}|\boldsymbol{H}_{\mathrm{s}N}(\boldsymbol{x},0)|^2+\int_\Omega 2w(\boldsymbol{m}_N(\boldsymbol{x},0))\\
&+\frac{k}{2}\int_\Omega\left(|\boldsymbol{m}_N(\boldsymbol{x},0)|^2-1\right)^2-\int_\Omega\boldsymbol{s}_N(\boldsymbol{x},0)\cdot\boldsymbol{m}_N(\boldsymbol{x},0).
\end{aligned}\tag{2.26}$$

Note that

$$\begin{aligned}
\int_\Omega\boldsymbol{m}_N\cdot\boldsymbol{s}_N\leq&\|\boldsymbol{m}_N\|_{L^6(\Omega)}\,\|\boldsymbol{s}_N\|_{L^{6/5}(\mathbb{R}^3)}\leq C\|\nabla\boldsymbol{m}_N\|_{L^2(\Omega)}\,\|\boldsymbol{s}_N\|_{L^{6/5}(\mathbb{R}^3)}\\
\leq&C\|\nabla\boldsymbol{m}_N\|_{L^2(\Omega)}\,\|\boldsymbol{s}_N\|_{L^1(\mathbb{R}^3)}^{3/4}\,\|\boldsymbol{s}_N\|_{L^3(\mathbb{R}^3)}^{1/4}\\
\leq&C\|\nabla\boldsymbol{m}_N\|_{L^2(\Omega)}\left(\sum_{j=1}^{\infty}\lambda_j\left\|\boldsymbol{\psi}_{jN}\right\|_{L^2(\mathbb{R}^3)}^2\right)^{\frac{3}{4}}\\
&\times\left(\sum_{j=1}^{\infty}\lambda_j\left\|\nabla\boldsymbol{\psi}_{jN}\right\|_{L^2(\mathbb{R}^3)}^2\right)^{\frac{1}{4}}\\
\leq&C\|\nabla\boldsymbol{m}_N\|_{L^2(\Omega)}\left(\sum_{j=1}^{\infty}\lambda_j\left\|\nabla\boldsymbol{\psi}_{jN}\right\|_{L^2(\mathbb{R}^3)}^2\right)^{\frac{1}{4}},
\end{aligned}\tag{2.27}$$

then together with (2.25) we reach that there exists a constant C, which may depend on the initial datum $\boldsymbol{\Phi}$ and $\boldsymbol{m}_0$ but is independent of N, such that for all $t > 0$

$$\begin{aligned}\frac{1}{2}\int_K \sum_{j=1}^{\infty} \lambda_j |\nabla \boldsymbol{\psi}_{jN}|^2 + \int_K |\nabla V_N|^2 + \frac{1}{2}\int_\Omega |\nabla \boldsymbol{m}_N|^2 + \int_{\mathbb{R}^3} |\boldsymbol{H}_{\mathrm{s}N}|^2 \\ + \frac{k}{2}\int_\Omega \left(|\boldsymbol{m}_N|^2 - 1\right)^2 + 2\alpha \int_0^t \int_\Omega |\partial_t \boldsymbol{m}_N|^2 \le C.\end{aligned} \tag{2.28}$$

Then, by (2.21) and (2.28), (2.19f) and (2.19g) follows from Sobolev interpolations. Furthermore, from (2.17) it follows that

$$\left| \int_K \partial_t \boldsymbol{\psi}_{jN}\, \theta_n \right| \le C \|\theta_n\|_{H^1(K)}, \quad \forall n \in \mathbb{N}, \tag{2.29}$$

therefore,

$$\left\{ \partial_t \boldsymbol{\psi}_{jN} \right\} \text{ is uniformly bounded in } H^{-1}(K). \tag{2.30}$$

□

It follows from Lemma 1 that up to subsequences

$$\boldsymbol{\Psi}_N \xrightarrow{N\to\infty} \boldsymbol{\Psi}^k \in L^\infty(\mathbb{R}^+, \mathcal{H}^1_{\boldsymbol{\lambda}}(K)) \text{ weak* }, \tag{2.31a}$$

$$\partial_t \boldsymbol{\Psi}_N \xrightarrow{N\to\infty} \partial_t \boldsymbol{\Psi}^k \in L^\infty(\mathbb{R}^+, \mathcal{H}^{-1}_{\boldsymbol{\lambda}}(K)) \text{ weak* }, \tag{2.31b}$$

$$\boldsymbol{m}_N \xrightarrow{N\to\infty} \boldsymbol{m}^k \in L^\infty(\mathbb{R}^+, H^1(\Omega)) \text{ weak* }, \tag{2.31c}$$

$$\partial_t \boldsymbol{m}_N \xrightarrow{N\to\infty} \partial_t \boldsymbol{m}^k \in L^2(\mathbb{R}^+, L^2(\Omega)) \text{ weakly }, \tag{2.31d}$$

$$w'(\boldsymbol{m}_N) \xrightarrow{N\to\infty} w'(\boldsymbol{m}^k) \in L^\infty(\mathbb{R}^+, L^r(\Omega)) \text{ weak* }, \quad 1 \le r \le 2, \tag{2.31e}$$

$$\boldsymbol{\rho}_N \xrightarrow{N\to\infty} \rho^k \in L^\infty(\mathbb{R}^+, L^r(K)) \text{ weak* }, \quad 1 \le r \le 3, \tag{2.31f}$$

$$\boldsymbol{s}_N \xrightarrow{N\to\infty} \boldsymbol{s}^k \in L^\infty(\mathbb{R}^+, L^r(K)) \text{ weak* }, \quad 1 \le r \le 3, \tag{2.31g}$$

$$V_N \xrightarrow{N\to\infty} V^k \in L^\infty(\mathbb{R}^+, L^6(K)) \text{ weak* }, \tag{2.31h}$$

$$\nabla V_N \xrightarrow{N\to\infty} \nabla V^k \in L^\infty(\mathbb{R}^+, L^2(K)) \text{ weak* }, \tag{2.31i}$$

$$|\boldsymbol{m}_N|^2 - 1 \xrightarrow{N\to\infty} |\boldsymbol{m}^k|^2 - 1 \in L^\infty(\mathbb{R}^+, L^2(\Omega)) \text{ weak* }, \tag{2.31j}$$

then by Aubin's lemma

$$\boldsymbol{\Psi}_N \xrightarrow{N\to\infty} \boldsymbol{\Psi}^k \in C([0,T], \mathcal{L}^2_{\boldsymbol{\lambda}}(K)) \text{ strongly }, \tag{2.31k}$$

by the Sobolev embedding theorem

$$\boldsymbol{m}_N \xrightarrow{N\to\infty} \boldsymbol{m}^k \in L^2([0,T], L^2(\Omega)) \text{ strongly }, \tag{2.31l}$$

and by the continuity of the map from $\boldsymbol{m}_N$ to $\boldsymbol{H}_{\mathrm{s}N}$

$$\boldsymbol{H}_{\mathrm{s}N} \xrightarrow{N\to\infty} \boldsymbol{H}^k_{\mathrm{s}} \in L^2([0,T], L^2(\mathbb{R}^3)) \text{ strongly }, \tag{2.31m}$$

and $\boldsymbol{H}^k_{\mathrm{s}} = -\nabla(\nabla N * \cdot \boldsymbol{m}^k)$.

Lemma 2 *The limit* $(\boldsymbol{\Psi}^k, \rho^k, \boldsymbol{s}^k)$ *satisfies* (2.5a) *and* (2.5b).

Proof Let $\tilde{\rho} = \sum_{j=1}^{\infty} \lambda_j |\boldsymbol{\psi}_j^k|^2$ and $\eta \in C_0^\infty(K)$, and because of (2.31k),

$$\begin{aligned}\left|\int_K (\boldsymbol{\rho}_N - \tilde{\rho})\eta\right| \le& \sum_{j=1}^{\infty} \lambda_j \int_K |\boldsymbol{\psi}_{jN} + \boldsymbol{\psi}_j^k||\boldsymbol{\psi}_{jN} - \boldsymbol{\psi}_j^k||\eta| \\ \le& C\left(\|\boldsymbol{\Psi}_N\|_{\mathcal{L}^2_{\boldsymbol{\lambda}}(K)} + \|\boldsymbol{\Psi}^k\|_{\mathcal{L}^2_{\boldsymbol{\lambda}}(K)}\right)\left(\|\boldsymbol{\Psi}_N - \boldsymbol{\Psi}^k\|_{\mathcal{L}^2_{\boldsymbol{\lambda}}(K)}\right) \\ \le& C\left(\|\boldsymbol{\Psi}_N - \boldsymbol{\Psi}^k\|_{\mathcal{L}^2_{\boldsymbol{\lambda}}(K)}\right) \xrightarrow{N\to\infty} 0.\end{aligned}$$

Then we get $\rho^k = \rho[\boldsymbol{\Psi}^k] = \sum_{j=1}^{\infty} \lambda_j |\boldsymbol{\psi}_j^k|^2$. A similar argument shows $\boldsymbol{s}^k = \boldsymbol{s}[\boldsymbol{\Psi}^k] = \sum_{j=1}^{\infty} \lambda_j (\boldsymbol{\psi}_j^{k\dagger} \hat{\boldsymbol{\sigma}} \boldsymbol{\psi}_j^k)$. □

Lemma 3 *The limit* (V^k, ρ^k) *satisfies* (2.9).

Proof It is easy to see that V^k is a weak solution of $-\Delta V^k = \rho^k$ on $K \times \mathbb{R}^+$. In addition, by (2.19f), since $\|\boldsymbol{\rho}_N\|_{L^2(K)}$ is uniformly bounded, we know that V_N are uniformly bounded in $H^2(K)$, so we know $V^k \in L^\infty(\mathbb{R}^+, H^2(K))$, which implies V is a strong solution. □

Lemma 4 *The limit* $(\boldsymbol{\Psi}^k, \boldsymbol{m}^k)$ *satisfies* (2.8) *and* (2.11) *weakly, i.e. for all* $\chi \in H^1([0,T] \times \Omega)$ *and* $\eta \in C([0,T], H^1(K))$*, it holds that*

$$\begin{aligned}\mathrm{i}\int_0^T \int_{\mathbb{R}^3} \partial_t \boldsymbol{\psi}_j^k \eta =& \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \nabla \boldsymbol{\psi}_j^k \cdot \nabla \eta + \int_0^T \int_{\mathbb{R}^3} V^k \boldsymbol{\psi}_j^k \eta \\ &- \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \boldsymbol{m}^k \cdot \hat{\boldsymbol{\sigma}} \boldsymbol{\psi}_j^k \eta, \\ \int_0^T \int_\Omega \alpha \partial_t \boldsymbol{m}^k \chi =& -\int_0^T \int_\Omega \left(\boldsymbol{m}^k \times \partial_t \boldsymbol{m}^k - \boldsymbol{H}_s^k - \frac{1}{2}\boldsymbol{s}^k + w'(\boldsymbol{m}^k)\right)\chi \\ &+ \int_0^T \int_\Omega k\left(|\boldsymbol{m}^k|^2 - 1\right)\boldsymbol{m}^k \chi + \nabla \boldsymbol{m}^k \cdot \nabla \chi.\end{aligned} \tag{2.32}$$

Furthermore, there is a constant C *such that*

$$\begin{aligned}\int_K \sum_{j=1}^{\infty} \lambda_j |\nabla \boldsymbol{\psi}_j^k|^2 + \int_K |\nabla V^k|^2 + \int_\Omega |\nabla \boldsymbol{m}^k|^2 + \int_{\mathbb{R}^3} |\boldsymbol{H}_s^k|^2 \\ + \frac{k}{2}\int_\Omega \left(|\boldsymbol{m}^k|^2 - 1\right)^2 + 2\alpha \int_0^t \int_\Omega |\partial_t \boldsymbol{m}^k|^2 \le C,\end{aligned} \tag{2.33}$$

uniformly in k.

Proof It is easy to see that (2.32) is true for all $\chi \in C^\infty([0,T] \times \Omega)$ and $\eta \in C([0,T], H^1(K))$ by passing the limit $N \to \infty$ in (2.17) and (2.18). Then by a density argument, (2.32) is also true for all $\chi \in H^1([0,T] \times \Omega)$. Taking the limit $N \to \infty$ in (2.25) gives the estimate (2.33) by $\lim_{N\to\infty} |\boldsymbol{m}_N(\boldsymbol{x}, 0)| = |\boldsymbol{m}_0(\boldsymbol{x})| = 1$. □

Limit as k tends to ∞.

From Lemma 4, in particular $\int_\Omega \left(|\boldsymbol{m}^k|^2-1\right)^2 \le C/k$, we can get, up to a subsequence,

$$\boldsymbol{m}^k \xrightarrow{k\to\infty} \boldsymbol{m} \text{ pointwise } a.e. \text{ with } |\boldsymbol{m}|=1. \tag{2.34}$$

In a similar way to (2.31), we can also get

$$\begin{aligned}
\boldsymbol{\Psi}^k &\xrightarrow{k\to\infty} \boldsymbol{\Psi} \in L^\infty(\mathbb{R}^+,\mathcal{H}^1_{\boldsymbol{\lambda}}(K)) \text{ weak* }, && (2.35a)\\
\partial_t\boldsymbol{\Psi}^k &\xrightarrow{k\to\infty} \partial_t\boldsymbol{\Psi} \in L^\infty(\mathbb{R}^+,\mathcal{H}^{-1}_{\boldsymbol{\lambda}}(K)) \text{ weak* }, && (2.35b)\\
\boldsymbol{\Psi}^k &\xrightarrow{k\to\infty} \boldsymbol{\Psi} \in C([0,T],\mathcal{L}^2_{\boldsymbol{\lambda}}(K)) \text{ strongly }, && (2.35c)\\
\boldsymbol{m}^k &\xrightarrow{k\to\infty} \boldsymbol{m} \in L^\infty(\mathbb{R}^+,H^1(\Omega)) \text{ weak* }, && (2.35d)\\
\partial_t\boldsymbol{m}^k &\xrightarrow{k\to\infty} \partial_t\boldsymbol{m} \in L^2(\mathbb{R}^+,L^2(\Omega)) \text{ weakly }, && (2.35e)\\
\boldsymbol{m}^k &\xrightarrow{k\to\infty} \boldsymbol{m} \in L^2([0,T],L^2(\Omega)) \text{ strongly }, && (2.35f)\\
\boldsymbol{m}^k &\xrightarrow{k\to\infty} \boldsymbol{m} \in L^4([0,T]\times\Omega) \text{ weakly }, && (2.35g)\\
|\boldsymbol{m}^k|^2-1 &\xrightarrow{k\to\infty} 0 \in L^2([0,T]\times\Omega) \text{ weakly and } a.e. , && (2.35h)\\
w'(\boldsymbol{m}^k) &\xrightarrow{k\to\infty} w'(\boldsymbol{m}) \in L^\infty(\mathbb{R}^+,L^r(\Omega)) \text{ weak* }, \quad 1\le r\le 2, && (2.35i)\\
\rho^k &\xrightarrow{k\to\infty} \rho \in L^\infty(\mathbb{R}^+,L^r(K)) \text{ weak* }, \quad 1\le r\le 3, && (2.35j)\\
\boldsymbol{s}^k &\xrightarrow{k\to\infty} \boldsymbol{s} \in L^\infty(\mathbb{R}^+,L^r(K)) \text{ weak* }, \quad 1\le r\le 3, && (2.35k)\\
V^k &\xrightarrow{k\to\infty} V \in L^\infty(\mathbb{R}^+,L^6(K)) \text{ weak* }, && (2.35l)\\
\boldsymbol{H}^k_{\mathrm{s}} &\xrightarrow{k\to\infty} \boldsymbol{H}_{\mathrm{s}} \in L^2([0,T],L^2(\mathbb{R}^3)) \text{ strongly }. && (2.35m)
\end{aligned}$$

Proof (Proof of Theorem 2) Let $\boldsymbol{\xi}\in C^\infty([0,T]\times\Omega)$, and $\boldsymbol{\chi}=\boldsymbol{m}^k\times\boldsymbol{\xi}$. As $\boldsymbol{\chi}\in H^1([0,T]\times\Omega)$, we get from (2.32) that

$$\begin{aligned}
&\int_0^T\int_\Omega \left(-\alpha\boldsymbol{m}^k\times\partial_t\boldsymbol{m}^k+|\boldsymbol{m}^k|^2\partial_t\boldsymbol{m}^k-(\boldsymbol{m}^k\cdot\partial_t\boldsymbol{m}^k)\boldsymbol{m}^k\right)\cdot\boldsymbol{\xi}\\
&=\int_0^T\int_\Omega \boldsymbol{m}^k\times\nabla\boldsymbol{m}^k\cdot\nabla\boldsymbol{\xi}-\boldsymbol{m}^k\times\left(\boldsymbol{H}^k_{\mathrm{s}}+\frac{1}{2}\boldsymbol{s}^k-w'(\boldsymbol{m}^k)\right)\cdot\boldsymbol{\xi}\,.
\end{aligned}\tag{2.36}$$

Since

$$\int_0^T\int_\Omega|\boldsymbol{m}^k|^2\partial_t\boldsymbol{m}^k\cdot\boldsymbol{\xi}=\int_0^T\int_\Omega(|\boldsymbol{m}^k|^2-1)\partial_t\boldsymbol{m}^k\cdot\boldsymbol{\xi}+\int_0^T\int_\Omega\partial_t\boldsymbol{m}^k\cdot\boldsymbol{\xi}\,,\tag{2.37}$$

we have

$$\int_0^T\int_\Omega|\boldsymbol{m}^k|^2\partial_t\boldsymbol{m}^k\cdot\boldsymbol{\xi}\xrightarrow{k\to\infty}\int_0^T\int_\Omega\partial_t\boldsymbol{m}\cdot\boldsymbol{\xi}\,.\tag{2.38}$$

On the other hand,

$$\int_0^T \int_\Omega (\boldsymbol{m}^k \cdot \partial_t \boldsymbol{m}^k)\boldsymbol{m}^k \cdot \boldsymbol{\xi} \xrightarrow{k\to\infty} \int_0^T \int_\Omega (\boldsymbol{m} \cdot \partial_t \boldsymbol{m})\boldsymbol{m} \cdot \boldsymbol{\xi} = 0\ . \tag{2.39}$$

Eventually we obtain that for all $\boldsymbol{\xi} \in C^\infty([0,T] \times \Omega)$ it holds

$$\begin{aligned}\int_0^T \int_\Omega \partial_t \boldsymbol{m} \cdot \boldsymbol{\xi} = \int_0^T \int_\Omega \left(\boldsymbol{m} \times \left(\alpha \partial_t \boldsymbol{m} + w'(\boldsymbol{m}) - \boldsymbol{H}_{\mathrm{s}} - \frac{1}{2}\boldsymbol{s} \right) \right) \cdot \boldsymbol{\xi} \\ + \int_0^T \int_\Omega \boldsymbol{m} \times \nabla \boldsymbol{m} \cdot \nabla \boldsymbol{\xi}.\end{aligned} \tag{2.40}$$

Since $|\boldsymbol{m}| = 1$ *a.e.* , by a density argument, we also obtain the above equation holds for all $\boldsymbol{\xi} \in H^1([0,T] \times \Omega)$. In the mean time, by passing the $k \to \infty$ limit in the Schrödinger equation in (2.32), we can obtain

$$\begin{aligned}\mathrm{i} \int_0^T \int_{\mathbb{R}^3} \partial_t \boldsymbol{\psi}_j \eta = \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \nabla \boldsymbol{\psi}_j \cdot \nabla \eta + \int_0^T \int_{\mathbb{R}^3} V \boldsymbol{\psi}_j \eta \\ - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \boldsymbol{m} \cdot \hat{\boldsymbol{\sigma}} \boldsymbol{\psi}_j \eta,\end{aligned} \tag{2.41}$$

for all $\eta \in C([0,T], H^1(K))$. This ends the proof of Theorem 2. □

The weak solutions have the following property:

Proposition 1 *Let* $(\boldsymbol{\Psi}, \boldsymbol{m}, V, \rho, \boldsymbol{s}, \boldsymbol{H}_s)$ *be one solution in Theorem 2, then*

$$\|\boldsymbol{\psi}_j(t)\|_{L^2(K)} = \|\boldsymbol{\varphi}_j\|_{L^2(K)}, \quad \|\boldsymbol{m}(t)\|_{L^2(\Omega)} = \|\boldsymbol{m}_0\|_{L^2(\Omega)}, \tag{2.42}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_K \sum_{j=1}^\infty \lambda_j |\nabla \boldsymbol{\psi}_j|^2 + \frac{\mathrm{d}}{\mathrm{d}t} \int_K |\nabla V|^2 + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \boldsymbol{m}|^2 \\ + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} |\boldsymbol{H}_s|^2 + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega 2w(\boldsymbol{m}) - \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \boldsymbol{s} \cdot \boldsymbol{m} + 2\alpha \int_\Omega |\partial_t \boldsymbol{m}|^2 = 0.\end{aligned} \tag{2.43}$$

Moreover, there exists a constant C *such that for all* $t > 0$*,*

$$\begin{aligned}\int_K \sum_{j=1}^\infty \lambda_j |\nabla \boldsymbol{\psi}_j|^2 + \int_K |\nabla V|^2 \\ + \int_\Omega |\nabla \boldsymbol{m}|^2 + \int_{\mathbb{R}^3} |\boldsymbol{H}_s|^2 + 2\alpha \int_0^t \int_\Omega |\partial_t \boldsymbol{m}|^2 \le C.\end{aligned} \tag{2.44}$$

2.2 Whole space

We then consider the Schrödinger equations in $\mathbb{R}^3$ in the SPLLG system and show the existence by representing the solution as a limit of the solutions of bounded-domain problems defined on a sphere whose radius goes to infinity [10].

We denote the sphere of radius R by $B_R = \{\boldsymbol{x} \in \mathbb{R}^3,\ |\boldsymbol{x}| < R\}$, and without loss of generality assume $\Omega \subset B_R$.

We consider the sequences $(\boldsymbol{\Psi}_R, \boldsymbol{m}_R, V_R, \boldsymbol{\rho}_R, \boldsymbol{s}_R, \boldsymbol{H}_{\mathrm{s}R})$, which are defined for $R > R_0$ and satisfy the following coupled system consisting of the Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t \boldsymbol{\Psi}_R = -\frac{1}{2}\Delta \boldsymbol{\Psi}_R + V\boldsymbol{\Psi}_R - \frac{1}{2}\boldsymbol{m}_R \cdot \hat{\boldsymbol{\sigma}}\boldsymbol{\Psi}_R, & (\boldsymbol{x}, t) \in B_R \times \mathbb{R}^+, \\ \boldsymbol{\Psi}_R(t = 0, \boldsymbol{x}) = \boldsymbol{\Phi}_R(\boldsymbol{x}), & x \in B_R, \\ \boldsymbol{\Psi}_R(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \partial B_R \times \mathbb{R}^+, \end{cases} \tag{2.45}$$

the Poisson equation

$$\begin{cases} -\Delta V_R = \boldsymbol{\rho}_R & (\boldsymbol{x}, t) \in B_R \times \mathbb{R}^+, \\ V_R(\boldsymbol{x}, t) = 0 & (\boldsymbol{x}, t) \in \partial B_R \times \mathbb{R}^+, \end{cases} \tag{2.46}$$

and the Landau-Lifshitz-Gilbert equation

$$\begin{cases} \partial_t \boldsymbol{m}_R = -\boldsymbol{m} \times \boldsymbol{H}_R + \alpha \boldsymbol{m}_R \times \partial_t \boldsymbol{m}_R, & (\boldsymbol{x}, t) \in \Omega \times \mathbb{R}^+, \\ \boldsymbol{m}_R(t = 0, \boldsymbol{x}) = \boldsymbol{m}_0(\boldsymbol{x}), & x \in \Omega, \\ \partial_\nu \boldsymbol{m}_R = \boldsymbol{0}, & (\boldsymbol{x}, t) \in \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{2.47}$$

where the effective field

$$\boldsymbol{H}_R = \Delta \boldsymbol{m}_R - w'(\boldsymbol{m}_R) + \boldsymbol{H}_{\mathrm{s}R} + \frac{1}{2}\boldsymbol{s}_R, \tag{2.48}$$

and $\boldsymbol{\rho}_R$, $\boldsymbol{s}_R$ and $\boldsymbol{H}_{\mathrm{s}R}$ given by

$$\boldsymbol{\rho}_R = \sum_{j=1}^{\infty} \lambda_j \, |\boldsymbol{\psi}_{jR}|^2, \tag{2.49a}$$

$$\boldsymbol{s}_R = \sum_{j=1}^{\infty} \lambda_j \mathrm{Tr}_{\mathbb{C}^2} \left(\hat{\boldsymbol{\sigma}} \big(\boldsymbol{\psi}_{jR} \boldsymbol{\psi}_{jR}^\dagger \big) \right), \tag{2.49b}$$

$$\boldsymbol{H}_{\mathrm{s}R} = -\nabla(\nabla N * \cdot \boldsymbol{m}_R), \tag{2.49c}$$

respectively. In (2.45) we have used the notation $\boldsymbol{\Psi}_R = \{\boldsymbol{\psi}_{jR}\}_{j\in\mathbb{N}}$ and $\boldsymbol{\Phi}_R = \{\boldsymbol{\varphi}_{jR}\}_{j\in\mathbb{N}}$, and (2.45) should be understood component-wisely for each $\boldsymbol{\psi}_{jR}$, $j \in \mathbb{N}$. We also assume $|\boldsymbol{m}_0(\boldsymbol{x})| \equiv 1$ for all $\boldsymbol{x} \in \Omega$ and set

$$\boldsymbol{\Psi}_R \equiv \boldsymbol{0}, \text{ in } \left(\mathbb{R}^3 \backslash B_R\right) \times \mathbb{R}^+, \quad \text{and} \quad \boldsymbol{m} \equiv \boldsymbol{0}, \text{ in } \left(\mathbb{R}^3 \backslash \bar{\Omega}\right) \times \mathbb{R}^+. \tag{2.50}$$

We assume that the initial $\boldsymbol{\Phi} = \{\boldsymbol{\varphi}_j\}_{j\in\mathbb{N}} \in \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3)$, and choose $\boldsymbol{\Phi}_R = \{\boldsymbol{\varphi}_{jR}\}_{j\in\mathbb{N}}$ as

$$\boldsymbol{\varphi}_{jR}(\boldsymbol{x}) = \begin{cases} \mathbf{0}, & j > R, \\ \boldsymbol{\varphi}_j(x)\sigma(\boldsymbol{x}/R), & j \le R, \end{cases} \tag{2.51}$$

where $\sigma(\boldsymbol{x}) \in C_0^\infty(B_1)$, $0 \le \sigma \le 1$, and $\sigma(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in B_{1/2}$.

Theorem 2 implies that the problem (2.45)-(2.49) has at least one weak solution ($\boldsymbol{\Psi}_R$, $\boldsymbol{m}_R$, V_R, $\boldsymbol{\rho}_R$, $\boldsymbol{s}_R$, $\boldsymbol{H}_{\mathrm{s}R}$). By Proposition 1 and the Gagliardo-Nirenberg interpolation inequality, we can get

$$\begin{aligned} &\int_{B_R} \sum_{j=1}^\infty \lambda_j |\nabla \boldsymbol{\psi}_{jR}|^2 + \int_{B_R} |\nabla V_R|^2 \\ &+ \int_\Omega |\nabla \boldsymbol{m}_R|^2 + \int_{\mathbb{R}^3} |\boldsymbol{H}_{\mathrm{s}R}|^2 + 2\alpha \int_0^t \int_\Omega |\partial_t \boldsymbol{m}_R|^2 \le C, \end{aligned} \tag{2.52}$$

where the constant C only depends on the initial conditions but not on time t and the radius R. Then

$$\|\boldsymbol{\rho}_R\|_{L^r(B_R)} + \|\boldsymbol{s}_R\|_{L^r(B_R)} \le C, \quad 1 \le r \le 3, \tag{2.53a}$$

$$\|\nabla\boldsymbol{\rho}_R\|_{L^s(B_R)} + \|\nabla\boldsymbol{s}_R\|_{L^s(B_R)} \le C, \quad 1 \le s \le \frac{3}{2}, \tag{2.53b}$$

$$\|V_R\|_{L^6(B_R)} \le C. \tag{2.53c}$$

Therefore, as $R \to \infty$, there exists a subsequence $\{\boldsymbol{\Psi}_R, \boldsymbol{m}_R, V_R, \boldsymbol{\rho}_R, \boldsymbol{s}_R, \boldsymbol{H}_{\mathrm{s}R}\}$ (not relabeled) such that

$$\boldsymbol{\Psi}_R \xrightarrow{R\to\infty} \boldsymbol{\Psi} \in L^\infty(\mathbb{R}^+, \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3)) \text{ weak* }, \tag{2.54a}$$

$$\boldsymbol{\rho}_R \xrightarrow{R\to\infty} \rho \in L^\infty(\mathbb{R}^+, L^r(\mathbb{R}^3)) \text{ weak* }, \quad 1 \le r \le 3, \tag{2.54b}$$

$$\boldsymbol{s}_R \xrightarrow{R\to\infty} \boldsymbol{s} \in L^\infty(\mathbb{R}^+, L^r(\mathbb{R}^3)) \text{ weak* }, \quad 1 \le r \le 3, \tag{2.54c}$$

$$V_R \xrightarrow{R\to\infty} V \in L^\infty(\mathbb{R}^+, L^6(\mathbb{R}^3)) \text{ weak* }, \tag{2.54d}$$

$$\nabla V_R \xrightarrow{R\to\infty} \nabla V \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \text{ weak* }, \tag{2.54e}$$

$$\boldsymbol{m}_R \xrightarrow{R\to\infty} \boldsymbol{m} \in L^\infty(\mathbb{R}^+, H^1(\Omega)) \text{ weak* }, \tag{2.54f}$$

$$\partial_t \boldsymbol{m}_R \xrightarrow{R\to\infty} \partial_t \boldsymbol{m} \in L^2(\mathbb{R}^+, L^2(\Omega)) \text{ weakly }, \tag{2.54g}$$

$$\boldsymbol{m}_R \xrightarrow{R\to\infty} \boldsymbol{m} \in L^2([0,T], L^2(\Omega)) \text{ strongly }, \tag{2.54h}$$

$$w'(\boldsymbol{m}_R) \xrightarrow{R\to\infty} w'(\boldsymbol{m}) \in L^\infty(\mathbb{R}^+, L^r(\Omega)) \text{ weak* }, \quad 1 \le r \le 2, \tag{2.54i}$$

and

$$\boldsymbol{H}_{\mathrm{s}R} \xrightarrow{R\to\infty} \boldsymbol{H}_{\mathrm{s}} = -\nabla(\nabla N * \cdot \boldsymbol{m}) \in L^2([0,T], L^2(\mathbb{R}^3)) \text{ strongly }. \tag{2.54j}$$

We then show the limit $(\boldsymbol{\Psi}, \boldsymbol{m}, V, \rho, \boldsymbol{s}, \boldsymbol{H}_{\mathrm{s}})$ satisfies the whole-space Schrödinger-Poisson-Landau-Lifshitz system (2.1)-(2.5). First we state the following convergence result:

Lemma 5 *For every $T > 0$ and bounded $K \subset \mathbb{R}^3$, there exists a subsequence such that*

$$\partial_t \boldsymbol{\Psi}_R \xrightarrow{R\to\infty} \partial_t \boldsymbol{\Psi} \in L^\infty((0,T), \mathcal{H}_{\boldsymbol{\lambda}}^{-1}(K)) \text{ weak* }, \tag{2.55a}$$

$$\boldsymbol{\Psi}_R \xrightarrow{R\to\infty} \boldsymbol{\Psi} \in C([0,T], \mathcal{L}_{\boldsymbol{\lambda}}^2(K)) \text{ strongly,} \tag{2.55b}$$

$$\boldsymbol{\rho}_R \xrightarrow{R\to\infty} \rho \in C([0,T], L^1(K)) \text{ strongly,} \tag{2.55c}$$

$$\boldsymbol{s}_R \xrightarrow{R\to\infty} \boldsymbol{s} \in C([0,T], L^1(K)) \text{ strongly.} \tag{2.55d}$$

Proof We omit the proof of this lemma and remark that this is essentially the same as Lemma 4.4 and 4.5 in [10]. □

By passing the $R \to \infty$ limit in (2.45) – (2.47), we obtain :

Lemma 6 *The limit $(\boldsymbol{\Psi}, \boldsymbol{m}, V, \boldsymbol{s}, \boldsymbol{H}_s)$ satisfies (2.7) for all $T > 0$, all $\chi \in H^1([0,T] \times \Omega)$, and all $\eta \in C([0,T], \mathcal{H}_{\boldsymbol{\lambda}}^1(K))$ for some bounded $K \subset \mathbb{R}^3$. And $\rho = \rho[\boldsymbol{\Psi}]$ and $\boldsymbol{s} = \boldsymbol{s}[\boldsymbol{\Psi}]$ satisfy (2.5a) and (2.5b) respectively.*

We refer to Lemma 4.10 in [10] for the Poisson potential:

Lemma 7 *The limit (V, ρ) satisfies (2.3), i.e.*

$$V(\boldsymbol{x}, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(\boldsymbol{y}, t)}{|\boldsymbol{x} - \boldsymbol{y}|} \, \mathrm{d}\boldsymbol{y}. \tag{2.56}$$

Lemma 6 and 7 imply that the limit $(\boldsymbol{\Psi}, \boldsymbol{m}, V, \rho, \boldsymbol{s}, \boldsymbol{H}_\mathrm{s})$ is a weak solution of the Schrodinger-Poisson-Landau-Lifshitz system, and we have proved Theorem 1.

3 Semiclassical limit: Assumptions and preliminaries

In this section, we introduce assumptions, conserved quantities and *a priori* estimates that are needed for taking the semiclassical limit of SPLLG system (1.1).

Assumption 1 *For fixed $\varepsilon \in (0, \varepsilon_0]$, we assume $\lambda_j^\varepsilon \geq 0$, $\forall j \in \mathbb{N}$, $\{\boldsymbol{\varphi}_j^\varepsilon\}_{j\in\mathbb{N}}$ is orthonormed in $L^2(\mathbb{R}^3; \mathbb{C}^2)$. $\lambda_j^\varepsilon \geq 0$, $\forall j \in \mathbb{N}$, $\{\boldsymbol{\varphi}_j^\varepsilon\}_{j\in\mathbb{N}}$ is orthonormal in $L^2(\mathbb{R}^3; \mathbb{C}^2)$*

Assumption 2 *There is a constant $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0]$ such that*

$$\sum_{j=1}^\infty \lambda_j^\varepsilon + \varepsilon^2 \sum_{j=1}^\infty \lambda_j^\varepsilon \|\nabla \boldsymbol{\varphi}_j^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \varepsilon^{-3} \sum_{j=1}^\infty (\lambda_j^\varepsilon)^2 \leq C, \tag{3.1}$$

for $\varepsilon \in (0, \varepsilon_o]$.

Assumption 3 *The initial condition of the LLG equation (1.6) is given by $\boldsymbol{m}^\varepsilon(\boldsymbol{x}, t = 0) = 0$ for $\boldsymbol{x} \in \bar{\Omega}^c$ and $\boldsymbol{m}^\varepsilon(\boldsymbol{x}, t = 0) = \boldsymbol{m}_0(x)$ for $\boldsymbol{x} \in \Omega$, where $\boldsymbol{m}_0 \in H^1(\Omega)$, $|\boldsymbol{m}_0| \equiv 1$, and $\partial_\nu \boldsymbol{m}_0 = 0$ on $\partial\Omega$.*

Remark 1 Assumptions 1 and 2 have been used in [30,25,5,6] for proving the semiclassical limit of the Schrödinger-Poisson system, and Assumption 3 was used in [1,11,18] for proving the existence of solutions to the LLG equation. The first two terms in the inequality (3.1) indicate that the total mass and total kinetic energy are bounded *resp.*, while the third term in (3.1) is a technical assumption used in proving the regularities of the physical observables.

We next introduce the mixed state density matrix,

$$Z^\varepsilon(\boldsymbol{x},\boldsymbol{y},t) = \sum_{j=1}^{\infty} \lambda_j^\varepsilon \boldsymbol{\psi}_j^\varepsilon(\boldsymbol{y},t)\boldsymbol{\psi}_j^{\varepsilon\dagger}(\boldsymbol{x},t), \tag{3.2}$$

and the p-norm,

$$|\!|\!| Z^\varepsilon |\!|\!|_p = \left(\sum_{j=1}^{\infty} |\lambda_j^\varepsilon|^p\right)^{\frac{1}{p}}, \quad \text{for any } p \geq 1. \tag{3.3}$$

Then the Wigner transform (1.14) can be rewritten as

$$W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_{\boldsymbol{y}}} Z^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}, \boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}, t\right) \mathrm{e}^{\mathrm{i}\boldsymbol{v}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,. \tag{3.4}$$

Note that the Wigner function W^ε is a 2×2 matrix and is connected to the densities and currents via its moments,

$$\rho^\varepsilon(\boldsymbol{x},t) = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}_{\mathbb{C}^2}\left(W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right)\mathrm{d}\boldsymbol{v}, \tag{3.5a}$$

$$\boldsymbol{j}^\varepsilon(\boldsymbol{x},t) = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \boldsymbol{v}\,\mathrm{Tr}_{\mathbb{C}^2}\left(W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right)\mathrm{d}\boldsymbol{v}, \tag{3.5b}$$

$$\boldsymbol{s}^\varepsilon(\boldsymbol{x},t) = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}_{\mathbb{C}^2}\left(\hat{\boldsymbol{\sigma}}W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right)\mathrm{d}\boldsymbol{v}, \tag{3.5c}$$

$$J_{\mathrm{s}}^\varepsilon(\boldsymbol{x},t) = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \boldsymbol{v}\otimes\mathrm{Tr}_{\mathbb{C}^2}\left(\hat{\boldsymbol{\sigma}}W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right)\mathrm{d}\boldsymbol{v}, \tag{3.5d}$$

and the kinetic energy

$$\begin{aligned} E_{\mathrm{kin}}^\varepsilon &= \int_{\mathbb{R}^3_{\boldsymbol{x}}} \frac{\varepsilon^2}{2}\sum_{j=1}^{\infty}\lambda_j^\varepsilon |\nabla\boldsymbol{\psi}_j^\varepsilon(\boldsymbol{x},t)|^2\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} \frac{|\boldsymbol{v}|^2}{2}\mathrm{Tr}_{\mathbb{C}^2}\left(W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right)\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.6}$$

Direct calculations from (1.1) show that the Wigner function (3.4) satisfies

$$\begin{aligned} \partial_t W^\varepsilon &= -\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} W^\varepsilon + \left(\Theta^\varepsilon[V^\varepsilon] + \frac{\mathrm{i}}{2}\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]\right)W^\varepsilon, \\ & \qquad W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t=0) = W_I^\varepsilon(\boldsymbol{x},\boldsymbol{v}), \end{aligned} \tag{3.7}$$

where the operator Θ^ε is given by

$$\Theta^\varepsilon[V^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v}) = \frac{1}{(2\pi)^3}\iint \frac{1}{\mathrm{i}\varepsilon}\left[V^\varepsilon\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right) - V^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\right)\right] \times W^\varepsilon(\boldsymbol{x},\boldsymbol{v}')\mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}', \tag{3.8}$$

and the operator Γ^ε is given by

$$\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v}) = \frac{1}{(2\pi)^3}\iint \left[M^\varepsilon\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right)W^\varepsilon(\boldsymbol{x},\boldsymbol{v}') - W^\varepsilon(\boldsymbol{x},\boldsymbol{v}')M^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\right)\right]\mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}', \tag{3.9}$$

with the matrix $M^\varepsilon = \hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^\varepsilon$.

The initial datum W_I^ε is the Wigner transform of the initial density matrix

$$Z_I^\varepsilon(\boldsymbol{x},\boldsymbol{y}) = \sum_{j=1}^{\infty}\lambda_j^\varepsilon\boldsymbol{\varphi}_j^\varepsilon(\boldsymbol{y})\boldsymbol{\varphi}_j^{\varepsilon\dagger}(\boldsymbol{x}), \tag{3.10}$$

which is

$$W_I^\varepsilon(\boldsymbol{x},\boldsymbol{v}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_{\boldsymbol{y}}} Z_I^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2},\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{v}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,. \tag{3.11}$$

In what follows, we give a list of conserved quantities that the SPLLG system preserves.

Conservation of the total mass.

$$\begin{aligned}\int_{\mathbb{R}^3_{\boldsymbol{x}}}\rho^\varepsilon(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}\mathrm{Tr}_{\mathbb{C}^2}\left(W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right)\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}\mathrm{Tr}_{\mathbb{C}^2}\left(W_I^\varepsilon(\boldsymbol{x},\boldsymbol{v})\right)\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\mathbb{R}^3_{\boldsymbol{x}}}\rho^\varepsilon(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} = \sum_{j=1}^{\infty}\lambda_j^\varepsilon.\end{aligned} \tag{3.12}$$

Conservation of the L^2-norm of W^ε.

$$\begin{aligned}\|W^\varepsilon(t)\|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}})} &:= \int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}\mathrm{Tr}_{\mathbb{C}^2}\left\{\left[W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right]^2\right\}\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}\mathrm{Tr}_{\mathbb{C}^2}\left\{\left[W_I^\varepsilon(\boldsymbol{x},\boldsymbol{v})\right]^2\right\}\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\\ &= \|W_I^\varepsilon\|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}})} = \frac{2}{(4\pi\varepsilon)^3}\sum_{j=1}^{\infty}(\lambda_j^\varepsilon)^2.\end{aligned} \tag{3.13}$$

This can be seen by left-multiplying $W^{\varepsilon\dagger} = W^\varepsilon$ on (3.7) and integrating

$$\begin{aligned} &-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}\left[W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right]^2\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ =&-2\mathrm{i}\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\Theta^\varepsilon[V^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ &+\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\Gamma^\varepsilon[M^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.14}$$

The second term on the right of the above equation is

$$\begin{aligned} &\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\Gamma^\varepsilon[M^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ =&\iiiint W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)M^\varepsilon\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right)W^\varepsilon(\boldsymbol{x},\boldsymbol{v}',t)\mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ &-\iiiint W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)W^\varepsilon(\boldsymbol{x},\boldsymbol{v}',t)M^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\right)\mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ =&\iiiint W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)M^\varepsilon\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right)W^\varepsilon(\boldsymbol{x},\boldsymbol{v}',t)\mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ &-\iiiint W^\varepsilon(\boldsymbol{x},\boldsymbol{v}',t)W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)M^\varepsilon\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right)\mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

By taking trace on both side of the above equation, the right hand side vanishes since $\mathrm{Tr}_{\mathbb{C}^2}(AB) = \mathrm{Tr}_{\mathbb{C}^2}(BA)$, and one has

$$\mathrm{Tr}\left(\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\Gamma^\varepsilon[M^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\right) = 0. \tag{3.15}$$

Essentially the same argument also yields

$$\mathrm{Tr}\left(\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\Theta^\varepsilon[V^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\right) = 0. \tag{3.16}$$

Therefore, taking trace on both side of (3.14) produces (3.13).
Energy dissipation. An extension of (2.43) implies that

$$\alpha\int_0^t\int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 + F_{\mathrm{SC}}(t) + F_{\mathrm{LL}}(t) = F_{\mathrm{SC}}(0) + F_{\mathrm{LL}}(0), \tag{3.17}$$

where we have defined the energy connected the Schrödinger equations as

$$\begin{aligned} F_{\mathrm{SC}} &= \frac{\varepsilon^2}{2}\int_{\mathbb{R}^3_{\boldsymbol{x}}}\sum_{j=1}^\infty \lambda_j^\varepsilon|\nabla\boldsymbol{\psi}_j^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^3_{\boldsymbol{x}}}|\nabla V^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} \\ &= \frac{1}{2}\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}|\boldsymbol{v}|^2\mathrm{Tr}_{\mathbb{C}^2}\left(W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right)\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^3_{\boldsymbol{x}}}|\nabla V^\varepsilon(\boldsymbol{x},t)|^2\,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{3.18}$$

and the by Landau-Lifshitz energy as

$$
\begin{aligned}
F_{\mathrm{LL}} = \frac{1}{2}\int_{\Omega} |\nabla \boldsymbol{m}^{\varepsilon}|^2 \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^3} |\boldsymbol{H}_{\mathrm{s}}^{\varepsilon}|^2 \,\mathrm{d}\boldsymbol{x} \\
+ \int_{\Omega} w(\boldsymbol{m}) \,\mathrm{d}\boldsymbol{x} - \frac{\varepsilon}{2}\int_{\Omega} \boldsymbol{s}^{\varepsilon} \cdot \boldsymbol{m}^{\varepsilon} \,\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.19}
$$

In the end, we shall give *a priori* estimates of densities and currents using the following classical interpolation lemma [30,25,2,5,6].

Lemma 8 *Let* $1 \le p \le \infty$*,* $q = (5p-3)/(3p-1)$*,* $s = (5p-3)/(4p-2)$*, and* $\theta = 2p/(5p-3)$*. Then and there exists a constant* $C > 0$ *such that*

$$
\|\rho^{\varepsilon}\|_{L^q(\mathbb{R}^3_{\boldsymbol{x}})} \le C \,|\!|\!| Z^{\varepsilon} |\!|\!|_p^{\theta} \left(\varepsilon^{-2} E_{kin}^{\varepsilon}\right)^{1-\theta}, \tag{3.20}
$$

$$
\|\boldsymbol{j}^{\varepsilon}\|_{L^s(\mathbb{R}^3_{\boldsymbol{x}})} \le C \,|\!|\!| Z^{\varepsilon} |\!|\!|_p^{\theta} \left(\varepsilon^{-2} E_{kin}^{\varepsilon}\right)^{1-\theta}, \tag{3.21}
$$

with E_{kin}^{ε} *given by* (3.6).

By (3.12)-(3.13) and Assumption 2, we conclude that there exists a constant C independent of ε such that

$$
\|\rho^{\varepsilon}\|_{L^{\infty}((0,\infty),L^1(\mathbb{R}^3_{\boldsymbol{x}}))} = \|\rho^{\varepsilon}(\cdot,0)\|_{L^1(\mathbb{R}^3_{\boldsymbol{x}})} \le C, \tag{3.22}
$$

$$
\|W^{\varepsilon}\|_{L^{\infty}((0,\infty),L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}))} = \|W_I^{\varepsilon}\|_{L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}})} \le C. \tag{3.23}
$$

By (1.4)-(1.5), and the Hölder's inequality, one has

$$
\|\nabla V^{\varepsilon}(t)\|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}})} \le \|V^{\varepsilon}(t)\|_{L^6(\mathbb{R}^3_{\boldsymbol{x}})} \|\rho^{\varepsilon}(t)\|_{L^{6/5}(\mathbb{R}^3_{\boldsymbol{x}})}. \tag{3.24}
$$

Then the Gagliardo-Nirenberg-Sobolev inequality $\|f\|_{L^q(\mathbb{R}^n)} \le C\|\nabla f\|_{L^p(\mathbb{R}^n)}$ for $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ implies

$$
\|\nabla V^{\varepsilon}(t)\|_{L^2(\mathbb{R}^3_{\boldsymbol{x}})} \le C \|\rho^{\varepsilon}(t)\|_{L^{6/5}(\mathbb{R}^3_{\boldsymbol{x}})}. \tag{3.25}
$$

Applying Lemma 8 brings

$$
\begin{aligned}
\|\rho^{\varepsilon}(t)\|_{L^{7/5}(\mathbb{R}^3_{\boldsymbol{x}})} \le & C \|W^{\varepsilon}(t)\|^{4/7}_{L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}})} \left(E_{\mathrm{kin}}^{\varepsilon}\right)^{3/7} \\
\le & C \left(\int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} |\boldsymbol{v}|^2 \mathrm{Tr}_{\mathbb{C}^2} \left(W^{\varepsilon}(\boldsymbol{x},\boldsymbol{v},t)\right) \mathrm{d}\boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \right)^{3/7}.
\end{aligned} \tag{3.26}
$$

Then applying the interpolation between $L^1(\mathbb{R}^3_{\boldsymbol{x}})$ and $L^{7/5}(\mathbb{R}^3_{\boldsymbol{x}})$ leads to

$$
\|\rho^{\varepsilon}(t)\|_{L^{6/5}(\mathbb{R}^3_{\boldsymbol{x}})} \le C \left(\int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} |\boldsymbol{v}|^2 \mathrm{Tr}_{\mathbb{C}^2} \left(W^{\varepsilon}(\boldsymbol{x},\boldsymbol{v},t)\right) \mathrm{d}\boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \right)^{1/4}, \tag{3.27}
$$

and therefore

$$
\|\nabla V^{\varepsilon}(t)\|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}})} \le C \left(\int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} |\boldsymbol{v}|^2 \mathrm{Tr}_{\mathbb{C}^2} \left(W^{\varepsilon}(\boldsymbol{x},\boldsymbol{v},t)\right) \mathrm{d}\boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \right)^{1/2}. \tag{3.28}
$$

For the coupling energy between spin and the magnetization, one has

$$\begin{aligned}\left|\int_{\Omega} \boldsymbol{s}^{\varepsilon} \cdot \boldsymbol{m}^{\varepsilon} \, \mathrm{d}\boldsymbol{x}\right| &\leq \int_{\Omega} |\boldsymbol{s}^{\varepsilon} \cdot \boldsymbol{m}^{\varepsilon}| \, \mathrm{d}\boldsymbol{x} \\ &\leq \int_{\mathbb{R}^3_{\boldsymbol{x}}} |\boldsymbol{s}^{\varepsilon}| \, \mathrm{d}\boldsymbol{x} \leq \sum_{j=1}^{\infty} \lambda_j^{\varepsilon} \|\boldsymbol{\psi}_j^{\varepsilon}(t)\|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}})} \leq C.\end{aligned} \tag{3.29}$$

Then by applying (3.28) and (3.29) to (3.17) we get

$$\alpha \int_0^t \int_{\Omega} |\partial_t \boldsymbol{m}^{\varepsilon}|^2 + \frac{1}{2} \int_{\Omega} |\nabla \boldsymbol{m}^{\varepsilon}(t)|^2 \, \mathrm{d}\boldsymbol{x} + E^{\varepsilon}_{\mathrm{kin}}(t) \leq \; C + F_{\mathrm{LL}}(0) + 2E^{\varepsilon}_{\mathrm{kin}}(0),$$

and then by Assumption 2, we have

$$E^{\varepsilon}_{\mathrm{kin}}(t) = \frac{1}{2} \int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} |\boldsymbol{v}|^2 \mathrm{Tr}_{\mathbb{C}^2}\big(W^{\varepsilon}(\boldsymbol{x}, \boldsymbol{v}, t)\big) \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \leq C, \tag{3.30}$$

$$\|V^{\varepsilon}\|_{L^{\infty}((0,\infty), L^6(\mathbb{R}^3_{\boldsymbol{x}}))} + \|\nabla V^{\varepsilon}\|_{L^{\infty}((0,\infty), L^2(\mathbb{R}^3_{\boldsymbol{x}}))} \leq C, \tag{3.31}$$

and

$$\alpha \int_0^t \int_{\Omega} |\partial_t \boldsymbol{m}^{\varepsilon}|^2 + \|\boldsymbol{m}^{\varepsilon}(t)\|_{L^2(\Omega)} + \|\nabla \boldsymbol{m}^{\varepsilon}(t)\|_{L^2(\Omega)} \leq C. \tag{3.32}$$

Then by Lemma 8, we have

$$\|\rho^{\varepsilon}\|_{L^{\infty}((0,\infty), L^q(\mathbb{R}^3_{\boldsymbol{x}}))} \leq C, \quad q \in [1, 7/5] \,, \tag{3.33a}$$

$$\|\boldsymbol{j}^{\varepsilon}\|_{L^{\infty}((0,\infty), L^s(\mathbb{R}^3_{\boldsymbol{x}}))} \leq C, \quad s \in [1, 7/6] \,. \tag{3.33b}$$

Similarly, one can also have the estimates for $\boldsymbol{s}^{\varepsilon}$ and $J_{\mathrm{s}}^{\varepsilon}$,

$$\|\boldsymbol{s}^{\varepsilon}\|_{L^{\infty}((0,\infty), L^q(\mathbb{R}^3_{\boldsymbol{x}}))} \leq C, \quad q \in [1, 7/5] \,, \tag{3.33c}$$

$$\|J_{\mathrm{s}}^{\varepsilon}\|_{L^{\infty}((0,\infty), L^s(\mathbb{R}^3_{\boldsymbol{x}}))} \leq C, \quad s \in [1, 7/6] \,. \tag{3.33d}$$

4 Semiclassical limit of the SPLLG system

In the section, we rigorously derive the semiclassical limit of the Schrödinger-Poisson-Landau-Lifshitz-Gilbert (SPLLG) system (1.1)-(1.6). Using (3.23), (3.33a)-(3.33d), (3.32), and (3.31), and applying the Banach-Alaoglu theorem, after restriction to a sub-sequence if necessary, we have

$$W_I^{\varepsilon} \xrightarrow{\varepsilon \to 0} W_I \text{ in } L^2(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}}) \text{ weakly}, \tag{4.1a}$$

$$W^{\varepsilon} \xrightarrow{\varepsilon \to 0} W \text{ in } L^{\infty}((0,\infty), L^2(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}})) \text{ weak* }, \tag{4.1b}$$

$$\rho^{\varepsilon} \xrightarrow{\varepsilon \to 0} \rho \text{ in } L^{\infty}((0,\infty), L^q(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak* }, q \in [1, 7/5] \,, \tag{4.1c}$$

$$\boldsymbol{j}^{\varepsilon} \xrightarrow{\varepsilon \to 0} \boldsymbol{j} \text{ in } L^{\infty}((0,\infty), L^s(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak* }, s \in [1, 7/6] \,, \tag{4.1d}$$

$$\boldsymbol{s}^\varepsilon \xrightarrow{\varepsilon\to 0} \boldsymbol{s} \text{ in } L^\infty((0,\infty), L^q(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak*}, q\in[1,7/5], \tag{4.1e}$$

$$J_{\mathrm{s}}^\varepsilon \xrightarrow{\varepsilon\to 0} J_{\mathrm{s}} \text{ in } L^\infty((0,\infty), L^s(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak*}, s\in[1,7/6], \tag{4.1f}$$

$$\boldsymbol{m}^\varepsilon \xrightarrow{\varepsilon\to 0} \boldsymbol{m} \text{ in } L^\infty((0,\infty), H^1(\Omega)) \text{ weak*}, \tag{4.1g}$$

$$\partial_t\boldsymbol{m}^\varepsilon \xrightarrow{\varepsilon\to 0} \partial_t\boldsymbol{m} \text{ in } L^2([0,T], L^2(\Omega)) \text{ weakly}, \tag{4.1h}$$

$$w'(\boldsymbol{m}^\varepsilon) \xrightarrow{\varepsilon\to 0} w'(\boldsymbol{m}) \text{ in } L^\infty(\mathbb{R}^+, L^r(\Omega)) \text{ weak*}, \quad 1\le r\le 2, \tag{4.1i}$$

$$V^\varepsilon \xrightarrow{\varepsilon\to 0} V \text{ in } L^\infty((0,\infty), L^6(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak*}, \tag{4.1j}$$

$$\nabla V^\varepsilon \xrightarrow{\varepsilon\to 0} \nabla V \text{ in } L^\infty((0,\infty), L^2(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak*}. \tag{4.1k}$$

Further more, from (3.32) by Aubin's lemma we get, up to a subsequence,

$$\boldsymbol{m}^\varepsilon \xrightarrow{\varepsilon\to 0} \boldsymbol{m} \text{ in } L^2([0,T], L^2(\Omega)) \text{ strongly}, \tag{4.1l}$$

and this together with the continuity of the map (1.13) from $\boldsymbol{m}^\varepsilon$ to $\boldsymbol{H}_{\mathrm{s}}^\varepsilon$ implies

$$\boldsymbol{H}_{\mathrm{s}}^\varepsilon \xrightarrow{\varepsilon\to 0} \boldsymbol{H}_{\mathrm{s}} \text{ in } L^2([0,T], L^2(\mathbb{R}^3_{\boldsymbol{x}})) \text{ strongly}, \tag{4.1m}$$

and

$$\boldsymbol{H}_{\mathrm{s}}(\boldsymbol{x}) = -\nabla\int_\Omega \nabla N(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{m}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},$$

Then we have the following lemma.

Lemma 9 *The limit observables can be calculated by taking moments to W:*

$$\rho(\boldsymbol{x},t) = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}_{\mathbb{C}^2}\big(W(\boldsymbol{x},\boldsymbol{v},t)\big)\,\mathrm{d}\boldsymbol{v},$$
$$\boldsymbol{j}(\boldsymbol{x},t) = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \boldsymbol{v}\mathrm{Tr}_{\mathbb{C}^2}\big(W(\boldsymbol{x},\boldsymbol{v},t)\big)\,\mathrm{d}\boldsymbol{v},$$
$$\boldsymbol{s}(\boldsymbol{x},t) = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}_{\mathbb{C}^2}\big(\hat{\boldsymbol{\sigma}}W(\boldsymbol{x},\boldsymbol{v},t)\big)\,\mathrm{d}\boldsymbol{v},$$
$$J_s(\boldsymbol{x},t) = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \boldsymbol{v}\otimes\mathrm{Tr}_{\mathbb{C}^2}\big(\hat{\boldsymbol{\sigma}}W(\boldsymbol{x},\boldsymbol{v},t)\big)\,\mathrm{d}\boldsymbol{v}.$$

The proof of this lemma is analogous to Lemma 3.1 in [25].

4.1 The limit of the Wigner-Poisson equation as $\varepsilon \to 0$

We denote $\phi = \phi(\boldsymbol{x},\boldsymbol{v},t)$ to be a C^∞-test function such that the support of $\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[\phi]$ is compact in $\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{y}}\times[0,\infty)$, where $\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}$ is the Fourier transform

$$\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[\phi](\boldsymbol{y}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} \phi(\boldsymbol{v})\mathrm{e}^{-\mathrm{i}\boldsymbol{y}\cdot\boldsymbol{v}}\,\mathrm{d}\boldsymbol{v}. \tag{4.2}$$

Multiplying equation (3.7) by ϕ and integrating by parts yield

$$\iiint \left(W^\varepsilon(\partial_t \phi + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \phi) + \left(\Theta^\varepsilon[V^\varepsilon] + \frac{\mathrm{i}}{2} \Gamma^\varepsilon[\boldsymbol{m}^\varepsilon] \right) W^\varepsilon \phi \right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}\, \mathrm{d}t = 0. \tag{4.3}$$

By (4.1), W^ε converges to W in the weak* sense, and taking the limit $\varepsilon \to 0$ gives

$$\lim_{\varepsilon \to 0} \iiint W^\varepsilon(\partial_t \phi + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \phi)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}\, \mathrm{d}t = \iiint W(\partial_t \phi + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \phi)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}\, \mathrm{d}t. \tag{4.4}$$

Next we study the limit of the Θ^ε and Γ^ε operators as $\varepsilon \to 0$,*resp.*.

The limit of the operator Θ^ε.

Lemma 10 *Let W^ε be the solution to the Wigner equation* (3.7) *coupled with the LLG equation* (1.6)*, then for any C^∞-test function $\phi = \phi(\boldsymbol{x}, \boldsymbol{v}, t)$ such that $\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[\phi]$ defined in* (4.2) *has compact support in $\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{y}} \times [0, \infty)$, we have*

$$\lim_{\varepsilon \to 0} \iiint \Theta^\varepsilon[V^\varepsilon] W^\varepsilon \phi\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}t = -\iiint W \nabla_{\boldsymbol{x}} V \cdot \nabla_{\boldsymbol{v}} \phi\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}t. \tag{4.5}$$

To prove this lemma, we first need to prove the following estimate.

Lemma 11 *We rewrite $\Theta^\varepsilon[V^\varepsilon]$ as*

$$\Theta^\varepsilon[V^\varepsilon] W^\varepsilon(\boldsymbol{x}, \boldsymbol{v}) = \frac{1}{(2\pi)^3} \iint \delta^\varepsilon[V^\varepsilon](\boldsymbol{x}, \boldsymbol{y}, t) W^\varepsilon(\boldsymbol{x}, \boldsymbol{v}', t) \mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot \boldsymbol{y}}\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{v}',$$

where

$$\delta^\varepsilon[V^\varepsilon](\boldsymbol{x}, \boldsymbol{y}, t) = \frac{1}{\mathrm{i}\varepsilon} \left[V^\varepsilon \left(\boldsymbol{x} - \frac{\varepsilon \boldsymbol{y}}{2} \right) - V^\varepsilon \left(\boldsymbol{x} + \frac{\varepsilon \boldsymbol{y}}{2} \right) \right].$$

then the symbols $\delta^\varepsilon[V^\varepsilon](\boldsymbol{x}, \boldsymbol{y}, t)$ can be written as

$$\delta^\varepsilon[V^\varepsilon](\boldsymbol{x}, \boldsymbol{y}, t) = \mathrm{i}\boldsymbol{y} \cdot \nabla_{\boldsymbol{x}} V^\varepsilon(\boldsymbol{x}, t) + R^\varepsilon(\boldsymbol{x}, \boldsymbol{y}, t),$$

where R^ε satisfies

$$\|R^\varepsilon\|_{L^\infty_{((0,\infty), L^2(B_R \times B_R))}} \le C(R) \varepsilon^{5/14}, \tag{4.6}$$

for every $R > 0$. Here B_R denotes the ball in $\mathbb{R}^3$ with radius R and center in origin.

Proof Direction calculations show that R^ε should be of the following form

$$R^\varepsilon(\boldsymbol{x},\boldsymbol{y},t) = \frac{\mathrm{i}}{2}\int_{-1}^{1} \boldsymbol{y}\cdot\left(\nabla_{\boldsymbol{x}} V^\varepsilon\left(x+\frac{\varepsilon s\boldsymbol{y}}{2},t\right) - \nabla_{\boldsymbol{x}} V^\varepsilon(\boldsymbol{x},t)\right)\mathrm{d}s.$$

Then by the estimates in [25], we know

$$\|R^\varepsilon(t)\|_{L^2(B_R\times B_R)} \le C_\sigma(R)\varepsilon^\sigma |\nabla_{\boldsymbol{x}} V^\varepsilon(t)|_{W^{\sigma,2}(B_R)}. \tag{4.7}$$

The embedding $W^{2,7/5}(B_{2R}) \Subset W^{1+\sigma,2}(B_{2R})$ with $\sigma = 5/14$, together with (3.31) and the standard localization argument for the Poisson equation, produces

$$\begin{aligned}\|V^\varepsilon(t)\|_{W^{1+\sigma,2}(B_{2R})} &\le C_\sigma(R)\|V^\varepsilon(t)\|_{W^{2,7/5}(B_{2R})}\\ &\le C_\sigma(R)\|\rho^\varepsilon(t)\|_{L^{7/5}(\mathbb{R}^3_{\boldsymbol{x}})} \le C_\sigma(R),\end{aligned} \tag{4.8}$$

which implies (4.6). □

Proof (Proof of Lemma 10) Notice that the Θ^ε part of the weak form of (4.3) can be written as

$$\begin{aligned}\iiint \Theta^\varepsilon[V^\varepsilon] W^\varepsilon\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t = &-\iiint W^\varepsilon\nabla_{\boldsymbol{x}} V^\varepsilon\cdot\nabla_{\boldsymbol{v}}\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\\ &+\iiint R^\varepsilon(\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[W^\varepsilon])(\overline{\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[\phi]})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}t,\end{aligned} \tag{4.9}$$

with $\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}$ defined in (4.2). Then by (4.1), Lemma 11 and taking the limit $\varepsilon\to 0$, one has

$$\lim_{\varepsilon\to 0}\iiint R^\varepsilon(\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[W^\varepsilon])(\overline{\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[\phi]})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}t = 0_{2\times 2}. \tag{4.10}$$

To pass to the limit of the term containing $\nabla_{\boldsymbol{x}} V^\varepsilon$ in (4.9), one only needs to show the strong convergence of V^ε in $C([0,T],H^1(B_R))$. Equation(4.8) implies

$$\|V^\varepsilon(t)\|_{L^\infty((0,\infty);W^{2,7/5}(B_R))} \le C(R) \tag{4.11}$$

for every $R>0$. By (1.4) and the continuity equation

$$\partial_t\rho^\varepsilon + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{j}^\varepsilon = 0, \tag{4.12}$$

one has

$$\partial_t V^\varepsilon(\boldsymbol{x},t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3_{\boldsymbol{y}}}\frac{\nabla\cdot\boldsymbol{j}^\varepsilon(\boldsymbol{y},t)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y} = \frac{1}{4\pi}\int_{\mathbb{R}^3_{\boldsymbol{y}}}\frac{(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{j}^\varepsilon(\boldsymbol{y},t)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,\mathrm{d}\boldsymbol{y}. \tag{4.13}$$

Then (3.33b) and Young's inequality for convolution yield

$$\|\partial_t V^\varepsilon\|_{L^\infty((0,\infty),L^r(B_R)} \le C(R),\ \text{with } r = \frac{21}{11}. \tag{4.14}$$

Since

$$W^{2,7/5}(B_R) \Subset W^{1+\sigma,7/5}(B_R) \subset H^1(B_R) \subset L^2(B_R) \subset L^r(B_R) \tag{4.15}$$

for $\sigma = 5/14$ and $r = 21/11$, by (4.11) and (4.14), one can apply the compactness result in [33] to conclude that, for every $R > 0$ and $T > 0$ there is a subsequence such that

$$V^\varepsilon \xrightarrow{\varepsilon \to 0} V \text{ in } C([0,T], H^1(B_R)) \text{ strongly.} \tag{4.16}$$

Therefore one can pass the limit of the Θ^ε part in (4.3) and prove Lemma 10. □

The limit of the operator Γ^ε.

Lemma 12 *Let W^ε and $\boldsymbol{m}^\varepsilon$ are the solutions to the Wigner equation (3.7) coupled with the LLG equation (1.6), and $|\boldsymbol{m}^\varepsilon| \equiv 1$ in Ω and is 0 in Ω^c, then for any C^∞-test function $\phi = \phi(\boldsymbol{x}, \boldsymbol{v}, t)$ such that $\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[\phi]$ defined in (4.2) has compact support in $\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{y}} \times [0, \infty)$, we have*

$$\lim_{\varepsilon \to 0} \iiint \Gamma^\varepsilon[\boldsymbol{m}^\varepsilon] W^\varepsilon \phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}t = \lim_{\varepsilon \to 0} \iiint [M, W] \phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}t, \tag{4.17}$$

where $M = \boldsymbol{m} \cdot \hat{\boldsymbol{\sigma}}$, $\boldsymbol{m}$ is the limit of $\boldsymbol{m}^\varepsilon$ in $L^2([0,T], L^2(\Omega))$, and $[\cdot,\cdot]$ denotes the commutator $[A, B] = AB - BA$.

One difficulty in proving this lemma is to deal with the jump discontinuities of $\boldsymbol{m}^\varepsilon$ across the boundary of Ω. We first prove the following lemma for a smooth $\boldsymbol{m}^\varepsilon$ in $\mathbb{R}^3$.

Lemma 13 *Suppose $\boldsymbol{m}^\varepsilon$ converge to $\boldsymbol{m}$ strongly in $L^2([0,T], L^2(\mathbb{R}^3_{\boldsymbol{x}}))$, and $\|\boldsymbol{m}^\varepsilon(t)\|_{H^1(\mathbb{R}^3_{\boldsymbol{x}})} \le C$. Suppose W^ε converge to W in $L^\infty((0,\infty), L^2(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}}))$ in the weak* sense, and $\|W^\varepsilon\|_{L^\infty((0,\infty),L^2(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}}))} \le C$. Then for any C^∞-test function $\phi = \phi(\boldsymbol{x}, \boldsymbol{v}, t)$ such that $\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[\phi]$ defined in (4.2) has compact support in $\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{y}} \times [0, \infty)$, we have*

$$\lim_{\varepsilon \to 0} \iiint \Gamma^\varepsilon[\boldsymbol{m}^\varepsilon] W^\varepsilon \phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}t = \iiint [M, W] \phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}t.$$

Proof To show this, we write $\Gamma^\varepsilon = \Gamma^\varepsilon_- - \Gamma^\varepsilon_+$, where

$$\Gamma^\varepsilon_+[\boldsymbol{m}^\varepsilon] W^\varepsilon(\boldsymbol{x}, \boldsymbol{v}) = \frac{1}{(2\pi)^3} \iint W^\varepsilon(\boldsymbol{x}, \boldsymbol{v}') M^\varepsilon \left(\boldsymbol{x} + \frac{\varepsilon \boldsymbol{y}}{2}\right) \mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot \boldsymbol{y}} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{v}',$$
$$\Gamma^\varepsilon_-[\boldsymbol{m}^\varepsilon] W^\varepsilon(\boldsymbol{x}, \boldsymbol{v}) = \frac{1}{(2\pi)^3} \iint M^\varepsilon \left(\boldsymbol{x} - \frac{\varepsilon \boldsymbol{y}}{2}\right) W^\varepsilon(\boldsymbol{x}, \boldsymbol{v}') \mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot \boldsymbol{y}} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{v}',$$

and

$$M^\varepsilon \left(\boldsymbol{x} + \frac{\varepsilon \boldsymbol{y}}{2}\right) = M^\varepsilon(\boldsymbol{x}) + \varepsilon R^\varepsilon(\boldsymbol{x}, \boldsymbol{y}),$$

where

$$R^{\varepsilon} = \frac{\boldsymbol{y}}{2} \cdot \int_0^1 \nabla_{\boldsymbol{x}} M^{\varepsilon} \left(\boldsymbol{x} + \frac{\varepsilon \boldsymbol{y} s}{2}, t \right) \, \mathrm{d}s.$$

Then we can estimate

$$\begin{aligned}
\|R^{\varepsilon}(t)\|_{L^2(B_R \times B_R)} =& \int_{B_R} \int_{B_R} \left| \frac{\boldsymbol{y}}{2} \cdot \int_0^1 \nabla_{\boldsymbol{x}} M^{\varepsilon} \left(\boldsymbol{x} + \frac{\varepsilon \boldsymbol{y} s}{2}, t \right) \, \mathrm{d}s \right|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \\
\leq & \int_{B_R} \int_{B_R} \int_0^1 \left| \frac{\boldsymbol{y}}{2} \cdot \nabla_{\boldsymbol{x}} M^{\varepsilon} \left(\boldsymbol{x} + \frac{\varepsilon \boldsymbol{y} s}{2}, t \right) \right|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \\
\leq & C(R) \int_{B_R} \int_{B_R} \int_0^1 \left| \nabla_{\boldsymbol{x}} M^{\varepsilon} \left(\boldsymbol{x} + \frac{\varepsilon \boldsymbol{y} s}{2}, t \right) \right|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \\
= & C(R) \int_0^1 \int_{B_R} \int_{B_R} \left| \nabla_{\boldsymbol{x}} M^{\varepsilon} \left(\boldsymbol{x} + \frac{\varepsilon \boldsymbol{y} s}{2}, t \right) \right|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}s \\
\leq & C(R) \int_0^1 \int_{B_R} \|\nabla_{\boldsymbol{x}} M^{\varepsilon} \, (t) \, \|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}})} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}s \\
\leq & C(R) \|\nabla_{\boldsymbol{x}} M^{\varepsilon} \, (t) \, \|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}})}.
\end{aligned} \tag{4.18}$$

Since $\|\boldsymbol{m}^{\varepsilon}(t)\|_{H^1(\mathbb{R}^3_{\boldsymbol{x}})} \leq C$, we get

$$\|R^{\varepsilon}(t)\|_{L^2(B_R \times B_R)} \leq C(R). \tag{4.19}$$

And since we have $\|W^{\varepsilon}\|_{L^{\infty}((0,\infty), L^2(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}}))} \leq C$ and $\|R^{\varepsilon}(t)\|_{L^2(B_R \times B_R)} \leq C(R)$, then if taking the limit $\varepsilon \to 0$, we get

$$\lim_{\varepsilon \to 0} \varepsilon \iiint \mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[W^{\varepsilon}] R^{\varepsilon} \overline{\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[\phi]} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}t = 0_{2\times 2}. \tag{4.20}$$

Thus we have

$$\lim_{\varepsilon \to 0} \iiint \varGamma^{\varepsilon}_{+}[\boldsymbol{m}^{\varepsilon}] W^{\varepsilon} \phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}t = \iiint WM\phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}t. \tag{4.21}$$

Similarly we have $\lim_{\varepsilon \to 0} \iiint \varGamma^{\varepsilon}_{-}[\boldsymbol{m}^{\varepsilon}] W^{\varepsilon} \phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}t = \iiint MW\phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}t$, and that completes the proof.

□

Proof (Proof of Lemma 12) We define

$$\boldsymbol{m}^{\varepsilon,\beta} = \boldsymbol{m}^{\varepsilon} * \varphi^{\beta}$$

where $\varphi^\beta(\boldsymbol{x}) = \varphi(\boldsymbol{x}/\beta)$ and φ is a positive mollifier.

$$\begin{aligned}
&\left|\iiint \left(\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon - [M, W]\right)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \\
\le &\left|\iiint \Gamma^\varepsilon\left[\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{\varepsilon,\beta}\right]W^\varepsilon\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \\
&+ \left|\iiint \left(\Gamma^\varepsilon\left[\boldsymbol{m}^{\varepsilon,\beta}\right]W^\varepsilon - \left[M^\beta, W\right]\right)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \\
&+ \left|\iiint \left[M^\beta - M, W\right]\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|,
\end{aligned} \tag{4.22}$$

where we have use the notation $M^\beta = M * \varphi^\beta$. By the property of the mollifier function, the third term on the right hand side of (4.22) can be bounded by

$$\left|\iiint \left[M^\beta - M, W\right]\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \le C_\beta, \tag{4.23}$$

where C_β is a constant that goes to zero when β goes to zero. Since $\boldsymbol{m}^\varepsilon \in H^1(\Omega)$ and $\boldsymbol{m}^\varepsilon \equiv 0$ in Ω^c, we have $\boldsymbol{m}^{\varepsilon,\beta} \in H^1(\mathbb{R}^3_{\boldsymbol{x}})$. Further more, since as $\varepsilon \to 0$, $\boldsymbol{m}^\varepsilon \to \boldsymbol{m}$ in $L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})$ strongly, we have as $\varepsilon \to 0$, $\boldsymbol{m}^{\varepsilon,\beta} \to \boldsymbol{m}^\beta := \boldsymbol{m} * \varphi^\beta$ in $L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})$ strongly. Then we can apply Lemma 13 to $\boldsymbol{m}^{\varepsilon,\beta}$ and W^ε to get the limit of the second term on the right hand side of (4.22)

$$\lim_{\varepsilon\to 0} \iiint \left(\Gamma^\varepsilon\left[\boldsymbol{m}^{\varepsilon,\beta}\right]W^\varepsilon - \left[M^\beta, W\right]\right)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t = 0. \tag{4.24}$$

For the first term on the right hand side of (4.22), we have

$$\begin{aligned}
&\left|\iiint \Gamma^\varepsilon\left[\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{\varepsilon,\beta}\right]W^\varepsilon\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \\
\le &\left|\iiint \Gamma^\varepsilon\left[\boldsymbol{m}^\varepsilon - \boldsymbol{m}\right]W^\varepsilon\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| + \left|\iiint \Gamma^\varepsilon\left[\boldsymbol{m} - \boldsymbol{m}^\beta\right]W^\varepsilon\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \\
&+ \left|\iiint \Gamma^\varepsilon\left[\boldsymbol{m}^\beta - \boldsymbol{m}^{\varepsilon,\beta}\right]W^\varepsilon\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \\
\le &\, C(R)T\|\boldsymbol{m}^\varepsilon - \boldsymbol{m}\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})}\|W^\varepsilon\|_{L^\infty((0,\infty),L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}))} \\
&+ C(R)T\|\boldsymbol{m} - \boldsymbol{m}^\beta\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})}\|W^\varepsilon\|_{L^\infty((0,\infty),L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}))} \\
&+ C(R)T\|\boldsymbol{m}^\beta - \boldsymbol{m}^{\varepsilon,\beta}\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})}\|W^\varepsilon\|_{L^\infty((0,\infty),L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}))} \\
\le &\, C(R)T\|\boldsymbol{m}^\varepsilon - \boldsymbol{m}\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})} + C(R)T\|\boldsymbol{m} - \boldsymbol{m}^\beta\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})} \\
&+ C(R)T\|\boldsymbol{m}^\beta - \boldsymbol{m}^{\varepsilon,\beta}\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})},
\end{aligned} \tag{4.25}$$

where R is the radius of the support of $\mathcal{F}_{\boldsymbol{v},\boldsymbol{y}}[\phi]$. Using the Young's inequality for convolution and the fact that $\int \varphi^\beta \,\mathrm{d}\boldsymbol{x} = 1$, one gets

$$\|\boldsymbol{m}^\beta - \boldsymbol{m}^{\varepsilon,\beta}\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})} \le C\|\boldsymbol{m}^\varepsilon - \boldsymbol{m}\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})}. \tag{4.26}$$

Then since $\boldsymbol{m}^\varepsilon$ converge to $\boldsymbol{m}$ strongly in $L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})$, we have the first and third terms on the right hand side of (4.25) converge to zero as $\varepsilon\to 0$. By the property of the mollifier function,

$$\|\boldsymbol{m}-\boldsymbol{m}^\beta\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})}\le C_\beta. \tag{4.27}$$

Thus we have

$$\overline{\lim_{\varepsilon\to 0}}\left|\iiint \Gamma^\varepsilon\left[\boldsymbol{m}^\varepsilon-\boldsymbol{m}^{\varepsilon,\beta}\right]W^\varepsilon\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|\le C_\beta. \tag{4.28}$$

Then the estimates (4.22), (4.23), (4.24) and (4.28) yield

$$\overline{\lim_{\varepsilon\to 0}}\left|\iiint\big(\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon-[M,W]\big)\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|\le C_\beta. \tag{4.29}$$

But the left hand side of above inequality is independent of β, we then have

$$\lim_{\varepsilon\to 0}\left|\iiint\big(\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon-[M,W]\big)\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|=0, \tag{4.30}$$

which completes the proof of Lemma 12. □

In summary, by (4.4), Lemma 10 and Lemma 12, one can take $\varepsilon\to 0$ in (4.3) to get the semiclassical limit of the Schrödinger equation (1.1),

$$\iiint\left\{W(\partial_t\phi+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\phi-\nabla_{\boldsymbol{x}}V\cdot\nabla_{\boldsymbol{v}}\phi)+\frac{\mathrm{i}}{2}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m},\,W]\phi\right\}\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t=0. \tag{4.31}$$

Next we shall study the limit $\varepsilon\to 0$ of the LLG equation (1.6).

4.2 The limit of the Landau-Lifshitz equation as $\varepsilon\to 0$

Multiplying (1.6) by a test function ϕ in $C^\infty((0,\infty)\times\Omega)$ with compact support yields

$$\iint\partial_t\boldsymbol{m}^\varepsilon\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\alpha\iint\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\,\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t-\iint\boldsymbol{m}^\varepsilon\times\boldsymbol{H}^\varepsilon_{\mathrm{eff}}\,\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{4.32}$$

According to (4.1l) and (4.1h), $\boldsymbol{m}^\varepsilon\to\boldsymbol{m}$ and $\partial_t\boldsymbol{m}^\varepsilon\rightharpoonup\partial_t\boldsymbol{m}^\varepsilon$ in $L^2([0,T],L^2(\Omega))$ strongly and weakly *resp.*, and thus taking the limit $\varepsilon\to 0$ of the left-hand-side and the first term on the right-hand-side of (4.32) produces

$$\lim_{\varepsilon\to 0}\iint\partial_t\boldsymbol{m}^\varepsilon\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\iint\partial_t\boldsymbol{m}\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \tag{4.33}$$

$$\lim_{\varepsilon\to 0}\iint\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\iint\boldsymbol{m}\times\partial_t\boldsymbol{m}\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{4.34}$$

According to (1.9), one has

$$\begin{aligned}\iint \boldsymbol{m}^\varepsilon \times \boldsymbol{H}_{\mathrm{eff}}^\varepsilon \,\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = &- \iint \boldsymbol{m}^\varepsilon \times w'(\boldsymbol{m}^\varepsilon)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &- \iint \boldsymbol{m}^\varepsilon \times \nabla \boldsymbol{m}^\varepsilon \cdot \nabla\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+ \iint \boldsymbol{m}^\varepsilon \times \boldsymbol{H}_{\mathrm{s}}^\varepsilon \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+ \frac{\varepsilon}{2}\iint \boldsymbol{m}^\varepsilon \times \boldsymbol{s}^\varepsilon \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned} \tag{4.35}$$

By (4.1l), (4.1i), (4.1m), and (4.1g), one has

$$\lim_{\varepsilon\to 0}\iint \boldsymbol{m}^\varepsilon \times w'(\boldsymbol{m}^\varepsilon)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \iint \boldsymbol{m} \times w'(\boldsymbol{m}^\varepsilon)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{4.36a}$$

$$\lim_{\varepsilon\to 0}\iint \boldsymbol{m}^\varepsilon \times (\boldsymbol{H}_{\mathrm{s}}^\varepsilon + \boldsymbol{H}_0)\,\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \iint \boldsymbol{m} \times (\boldsymbol{H}_{\mathrm{s}} + \boldsymbol{H}_0)\,\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \tag{4.36b}$$

$$\lim_{\varepsilon\to 0}\iint \boldsymbol{m}^\varepsilon \times \nabla \boldsymbol{m}^\varepsilon \cdot \nabla\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \iint \boldsymbol{m} \times \nabla \boldsymbol{m} \cdot \nabla\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{4.36c}$$

Notice that (3.29) implies

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\iint \boldsymbol{m}^\varepsilon \times \boldsymbol{s}^\varepsilon \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \mathbf{0}, \tag{4.37}$$

Together with (4.36) and (4.37), we get from (4.35) that

$$\lim_{\varepsilon\to 0}\iint \boldsymbol{m}^\varepsilon \times \boldsymbol{H}_{\mathrm{eff}}^\varepsilon \,\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \iint \boldsymbol{m} \times \boldsymbol{H}_{\mathrm{eff}} \,\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \tag{4.38}$$

where $\boldsymbol{H}_{\mathrm{eff}} = \nabla w(\boldsymbol{m}) + \Delta \boldsymbol{m} + \boldsymbol{H}_{\mathrm{s}}$. Then by (4.33), (4.34), and (4.38), one can take the $\varepsilon \to 0$ limit in the LLG equation (1.6).

We summarize all results of the semiclassical limit of the SPLLG system (1.1)-(1.6) in the following theorem.

Theorem 3 *Under Assumptions 1 and 2, there exists a sequence of solutions $(W^\varepsilon, \boldsymbol{m}^\varepsilon)$ to the Wigner-Poisson-Landau-Lifshitz system* (3.7)*,* (1.4)*,* (1.6)*, and* (1.13) *such that*

$$\begin{aligned} W_I^\varepsilon &\xrightarrow{\varepsilon\to 0} W_I \ \text{ in } L^2(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}}) \text{ weakly}, \\ W^\varepsilon &\xrightarrow{\varepsilon\to 0} W \ \text{ in } L^\infty((0,\infty), L^2(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}})) \text{ weak*} \ , \\ V^\varepsilon &\xrightarrow{\varepsilon\to 0} V \ \text{ in } L^\infty((0,\infty), L^6(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak*} \ , \\ \nabla V^\varepsilon &\xrightarrow{\varepsilon\to 0} \nabla V \ \text{ in } L^\infty((0,\infty), L^2(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak*} \ , \\ \boldsymbol{m}^\varepsilon &\xrightarrow{\varepsilon\to 0} \boldsymbol{m} \ \text{ in } L^\infty((0,\infty), H^1(\Omega)) \text{ weak*} \ , \\ \rho^\varepsilon &\xrightarrow{\varepsilon\to 0} \rho \ \text{ in } L^\infty((0,\infty), L^q(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak*} \ , q \in [1, 7/5]\,, \end{aligned}$$

$$\boldsymbol{s}^{\varepsilon} \xrightarrow{\varepsilon\to 0} \boldsymbol{s} \text{ in } L^{\infty}((0,\infty), L^{s}(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak*} \ , s \in [1, 7/6]\,,$$
$$\boldsymbol{H}_s^{\varepsilon} \xrightarrow{\varepsilon\to 0} \boldsymbol{H}_s \text{ in } L^{\infty}((0,\infty), L^{2}(\Omega)) \text{ weak*} \ .$$

and for all $T > 0$,

$$\boldsymbol{m}^{\varepsilon} \xrightarrow{\varepsilon\to 0} \boldsymbol{m} \text{ in } L^{2}([0,T], L^{2}(\Omega)) \text{ strongly,}$$
$$\boldsymbol{H}_s^{\varepsilon} \xrightarrow{\varepsilon\to 0} \boldsymbol{H}_s \text{ in } L^{2}([0,T], L^{2}(\mathbb{R}^3_{\boldsymbol{x}})) \text{ strongly.}$$

Here W is a weak solution of the following Wigner equation

$$\partial_t W = -\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} W + \nabla_{\boldsymbol{x}} V \cdot \nabla_{\boldsymbol{v}} W + \frac{\mathrm{i}}{2}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m},\, W],$$

$\boldsymbol{m}$ is the weak solution of

$$\partial_t \boldsymbol{m} = -\boldsymbol{m}\times\boldsymbol{H}_{\mathit{eff}} + \alpha\boldsymbol{m}\times\partial_t\boldsymbol{m},$$

and the potential V, magnetic fields $\boldsymbol{H}_{\mathit{eff}}$ and $\boldsymbol{H}_s$ and densities ρ and $\boldsymbol{s}$ are given by

$$V = -N*\rho, \quad \boldsymbol{H}_{\mathit{eff}} = -w'(\boldsymbol{m}) + \Delta\boldsymbol{m} + \boldsymbol{H}_s,$$
$$\boldsymbol{H}_s = -\nabla\left(\nabla N * \cdot\boldsymbol{m}\right), \quad \rho = \int_{\mathbb{R}^3_{\boldsymbol{v}}} W \,\mathrm{d}\boldsymbol{v}, \quad \boldsymbol{s} = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}_{\mathbb{C}^2}(\hat{\boldsymbol{\sigma}} W)\,\mathrm{d}\boldsymbol{v},$$

with N given in (1.5).

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