Negligible Subsets in the Space Of Homeomorphisms

Raymond Y Wong

We show that for various compact metric spaces X, the space of homeomorphisms H(X) is homeomorphic to H(X)\K, where $K = \bigcup_{i>0} K_i \subset H(X)$ with each K_i is either (1) closed and equi-uniformly continuous or (2) topologically complete.

Our motivation for the study of negligible subsets is that, for a compact piecewise linear n-manifold M, it has been a long standing problem as to whether the space of homeomorphisms H(M) is an absolute neighborhood retract (The Homeomorphism Group Problem). It turns out ([GH]) that there is a dense G_δ subset $G \subset H(M)$ which is homeomorphic to a s-manifold, where s is the countable infinite product of open interval (-1, 1), and such that the complement H(M)\G is a countable union of closed sets $\{K_i\}$ each of which is a Z-set in the sense of [An] (it means that, for any homotopically trivial open set U in H(M), $U \setminus K_i$ remains homotopically trivial). It is therefore natural to ask whether the union $\bigcup_{i>0} K_i$ may be deleted from H(M). If the answer is yes, then H(M) is homeomorphic to G.

Notation. For a compact metric space (X, d), let C(X) denote the space of continuous functions of X into X. The metric defined on C(X) is $\rho(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}$. Without lost of generality, we may assume $\rho(f, g) \leq 1$. Let H(X) denote the subspace consisting of homeomorphisms of X onto X. H(X) is a topological group via compositions. Denote H*(X) = the closure of H(X) in C(X). To say $K_i \subset H(X)$ is

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equi- uniformly continuous we mean that, for any given $\varepsilon > 0$, there exists a $\delta_i > 0$, such that for all f ε K, $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta_i$. Finally K H(X) is said to be negligible if H(X) is homeomorphic to H(X)\K.

Theorem 1. Let X R^m be the m unit-ball of the Euclidean m-space R^m. Denote by $H_\partial(X)$ the subspace consisting of all f H(X) which is the identity on the boundary ∂X of X. Suppose K = $_{i>0}$ K_i $H_\partial(X)$, where each K_i is closed and equi-uniformly continuous, then $H_\partial(X)$ is homeomorphic to $H_\partial(X)\backslash K$.

Proof. The proof require the concept of Morse's μ-length of paths ([Mo]): Let f : $[a, b] \rightarrow X$ be a path in a metric space (X, d) where a < b. For each positive integer n, let

$A_n = \{(t_0, t_1, \cdots, t_n) \mid a \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq b\}$. For each $(t_0, t_1, \cdots, t_n)$ A_n, define $\delta(f ; t_0, t_1, \cdots, t_n) = \min\{d((f(t_i), f(t_{i-1})) \mid i = 1, 2, \cdots, n\}$ and

$$\mu_n(f) = \sup\{\delta(f ; t_0, t_1, \cdots, t_n) \mid (t_0, t_1, \cdots, t_n) \;\; A_n\}.$$

The μ-length of f is $$\mu(f) = \sum_{n=1}^{\infty} (1/2^n)\, \mu_n(f).$$

We first consider a single closed and equi-uniformly continuous subset K $H_\partial(X)$. Let J = [a, b] X denote an interval with a < b and with a, b ∂X. For each f K, by Lemma 1.3 of [Sa-Wo], there is a point t_f, $a < t_f < b$, depending continuously on f, such that $\mu(f|_{[a, t_f]}) = \mu(f|_{[t_f, b]})$.

For any subset A X, Denote Mesh(A) = sup{d(x, y) | x, y ε A}. The main goal of the following lemma is to establish (d).

Lemma 1. (a) For any a < r < b, $\mu(f|_{[a, r]}) \leq \text{Mesh}(f([a, r]))$.

(b) For any a < r < r' < b, $\mu(f|_{[a, r]}) < \mu(f|_{[a, r']})$.

(c) Let E denote the identity map on X. then

$$\mu(E|_{[a, b]}) \leq \mu(f|_{[a, b]}) \leq 2\mu(f|_{[a, t_f]})$$

(d) there is a point r, a < r < b, such that $t_f > r$ for all f K.

Proof. (a) For any n, $\delta(f|_{[a,r]}; t_0, t_1, \ldots, t_n) \leq$ Mesh(f([a, r]). Hence $\mu_n(f) \leq$ Mesh(f([a, r]). It follows that $\mu(f|_{[a,r]}) \leq$ Mesh(f([a, r]).

The first inequality of (c) is true since a, b ∂X and each f K fixes the end-points a, b. The rest of the proof (b) and (c) is a straightforward application of the definition of the μ-length of f and the triangle inequality of the metric d.

(d) Let $\varepsilon = (1/3)\mu(E|_{[a,b]})$. Since K is equi-uniformly continuous, there exists $\delta > 0$, such that for all f ε K, $d(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$. Consider a point r, a < r < b, such that $d(a, r) < \delta$. Since the metric d is Euclidean, $d(x, y) \leq d(a, r) < \delta$ for all x, y [a, b]. It follows for all f K, $d(f(x), f(y)) < \varepsilon$. Thus $\mu(f|_{[a,r]}) \leq$ Mesh(f([a, r]) $\leq \varepsilon$ by Lemma 1(a). Using Lemma1(c), we have $\mu(f|_{[a,r]}) < \mu(f|_{[a,t_f]})$. Hence $r < t_f$.

Inside the m-ball X we can construct a countable, mutually disjoint collection of intervals $[a_i, b_i]$ (actually a line segment being regarded as an interval), with a_i, b_i ∂X and the diameter$(b_i - a_i)$ converges to 0. Mimicking the construction of f K on [a, b], we consider the construction of f K on each $[a_i, b_i]$. Using lemma 1(d), we can choose, for each fix i, a point r_i, $a_i < r_i < b_i$, such that $t_{i\,f} > r_i$ for all f ε K. Recall that $t_{i\,f}$ is the point in the interval (a_i, b_i) for which $\mu(f|_{[a_i, t_{if}]}) = \mu(f|_{[t_{if}, b_i]})$. Denote $s = (a_i, b_i)^\infty$.

Let $H_0(X) = \{ f \;\; H_\partial(X) \mid$ for all i, $\mu(f|_{[a_i, m_i]}) = \mu(f|_{[m_i, b_i]})$, where m_i = mid-point of $[a_i, b_i]\}$. As demonstrated in [Sa-Wo], there is a homeomorphism $\phi : H_\partial(X) \to H_0(X) \times s$ taking f $H_\partial(X)$ to the point $\phi(f) = (f', t_f)$, where f' ε $H_0(X)$ and $t_f = (t_{i\,f})$ s.

Thus for f K, $\phi(f) = (f', t_f)$ with each $t_{i\,f} > r_i > a_i$. In other words, the image $\phi(K)$ is a closed set whose projection into each factor (a_i, b_i) is contained in $[r_i, b_i)$. By the techniques of infinite-dimensional (I-D) topology ([An]), there is homeomorphism $\Gamma : (H_0(X) \times s)\backslash K \to H_0(X) \times s$ and the homeomorphism changes only the s-coordinates of each point.

Now suppose K = $_{i>0} K_i$. The fact that we can delete the infinite sequence $\{K_i\}$ from $H_0(X) \times s$ is also a result of I-D topology techniques. Basically we write $s = s_1 \times s_2 \times \cdots$, an infinite product with each s_i a copy of s. We then delete each K_i from $H_0(X) \times s$ changing only the s_i-coordinates of each point. Collectively we can construct a homeomorphism taking $(H_0(X) \times s)\backslash K$ onto $H_0(X) \times s$.

Theorem 2. Let X be a compact metric space containing a closed neighborhood N homeomorphic to some k-simplex. For K H(X), H(X) is homeomorphic to H(X)\K provided K is a countable union of topologically complete subsets $_{i>0} K_i$.

Remark. Employing Bessaga's ([Be]) approach, in [Do], Dobrowolski shows that for a compact subset K E, K is negligible in E for a list of spaces E. Including in the list is E = H(X), where X is a locally compact space containing a bicollared set. Using different approach, a similar result on H(X) was proved by Mason ([Ma]). A key step in our argument required each K_i to be topologically complete, a condition weaker than compactness but not as weak as being locally homotopically trivial, which is required to settle the Homeomorphism Group Problem.

Proof. The proof is the results of the following three steps.

(1) *Construction of a pinching map* λ. The main idea is to construct a sequence of paths each starting from a point (identity) in H(X) and ending with a point (a pinching map) in H*(X)\H(X). We use these paths to push $\{K_i\}$ out of H(X). First of all, inside N we may assume it contains an interval J = [a, b] with 0 < b – a < 1. We choose a point r, a < r < b. Starting with identity, the idea is to shrink the interval [r, b] onto the point b. In other words, we construct a path $\{\lambda_t\}$ satisfying (i) each λ_t is the identity outside N, (ii) λ_0 = identity, $\lambda_t([a, b]) = [a, b]$, $\lambda_t|_N$ is a canonical piecewise linear map that shrinks the interval [r, b] onto the interval $[(1 - t)r + t\,b,\ b]$ and (iii) for $t < 1$, each λ_t H(X) and $\lambda_1$collapses the entire interval [r, b] onto b.

Next along J we construct a countable, mutually disjoint collection of k-simplices $\{N_i\}$, each a smaller version of N, with diameter(N_i) converges to 0. For each i

> 0, we choose an interval $[a_i, b_i]$ N_i with $a_i < b_i$. We also choose, for each i, a point r_i, $a_i < r_i < b_i$. We then construct a sequence of paths $\{\lambda_{i\,t}\}_{i\,,}$ such that for each i, the homotopy $\{\lambda_{i\,t}\}$ take place in N_i completely analogous to the homotopy $\{\lambda_t\}$ in N. Define $\lambda : [0, 1]^\infty \rightarrow H^*(X)$ as follow. For each point $t = (t_i)_{i>0}$, $\lambda(t)$ is the map which is the identity outside $_{>0}$ N_i and such that for each i, $\lambda(t)|_{N_i} = \lambda_{i\,t_i}|_{N_i}$. Clearly λ satisfies the following

Lemma 2. (i) Denote $Q = \lambda([0, 1]^\infty)$. Then Q $H^*(X)$ and $\lambda([0, 1)^\infty)$ $H(X)$,

(ii) for $\beta = [0, 1]^\infty \setminus [0, 1)^\infty$, $\lambda(\beta)$ $H^*(X)\setminus H(X)$,

(iii) for all $t = (t_i)_{i>0}$, $t' = (t'_i)_{i>0}$, $\rho(\lambda(t), \lambda(t')) = \sup_{i>0}| t_i - t_i'|$,

(iv) λ is an imbedding and therefore the images Q is an absolute retract.

(2) *Construction of a contractive map* φ. We say a map $\phi : C(X) \rightarrow C(X)$ is <u>contractive</u> if there is some number $0 \leq r < 1$, such that $\rho(\phi(f), \phi(g)) \leq r\rho(f, g)$ for all f, g ε C(X). Now let K = $_{i>0}K_i$ H(X) be given such that each K_i is a complete subset of H(X). Let $r = b - a < 1$. For any f ε $H^*(X)$, denote $t_i = 1 - r\rho(f, K_i)$. Thus $0 \leq t_i \leq 1$ (recall that we assume the metric ρ is bounded by 1) and that $t_i = 1$ if and only if f ε K_i .Let $t = (t_i)_{i>0}$. Define $\phi : H^*(X) \rightarrow H^*(X)$ by $\phi(f) = \lambda(t)$.

Lemma 3. (i) $\phi(H^*(X))$ Q $(= \lambda([0, 1]^\infty))$,

(ii) $\phi(K)$ $H^*(X)\setminus H(X)$,

(iii) $\phi(H(X)\setminus K)$ H(X) and

(iv) ϕ satisfies $\rho(\phi(f), \phi(g)) \leq r\rho(f, g)$, for all f, g   H*.

Proof. (i)-(iii) is clear. To verify (iv), denote $t = (t_i)_{i>0}$ and $t' = (t_i')_{i>0}$, where $t_i = 1 - r\rho(f, K_i)$ and $t_i' = 1 - r\rho(g, K_i)$. By Lemma 1(iii), $\rho(\phi(f), \phi(g)) = \rho(\lambda_t, \lambda_{t'}) = \sup_{i>0}| t_i - t_i'| = \sup_{i>0} r\,|\rho(f, K_i) - \rho(g, K_i)|$. Since $|\rho(f, K_i) - \rho(g, K_i)| \leq \rho(f, g)$ for all i, $\rho(\phi(f), \phi(g)) \leq r\rho(f, g)$.

(3) *A homeomorphism of H(X)\K onto H(X)*. Let $\phi : H^*(X) \rightarrow H^*(X)$ be defined as in (2) above. Define $\Gamma : H(X)\setminus K \rightarrow H(X)$ by $\Gamma(f) = \phi(f)^{-1}f$. We will show that Γ is a

homeomorphism onto H. The condition that each K_i to be topologically complete is a key requirement to show that Γ is surjective.

First of all, the map Γ is well-defined since for $f \quad K$, $\phi(f) \quad H(X)$ (Lemma 3(iii)), so $\phi(f)^{-1}$ exists. Composition and inverse operations in H(X) are continuous, so Γ is continuous. Secondly, it is straightforward to verify that ρ is invariant under right multiplication; that is, for any $h \quad H(X)$, $\rho(f, g) = \rho(fh, gh)$ for all $f, g \quad H(X)$.

To show Γ is one-to-one, suppose $\Gamma(f) = \Gamma(g)$ for $f, g \quad H(X)\backslash K$. Then $\phi(f)^{-1}f = \phi(g)^{-1} g$, or $\phi(g)\phi(f)^{-1} = gf^{-1}$. Denote e = identity. We have $\rho(\phi(g), \phi(f)) = \rho(\phi(g)\phi(f)^{-1}, e) = \rho(gf^{-1}, e) = \rho(g, f)$. Since ϕ is a contractive map (Lemma 3(iv)), f = g.

To show Γ is onto, let $g_0 \quad H(X)$ be given. Consider the map $\eta : H^*(X) \to H^*(X)$ defined by $\eta(f) = \phi(f)g_0$. By Lemma 3(i), the images $\eta(H^*(X)) = \phi(H^*(X))g_0 \quad Qg_0$. It follows that the restriction $\eta|_{Qg_0} : Qg_0 \to Qg_0$. Since Qg_0 is an absolute retract by Lemma 2(iv), η have a fixed point, say $f_0 \quad Qg_0$ such that $\eta(f_0) = f_0$.

We want to assert $f_0 \quad \lambda(\beta)g_0$ by showing that any $f \quad \lambda(\beta)g_0$ is not a fixed point of $\eta|_{Qg_0}$. Given any i > 0. Since $\lambda(\beta)g_0 \quad H^*(X)\backslash H(X)$ and $K_i \quad H(X)$ is topologically complete, K_i is a closed relative to H*. Thus any $f \quad \lambda(\beta)g_0$ must have a positive distance from K_i. Let $t_i = 1 - r\rho(f, K_i)$ and denote $t = (t_i)_{i>0}$. It follows $t \quad [0, 1)^{\infty}$. By Lemma 2(i), $\eta(f) = \phi(f)g_0 = \lambda(t)g_0 \quad H(X)$. In other words, $\eta(f) \neq f$ and so $f_0 \quad H(X)$.

Next we assert that $f_0 \quad K$. For if $f_0 \quad K$, then $f_0 = \eta(f_0) = \phi(f_0)g_0 \quad \lambda(\beta)g_0$, a contradiction. Thus $f_0 \quad H(X)\backslash K$ and $f_0 = \eta(f_0) = \phi(f_0)g_0$, or $\Gamma(f_0) = \phi(f_0)^{-1}f_0 = g_0$.

The verification that Γ^{-1} is continuous is rather straightforward and will be omitted. Thus $\Gamma : H(X)\backslash K \to H(X)$ is a homeomorphism and the proof of Theorem 2 is complete.

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E-mail: wong@math.ucsb.edu

Math Department, University of California, Santa Barbara, CA 93106