122B Midterm Spring 2009

May 6, Wednesday

50 points total Name Solutions

Your Total

1 2 3 4 5

1 (10 pts) Find the residues of the following functions at the points indicated:
a) $f(z) = \frac{\sin z+1}{z^4}$ at the point $z = 0$

$z = 0$ is a pole of order 4

So $\operatorname{Res}_{z=0} \left(\frac{\sin z+1}{z^4}\right) = \frac{(\sin z+1)'''\big|_{z=0}}{3!} = \boxed{-\frac{1}{6}}$

b) $f(z) = (z+1)/(z^2+2z-3)$ at the point $z = 1$.

$f(z) = \frac{z+1}{(z-1)(z+3)}$. $z=0$ is a pole of order 1

$\operatorname{Res}_{z=1} f(z) = \frac{2}{4} = \boxed{\frac{1}{2}}$

2 (10 pts) Write the Laurent expansion of the function $f(z) = \frac{1}{z(1+z^2)}$ that is valid for a) $0 < |z| < 1$; b) $1 < |z|$.

a) $f(z) = \frac{1}{z} \frac{1}{1-(-z^2)} = \frac{1}{z} \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n-1}$ $\quad 0 < |z| < 1$

b) $f(z) = \frac{1}{z} \cdot \frac{1}{z^2\left(1-\left(-\frac{1}{z^2}\right)\right)}$

$= \frac{1}{z^3} \sum_{n=0}^{\infty} \left(-\frac{1}{z^2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n+3}}$ $\quad |z| > 1$

3 (10 pts) a) Find the residues of the function

$$f(z) = \frac{e^z}{\sin z}$$

at each of its poles

$f(z)$ has poles at $z = k\pi$, $k = 0, \pm 1, \pm 2, \cdots$

They are simple poles

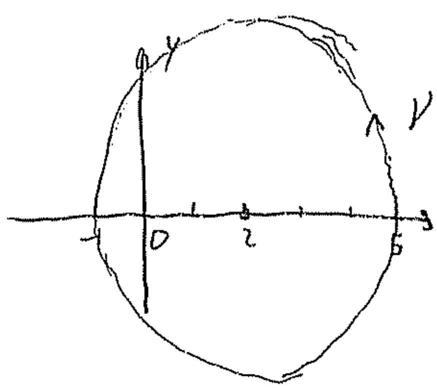
$$\operatorname*{Res}_{z=k\pi} f(z) = \frac{e^{k\pi}}{\cos k\pi} = (-1)^k e^{k\pi}$$

$k = 0, \pm 1, \pm 2, \cdots$

b) Use the residue theorem to evaluate the contour integral

$$\int_\gamma \frac{e^z}{\sin z} dz,$$

where γ is the circle of radius 3 centered at the point $z = 2$.



The singular points inside the contour are 0, π

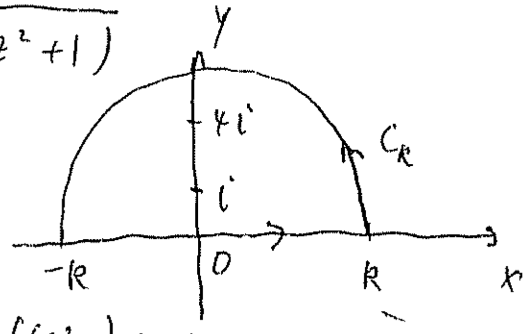
By Residue thm

$$\int_\gamma \frac{e^z}{\sin z} dz = 2\pi i \left(1 + (-e^\pi)\right)$$
$$= 2\pi i (1 - e^\pi)$$

4 (10 pts) Use the residue theorem to evaluate the following integral

$$\int_0^\infty \frac{x^2}{(x^2+16)(x^2+1)}\,dx$$

Solution: Let $f(z) = \dfrac{z^2}{(z^2+16)(z^2+1)}$



Then $\displaystyle\int_{-R}^{R} \frac{x^2}{(x^2+16)(x^2+1)}\,dx$

$$+ \int_{C_R} f(z)\,dz = \left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=4i} f(z)\right) 2\pi i$$

$z = i,\ 4i$ are simple poles

$$\operatorname*{Res}_{z=i} f(z) = -\frac{1}{30i} \qquad \operatorname*{Res}_{z=4i} f(z) = \frac{2}{15i}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^2}{(R^2-16)(R^2-1)} \cdot \pi R \longrightarrow 0 \text{ as } R \to \infty$$

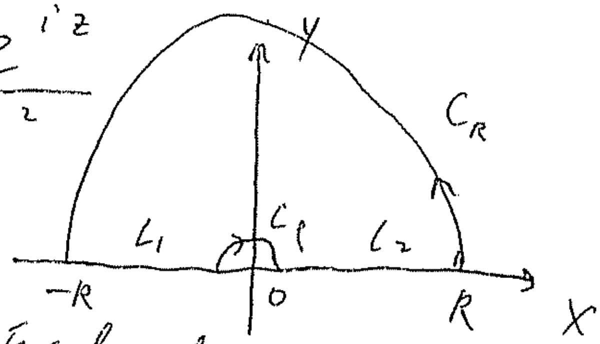
So $$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+16)(x^2+1)}\,dx = 2\pi i\left(\frac{2}{15i} - \frac{1}{30i}\right)$$
$$= \frac{\pi}{5}$$

and $$\int_0^\infty \frac{x^2}{(x^2+16)(x^2+1)}\,dx = \boxed{\frac{\pi}{10}}$$

5 (10 pts) Use the residue theorem to evaluate the integral

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx$$

Solution: Let $f(z) = \dfrac{1-e^{iz}}{z^2}$



$f(z)$ has simple pole at $z=0$ and analytic inside the contour

So $\displaystyle\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz + \int_{C_R} f(z)\,dz = 0$

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = -\pi i \operatorname*{Res}_{z=0} f(z) = \pi i^2 = -\pi$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{2}{R^2}\cdot \pi R \to 0 \quad \text{as } R\to\infty$$

$$\int_{L_1} f(z)\,dz = \int_\rho^R \frac{1-e^{-ir}}{r^2}\,dr, \quad \int_{L_2} f(z)\,dz = \int_\rho^R \frac{1-e^{ir}}{r^2}\,dr$$

So $\displaystyle\int_0^\infty \frac{2-e^{-ir}-e^{ir}}{r^2}\,dr - \pi = 0$

$$\int_0^\infty \frac{2-2\cos r}{r^2}\,dr = \pi$$

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \boxed{\frac{\pi}{2}}$$