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ON THE FUNDAMENTAL GROUPS OF MANIFOLDS WITH ALMOST-NONNEGATIVE RICCI CURVATURE

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ABSTRACT. We give an upper bound on the growth of $\pi_1(M)$ for a class of manifolds M with Ricci curvature $\mathrm{Ric}_M \geq -\varepsilon$, diameter $d(M) = 1$, and volume $\mathrm{vol}(M) \geq v$.

In [4], Milnor proved that every finitely generated subgroup of the fundamental group of a manifold M^n with nonnegative Ricci curvature is of polynomial growth with degree $\leq n$. It is conjectured by Gromov [2] that the fundamental group of a near-elliptic manifold (in the sense of Gromov) is of polynomial growth. The purpose of this note is to present the following theorem.

Theorem 1. *For any constant $v > 0$, there exists $\varepsilon = \varepsilon(n, v) > 0$ such that if a complete manifold M^n admits a metric satisfying the conditions $\mathrm{Ric}_M \geq -\varepsilon$, $d(M) = 1$, and $\mathrm{vol}(M) \geq v$, then the fundamental group of M is of polynomial growth with degree $\leq n$.*

Our proof depends essentially on a recent result of M. Anderson [1].

Theorem 2 (M. Anderson). *In the class of compact n-dimensional Riemannian manifolds M such that $\mathrm{Ric}_M \geq (n-1)H$, $\mathrm{vol}(M) \geq v$, and $d(M) \leq D$, there are only finitely many isomorphism classes of $\pi_1(M)$.*

Proof of Theorem 1. Choose a base point $\tilde{x}_0$ in the universal covering $\widetilde{M} \xrightarrow{p} M$, and let $x_0 = p(\tilde{x}_0)$ and $g_1, \dots, g_r$ be a set of generators of the fundamental group $\pi_1(M)$ viewed as deck transformations in the isometry group of $\widetilde{M}$. Denote $\Gamma(s) = \{$ distinct words in $\pi_1(M)$ of length $\leq s\}$, $\gamma(s) = \#\Gamma(s)$, and $l = \max_{1\leq i\leq r}\{d(\tilde{x}_0, g_i(\tilde{x}_0))\}$.

Choose a fundamental domain F of $\pi_1(M)$ which contains $\tilde{x}_0$; then

$$\bigcup_{g\in\Gamma(s)} g(F) \subset B_{sl+d}(\tilde{x}_0),$$

where $d = d(M) = 1$. Therefore,

$$\gamma(s)\cdot\mathrm{vol}(M) \leq \mathrm{vol}\left(B_{sl+1}(\tilde{x}_0)\right). \tag{1}$$

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Now suppose, on the contrary, that for any $\varepsilon > 0$, there is a manifold M^n with a metric satisfying $\mathrm{Ric}_M \geq -\varepsilon$, $d(M) = 1$, $\mathrm{vol}(M) \geq v$, and $\pi_1(M)$ is not of polynomial growth with degree $\leq n$. By the proof of Theorem 2, $\pi_1(M)$ has a presentation which obeys the following:

(1) The number of generators $g_1, \ldots, g_N$ is uniformly bounded with $N \leq N(v/D^n, HD^2)$,
(2) $d(g_i(\tilde{x}_0), \tilde{x}_0) \leq 3D$,
(3) every relation is of the form $g_i g_j = g_k$.

The statements (2) and (3) have already been proved by Gromov [3]. By our assumption, $\pi_1(M)$ is not of polynomial growth with degree $\leq n$. In particular, when taking the above generators, we can find real numbers s_i for all i such that

$$\gamma(s_i) > i s_i^n . \tag{2}$$

It is crucial that this relation is independent of ε, as follows from (1) and (3).

On the other hand, by (1) we have

$$\gamma(s) \leq \frac{1}{v} \int_0^{3s+1} \left(\frac{\sinh \sqrt{\varepsilon t}}{\sqrt{\varepsilon}} \right)^{n-1} dt .$$

For any fixed, sufficiently large s_0, there is $\varepsilon_0 = \varepsilon(s_0)$ such that for all $s \leq s_0$, $\varepsilon \leq \varepsilon_0$,

$$\gamma(s) \leq \frac{6^n}{nv} s^n . \tag{3}$$

Now take $i_0 > 6^n/nv$. Then for $\varepsilon < \varepsilon(s_{i_0})$, using (2) and (3), we get a contradiction.

We would like to mention that Peter Peterson, working from a different orientation and with different technique, has obtained a slightly weaker result. Instead of a lower volume bound, he imposes a lower bound on the contractibility radius and arrives at the same conclusion.

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