Comparison Geometry for the Bakry-Emery Ricci Tensor

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Abstract

For Riemannian manifolds with a measure $(M, g, e^{-f} dvol_g)$ we prove mean curvature and volume comparison results when the ∞-Bakry-Emery Ricci tensor is bounded from below and f is bounded or $\partial_r f$ is bounded from below, generalizing the classical ones (i.e. when f is constant). This leads to extensions of many theorems for Ricci curvature bounded below to the Bakry-Emery Ricci tensor. In particular, we give extensions of all of the major comparison theorems when f is bounded. Simple examples show the bound on f is necessary for these results.

1 Introduction

In this paper we study smooth metric measure spaces $(M^n, g, e^{-f} dvol_g)$, where M is a complete n-dimensional Riemannian manifold with metric g, f is a smooth real valued function on M, and $dvol_g$ is the Riemannian volume density on M. In this paper by the Bakry-Emery Ricci tensor we mean

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f.$$

This is often also referred to as the ∞-Bakry-Emery Ricci Tensor. Bakry and Emery [4] extensively studied (and generalized) this tensor and its relationship to diffusion processes. The Bakry-Emery tensor also occurs naturally in many different subjects, see e.g. [23] and [29, 1.3]. The equation $\mathrm{Ric}_f = \lambda g$ for some constant λ is exactly the gradient Ricci soliton equation, which plays an important role in the theory of Ricci flow. Moreover Ric_f has a natural extension to metric measure spaces [22, 37, 38].

When f is a constant function, the Bakry-Emery Ricci tensor is the Ricci tensor so it is natural to investigate what geometric and topological results for the Ricci tensor extend to the Bakry-Emery Ricci tensor. Interestingly, Lichnerowicz [20] studied this problem at least 10 years before the work of Bakry and Emery. This has also been actively investigated recently and there are a number of interesting results in this direction which we will discuss below. Also see [8] for another modification of the Ricci tensor and Appendix A for a discussion of the N-Bakry-Emery Ricci tensor Ric_f^N (see (1.6) for the definition). We thank Matthew Gursky for making us aware of [20].

Although there is a Bochner formula for the Bakry-Emery Ricci tensor [23] (see also (5.10)), the other basic geometric tools for Ricci curvature do not extend for a completely general function f. In Section 7 we give a quick overview with examples where the Myers' theorem, Bishop-Gromov's volume comparison, Cheeger-Gromoll's splitting theorem, and Abresch-Gromoll's excess estimate are not true even though Ric_f has the appropriate lower bound. In this paper we are concerned with finding conditions on f that imply versions of these theorems. In particular, we give versions of these theorems when f or the radial derivative of f is bounded.[1] These results give new tools for

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[1] After writing the original version of this paper, we learned that Lichnerowicz had proven the splitting theorem for f bounded. We think this result is very interesting and does not seem to be well known in the literature, so we have retained our complete proof here.

studying the Bakry-Emery tensor and lead to generalizations of many of the classical topological and geometric theorems for manifolds with a lower Ricci curvature bound, and generalize all previous topological results for the Bakry-Emery tensor.

For Ricci curvature all of the theorems listed above can be proven from the mean curvature (or Laplacian) comparison, see [46]. Recall that the mean curvature measures the relative rate of change of the volume element. Therefore, for the measure $e^{-f}dvol$, the associated mean curvature is $m_f = m - \partial_r f$, where m is the mean curvature of the geodesic sphere with inward pointing normal vector. Note that $m_f = \Delta_f(r)$, where r is the distance function and $\Delta_f = \Delta - \nabla f \cdot \nabla$ is the naturally associated (f-)Laplacian which is self-adjoint with respect to the weighted measure.

In this paper we prove three mean curvature comparisons. The first (see Theorem 2.1) is the most general and is quite simple to prove. Still, it has some interesting applications for manifolds with positive Bakry-Emery tensor (Corollaries 4.1 and 4.2). The other two are more delicate and have many applications.

Theorem 1.1 (Mean Curvature Comparison.) *Let $p \in M^n$ Assume $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$,*

a) if $\partial_r f \geq -a$ $(a \geq 0)$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/2\sqrt{H}$) then

$$m_f(r) - m_H(r) \leq a \tag{1.1}$$

along that minimal geodesic segment from p. Equality holds if and only if the radial sectional curvatures are equal to H and $f(t) = f(p) - at$ for all $t < r$.

b) if $|f| \leq k$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/4\sqrt{H}$) then

$$m_f(r) \leq m_H^{n+4k}(r) \tag{1.2}$$

along that minimal geodesic segment from p. In particular when $H = 0$ we have

$$m_f(r) \leq \frac{n+4k-1}{r} \tag{1.3}$$

Here m_H^{n+4k} is the mean curvature of the geodesic sphere in M_H^{n+4k}, the simply connected model space of dimension $n+4k$ with constant curvature H and m_H is the mean curvature of the model space of dimension n. See (2.20) in Section 2 for the case $H > 0$ and $r \in [\frac{\pi}{4\sqrt{H}}, \frac{\pi}{2\sqrt{H}}]$ in part b.

As in the classical case, these mean curvature comparisons have many applications. First, we have volume comparison theorems. Let $\mathrm{Vol}_f(B(p,r)) = \int_{B(p,r)} e^{-f} dvol_g$, the weighted (or f-)volume and $\mathrm{Vol}_H^n(r)$ be the volume of the radius r-ball in the model space M_H^n.

Theorem 1.2 (Volume Comparison.) *Let $(M^n, g, e^{-f}dvol_g)$ be complete smooth metric measure space with $\mathrm{Ric}_f \geq (n-1)H$. Fix $p \in M^n$.*

a) If $\partial_r f \geq -a$ along all minimal geodesic segments from p then for $R \geq r > 0$ (assume $R \leq \pi/2\sqrt{H}$ if $H > 0$) ,

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq e^{aR}\frac{\mathrm{Vol}_H^n(R)}{\mathrm{Vol}_H^n(r)}. \tag{1.4}$$

Moreover, equality holds if and only if the radial sectional curvatures are equal to H and $\partial_r f \equiv -a$. In particular if $\partial_r f \geq 0$ and $\mathrm{Ric}_f \geq 0$ then M has f-volume growth of degree at most n.

b) If $|f(x)| \leq k$ then for $R \geq r > 0$ (assume $R \leq \pi/4\sqrt{H}$ if $H > 0$),

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq \frac{\mathrm{Vol}_H^{n+4k}(R)}{\mathrm{Vol}_H^{n+4k}(r)}. \tag{1.5}$$

In particular, if f is bounded and $\mathrm{Ric}_f \geq 0$ then M has polynomial f-volume growth.

Remark 1 When $\mathrm{Ric}_f \geq 0$ the condition f is bounded or $\partial_r f \geq 0$ is necessary to show polynomial f-volume growth as shown by Example 7.4. Similar statements are true for the volume of tubular neighborhood of a hypersurface. See Section 3 for another version of volume comparison which holds for all $r > 0$ even when $H > 0$.
Remark 2 To prove the theorem we only need a lower bound on Ric_f along the radial directions. Given any manifold M^n with Ricci curvature bounded from below one can always choose suitable f to get any lower bound for Ric_f along the radial directions. For example if $\mathrm{Ric} \geq -1$ and $p \in M$, if we choose $f(x) = r^2 = d^2(p,x)$, then $\mathrm{Ric}_f(\partial_r, \partial_r) \geq 1$. Also see Example 7.3.
Remark 3 Volume comparison theorems have been proven for manifolds with N-Bakry Emery Ricci tensor bounded below. See Qian [34], Bakry-Qian [6], Lott [23], and Appendix A. The N-Bakry Emery Ricci tensor is

$$\mathrm{Ric}_f^N = \mathrm{Ric}_f - \frac{1}{N} df \otimes df \quad \text{for } N > 0. \tag{1.6}$$

For example, Qian shows that if $\mathrm{Ric}_f^N \geq 0$ then $\mathrm{Vol}_f(B(p,r))$ is of polynomial growth with degree $\leq n + N$. Note that $\mathrm{Ric}_f = \mathrm{Ric}_f^\infty$ so one does not expect polynomial volume growth for $\mathrm{Ric}_f \geq 0$. Since $\mathrm{Ric}_f^N \geq 0$ implies $\mathrm{Ric}_f \geq 0$ our result greatly improves the volume comparison result of Qian when N is big and f is bounded, or when $\partial_r f \geq 0$.

The mean curvature and volume comparison theorems have many other applications. We highlight two extensions of theorems of Calabi-Yau [44] and Myers' to the case where f is bounded.

Theorem 1.3 *If M is a noncompact, complete manifold with $\mathrm{Ric}_f \geq 0$ for some bounded f then M has at least linear f-volume growth.*

Theorem 1.4 (Myers' Theorem) *If M has $\mathrm{Ric}_f \geq (n-1)H > 0$ and $|f| \leq k$ then M is compact and $\mathrm{diam}_M \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}$.*

Examples 7.1 and 7.2 show that the assumption of bounded f is necessary in both theorems. Qian [34] has proven versions of both theorems for Ric_f^N. For other Myers' theorems see [12, 45, 19, 26].

The paper is organized as follows. In the next section we state and prove the mean curvature comparisons. In Sections 3 and 4 we prove the volume comparison theorems and discuss their applications, including Theorem 1.3. In Section 5 we apply the mean curvature comparison to prove the splitting theorem for the Bakry-Emery tensor that is originally due to Lichnerowicz. In Section 6 we discuss some other applications of the mean curvature comparison including the Myers' theorem and an extension Abresch-Gromoll's excess estimate to Ric_f. In Section 7 we discuss examples and questions. Finally in Appendix A we state the mean curvature comparison for Ric_f^N. This is a special case of an estimate in [6], but we have written the result in more Riemannian geometry friendly language. This gives other proofs of the comparison theorems for Ric_f^N mentioned above.

After posting the original version of this paper we learned from Fang, Li, and Zhang about their work which is closely related to some of our work here. We thank them for sharing their work with us. Their paper is now posted, see [10]. Motivated from their paper we were able to strengthen the original version of Theorem 1.1 and Theorem 1.2 and give a new proof to Theorem 1.1. This proof of the mean curvature comparison seems to us to be new even in the classical Ricci curvature case. We have moved our original proof using ODE methods to an appendix because we feel it might be useful in other applications.

From the work of [32] one expects that the volume comparison and splitting theorem can be extended to the case that Ric_f is bounded from below in the integral sense. We also expect similar versions for metric measure spaces. These will be treated in separate paper.

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2 Mean Curvature Comparisons

In this section we prove the mean curvature comparison theorems. First we give a rough estimate on m_f which is useful when $\mathrm{Ric}_f \geq \lambda g$ and $\lambda > 0$.

Theorem 2.1 (Mean Curvature Comparison I.) *If* $\mathrm{Ric}_f(\partial_r, \partial_r) \geq \lambda$ *then given any minimal geodesic segment and* $r_0 > 0$*,*

$$m_f(r) \leq m_f(r_0) - \lambda(r - r_0) \quad \text{for } r \geq r_0. \tag{2.1}$$

Equality holds for some $r > r_0$ *if and only if all the radial sectional curvatures are zero,* $\mathrm{Hess} r \equiv 0$*, and* $\partial_r^2 f \equiv \lambda$ *along the geodesic from* r_0 *to* r*.*

Proof: Applying the Bochner formula

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u) \tag{2.2}$$

to the distance function $r(x) = d(x,p)$, we have

$$0 = |\mathrm{Hess}\, r|^2 + \frac{\partial}{\partial r}(\Delta r) + \mathrm{Ric}(\nabla r, \nabla r). \tag{2.3}$$

Since $\mathrm{Hess}\, r$ is the second fundamental from of the geodesic sphere and Δr is the mean curvature, with the Schwarz inequality, we have the Riccati inequality

$$m' \leq -\frac{m^2}{n-1} - \mathrm{Ric}(\partial r, \partial r). \tag{2.4}$$

And equality holds if and only if the radial sectional curvatures are constant. Since $m_f' = m' - \mathrm{Hess} f\,(\partial r, \partial r)$, we have

$$m_f' \leq -\frac{m^2}{n-1} - \mathrm{Ric}_f(\partial r, \partial r). \tag{2.5}$$

If $\mathrm{Ric}_f \geq \lambda$, we have

$$m_f' \leq -\lambda.$$

This immediately gives the inequality (2.1).

To see the equality statement, suppose $m_f' \equiv -\lambda$ on an interval $[r_0, r]$, then from (2.5) we have $m \equiv 0$ and

$$m_f' = -\partial_r^2 f = -\mathrm{Ric}_f(\partial_r, \partial_r) = -\lambda. \tag{2.6}$$

So we also have $\mathrm{Ric}(\partial_r, \partial_r) = 0$. Then by (2.3) $\mathrm{Hess}\, r = 0$, which implies the sectional curvatures must be zero. ■

Proof of Theorem 1.1. Let $\mathrm{sn}_H(r)$ be the solution to

$$\mathrm{sn}_H'' + H\mathrm{sn}_H = 0$$

such that $\mathrm{sn}_H(0) = 0$ and $\mathrm{sn}_H'(0) = 1$. Then

$$m_H^n = (n-1)\frac{\mathrm{sn}_H'}{\mathrm{sn}_H}. \tag{2.7}$$

So we have

$$
\begin{aligned}
\left(\mathrm{sn}_H^2 m\right)' &= 2\mathrm{sn}_H'\mathrm{sn}_H m + sn_H^2 m' && (2.8)\\
&\le 2\mathrm{sn}_H'\mathrm{sn}_H m - \frac{\mathrm{sn}_H^2 m^2}{n-1} - \mathrm{sn}_H^2\mathrm{Ric}(\partial_r,\partial_r) && (2.9)\\
&= -\left(\frac{\mathrm{sn}_H m}{\sqrt{n-1}} - \sqrt{n-1}\mathrm{sn}_H'\right)^2 + (n-1)(\mathrm{sn}_H')^2 - \mathrm{sn}_H^2\mathrm{Ric}(\partial_r,\partial_r) && (2.10)\\
&\le (n-1)(\mathrm{sn}_H')^2 - (n-1)H\mathrm{sn}_H^2 + \mathrm{sn}_H^2\partial_r\partial_r f. && (2.11)
\end{aligned}
$$

Here in the 2nd line we have used (2.4), and in the last we used the lower bound on Ric_f.

On the other hand (2.7) implies that

$$(\mathrm{sn}_H^2 m_H)' = (n-1)(\mathrm{sn}_H')^2 - (n-1)H\mathrm{sn}_H^2.$$

Therefore we have

$$\left(\mathrm{sn}_H^2 m\right)' \le \left(\mathrm{sn}_H^2 m_H\right)' + \mathrm{sn}_H^2\partial_t\partial_t f. \tag{2.12}$$

Integrating from 0 to r yields

$$\mathrm{sn}_H^2(r)m(r) \le \mathrm{sn}_H^2(r)m_H(r) + \int_0^r \mathrm{sn}_H^2(t)\partial_t\partial_t f(t)dt. \tag{2.13}$$

When f is constant (the classical case) this gives the usual mean curvature comparison. This quick proof does not seem to be in the literature.

Proof or Part a. Using integration by parts on the last term we have

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r) - \int_0^r (\mathrm{sn}_H^2(t))'\partial_t f(t)dt. \tag{2.14}$$

Under our assumptions $(\mathrm{sn}_H^2(t))' = 2\mathrm{sn}_H'(t)\mathrm{sn}_H(t) \ge 0$ so if $\partial_t f(t) \ge -a$ we have

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r) + a\int_0^r (\mathrm{sn}_H^2(t))'dt = \mathrm{sn}_H^2(r)m_H(r) + \mathrm{sn}_H^2(r)a \tag{2.15}$$

This proves the inequality.

To see the rigidity statement suppose that $\partial_t f \ge -a$ and $m_f(r) = m_H(r) + a$ for some r. Then from (2.14) we see

$$a\mathrm{sn}_H^2 \le \int_0^r (\mathrm{sn}_H^2(t))'\partial_t f(t)dt \le a\mathrm{sn}_H^2. \tag{2.16}$$

So that $\partial_t f \equiv -a$. But then $m(r) = m_f - a = m_H(r)$ so that the rigidity follows from the rigidity for the usual mean curvature comparison.

Proof of Part b. Integrate (2.14) by parts again

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r) - f(r)(\mathrm{sn}_H^2(r))' + \int_0^r f(t)(sn_H^2)''(t)dt. \tag{2.17}$$

Now if $|f| \le k$ and $r \in (0, \frac{\pi}{4\sqrt{H}}]$ when $H > 0$, then $(sn_H^2)''(t) \ge 0$ and we have

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r) + 2k(\mathrm{sn}_H^2(r))'. \tag{2.18}$$

From (2.7) we can see that

$$(\mathrm{sn}_H^2(r))' = 2\mathrm{sn}_H'\mathrm{sn}_H = \frac{2}{n-1}m_H\mathrm{sn}_H^2.$$

so we have

$$m_f(r) \le \left(1 + \frac{4k}{n-1}\right) m_H(r) = m_H^{n+4k}(r). \tag{2.19}$$

■

Now when $H > 0$ and $r \in [\frac{\pi}{4\sqrt{H}}, \frac{\pi}{2\sqrt{H}}]$,

$$\begin{aligned} \int_0^r f(t)(sn_H^2)''(t)dt &\le k\left(\int_0^{\frac{\pi}{4\sqrt{H}}} (sn_H^2)''(t)dt - \int_{\frac{\pi}{4\sqrt{H}}}^r (sn_H^2)''(t)dt\right) \\ &= k\left(\frac{2}{\sqrt{H}} - sn_H(2r)\right). \end{aligned}$$

Hence

$$m_f(r) \le \left(1 + \frac{4k}{n-1} \cdot \frac{1}{\sin(2\sqrt{H}r)}\right) m_H(r). \tag{2.20}$$

This estimate will be used later to prove the Myers' theorem in Section 5.

Remark In the case $H = 0$, we have $sn_H(r) = r$ so (2.17) gives the estimate in [10] that

$$m_f(r) \le \frac{n-1}{r} - \frac{2}{r}f(r) + \frac{2}{r^2}\int_0^r f(t)dt. \tag{2.21}$$

Remark The exact same argument gives mean curvature comparison for the mean curvature of distance sphere of hypersurfaces with Ric_f lower bound.

3 Volume Comparisons

In this section we prove the volume comparison theorems.

For $p \in M^n$, use exponential polar coordinate around p and write the volume element $d\,vol = \mathcal{A}(r,\theta)dr \wedge d\theta_{n-1}$, where $d\theta_{n-1}$ is the standard volume element on the unit sphere $S^{n-1}(1)$. Let $\mathcal{A}_f(r,\theta) = e^{-f}\mathcal{A}(r,\theta)$. By the first variation of the area (see [46])

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = (\ln(\mathcal{A}(r,\theta)))' = m(r,\theta). \tag{3.1}$$

Therefore

$$\frac{\mathcal{A}_f'}{\mathcal{A}_f}(r,\theta) = (\ln(\mathcal{A}_f(r,\theta)))' = m_f(r,\theta). \tag{3.2}$$

And for $r \ge r_0 > 0$

$$\frac{\mathcal{A}_f(r,\theta)}{\mathcal{A}_f(r_0,\theta)} = e^{\int_{r_0}^r m_f(r,\theta)}. \tag{3.3}$$

The volume comparison theorems follow from the mean curvature comparisons through this equation.

First applying the mean curvature estimate Theorem 2.1 we have the following basic volume comparison theorem.

Theorem 3.1 (Volume Comparison I) *Let $\mathrm{Ric}_f \ge \lambda$ then for any r there are constants A, B, and C such that*

$$\mathrm{Vol}_f(B(p,R)) \le A + B\int_r^R e^{-\lambda t^2 + Ct}dt.$$

The version of Theorem 3.1 for tubular neighborhoods of hypersurfaces is very similar and has been proven by Morgan [25], also see [26]. As Morgan points out, the theorem is optimal and the constants can not be uniform as the Gaussian soliton shows, see Example 7.1.

Proof: Using the mean curvature estimate (2.1)

$$\int_{r_0}^{r} m_f(r) \leq m_f(r_0)r - \frac{1}{2}\lambda r^2.$$

Hence

$$\mathcal{A}_f(r,\theta) \leq \mathcal{A}_f(r_0,\theta)e^{m_f(r_0,\theta)r - \frac{1}{2}\lambda r^2}.$$

Now let $A(p, r_0, r)$ be the the annulus $A(p, r_0, r) = B(p, r) \setminus B(p, r_0)$. Then

$$\begin{aligned}
\mathrm{Vol}_f(A(p,r_0,r)) &= \int_{r_0}^{r}\int_{S^{n-1}} \mathcal{A}_f(s,\theta)d\theta ds && (3.4)\\
&\leq \int_{r_0}^{r}\int_{S^{n-1}} \mathcal{A}_f(r_0,\theta)e^{m_f(r_0,\theta)r-\frac{1}{2}\lambda r^2}d\theta ds && (3.5)\\
&\leq A_f(r_0)\int_{r_0}^{r} e^{Cr-\frac{1}{2}(n-1)\lambda r^2}ds. && (3.6)
\end{aligned}$$

Where $A_f(r_0)$ is the surface area of the geodesic sphere induced from the f-volume element and C is a constant such that $C \geq m_f(r_0, \theta)$ for all θ where it is defined. Since $\mathrm{Vol}_f(B(p, r)) = \mathrm{Vol}_f(\mathrm{Vol}(B(p, r_0)) + \mathrm{Vol}_f(A(p, r_0, r))$ this proves the theorem. ■

We also have a rigidity statement for (3.5). That is, if the inequality (3.5) is an equality then we must have equalities in the mean curvature comparison along all the geodesics, this implies that $\mathrm{Hess}\, r \equiv 0$ which implies that

$$A(p, r_0, r) \cong S(p, r_0) \times [r_0, r] \tag{3.7}$$

where $S(p, r_0)$ is the geodesic sphere with radius r_0. Moreover, $f(x, t) = f(x) + \partial_r f(x)(r - r_0) + \frac{\lambda}{2}(r - r_0)^2$.

Now we prove Theorem 1.2 using Theorem 1.1.

Proof of Theorem 1.2: For Part a) we compare with a model space, however, we modify the measure according to a. Namely, the model space will be the pointed metric measure space $M^n_{H,a} = (M^n_H, g_H, e^{-h}dvol, O)$ where (M^n_H, g_H) is the n-dimensional simply connected space with constant sectional curvature H, $O \in M^n_H$, and $h(x) = -a \cdot d(x, O)$. We make the model a pointed space because the space only has $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$ in the radial directions from O and we only compare volumes of balls centered at O.

Let $\mathcal{A}^a_H$ be the h-volume element in $M^n_{H,a}$. Then $\mathcal{A}^a_H(r) = e^{ar}\mathcal{A}_H(r)$ where $\mathcal{A}_H$ is the Riemannian volume element in M^n_H. By the mean curvature comparison we have $(\ln(\mathcal{A}_f(r,\theta))' \leq a + m_H = (\ln(\mathcal{A}^a_H))'$ so for $r < R$,

$$\frac{\mathcal{A}_f(R,\theta)}{\mathcal{A}_f(r,\theta)} \leq \frac{\mathcal{A}^a_H(R,\theta)}{\mathcal{A}^a_H(r,\theta)}. \tag{3.8}$$

Namely $\frac{\mathcal{A}_f(r,\theta)}{\mathcal{A}^a_H(r,\theta)}$ is nonincreasing in r. Using Lemma 3.2 in [46], we get for $0 < r_1 < r$, $0 < R_1 < R$, $r_1 \leq R_1$, $r \leq R$,

$$\frac{\int_{R_1}^{R} \mathcal{A}_f(t,\theta)dt}{\int_{r_1}^{r} \mathcal{A}_f(t,\theta)dt} \leq \frac{\int_{R_1}^{R} \mathcal{A}^a_H(t,\theta)dt}{\int_{r_1}^{r} \mathcal{A}^a_H(t,\theta)dt}. \tag{3.9}$$

Integrating along the sphere direction gives

$$\frac{\mathrm{Vol}_f(A(p,R_1,R))}{\mathrm{Vol}_f(A(p,r_1,r))} \leq \frac{\mathrm{Vol}_H^a(R_1,R)}{\mathrm{Vol}_H^a(r_1,r)}. \tag{3.10}$$

Where $\mathrm{Vol}_H^a(r_1,r)$ is the h-volume of the annulus $B(O,r)\setminus B(O,r_1)\subset M_H^n$. Since $\mathrm{Vol}_H(r_1,r)\leq \mathrm{Vol}_H^a(r_1,r)\leq e^{ar}\mathrm{Vol}_H(r_1,r)$ this gives (1.4) when $r_1=R_1=0$ and proves Part b).

In the model space the radial function h is not smooth at the origin. However, clearly one can smooth the function to a function with $\partial_r h\geq -a$ and $\partial_r^2 h\geq 0$ such that the h-volume taken with the smoothed h is arbitrary close to that of the model. Therefore, the inequality (3.10) is optimal. Moreover, one can see from the equality case of the mean curvature comparison that if the annular volume is equal to the volume in the model then all the radial sectional curvatures are H and f is exactly a linear function.

Proof of Part b): In this case let $\mathcal{A}_H^{n+4k}$ be the volume element in the simply connected model space with constant curvature H and dimension $n+4k$.

Then from the mean curvature comparison we have $\ln(\mathcal{A}_f(r,\theta))'\leq \ln(\mathcal{A}_H^{n+4k}(r))'$. So again applying Lemma 3.2 in [46] we obtain

$$\frac{\mathrm{Vol}_f(A(p,R_1,R))}{\mathrm{Vol}_f(A(p,r_1,r))} \leq \frac{Vol_H^{n+4k}(R_1,R)}{Vol_H^{n+4k}(r_1,r)}. \tag{3.11}$$

With $r_1=R_1=0$ this implies the relative volume comparison for balls

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq \frac{Vol_H^{n+4k}(R)}{Vol_H^{n+4k}(r)}. \tag{3.12}$$

Equivalently

$$\frac{\mathrm{Vol}_f(B(p,R))}{V_H^{n+4k}(R)} \leq \frac{\mathrm{Vol}_f(B(p,r))}{V_H^{n+4k}(r)}. \tag{3.13}$$

Since $n+4k>n$ we note that the right hand side blows up as $r\to 0$ so one does not obtain a uniform upper bound on $\mathrm{Vol}_f(B(p,R))$. Indeed, it is not possible to do so since one can always add a constant to f and not effect the Bakry-Emery tensor.

By taking $r=1$ we do obtain a volume growth estimate for $R>1$

$$\mathrm{Vol}_f(B(p,R)) \leq \mathrm{Vol}_f(B(p,1))Vol_H^{n+4k}(R). \tag{3.14}$$

Note that, from Part a) $\mathrm{Vol}_f(B(p,1))\leq e^{-f(p)}e^a\omega_n$ if $\partial_r f\geq -a$ on $B(p,1)$. ∎

In the next section we collect the applications of the volume comparison theorems.

4 Applications of the volume comparison theorems.

In the case where $\lambda>0$ Theorem 3.1 gives two very interesting corollaries. The first is also observed in [25].

Corollary 4.1 *If M is complete and $Ric_f\geq\lambda>0$ then $\mathrm{Vol}_f(M)$ is finite and M has finite fundamental group.*

We note the finiteness of volume is true in the setting of more general diffusion operators [4] and more general metric measure spaces [15]. Using a different approach the second author has proven that the fundamental group is finite for spaces satisfying $\mathrm{Ric}+\mathcal{L}_X g\geq\lambda>0$ for some vector field

X [42]. This had earlier been shown under the additional assumption that the Ricci curvature is bounded by Zhang [45]. See also [27]. When M is compact the finiteness of fundamental group was first shown by X. Li [19, Corollary 3] using a probabilistic method. Also see [45, 12, 34, 23]. We would like to thank Prof. David Elworthy for bringing the article [19] to our attention.

The second corollary is the following Liouville Theorem, which is a strengthening of a result of Naber [27].

Corollary 4.2 *If M is complete with $Ric_f \geq \lambda > 0$, $u \geq 0$, $\Delta_f(u) \geq 0$, and there is $\alpha < \lambda$ such that $u(x) \leq e^{\alpha d(p,x)^2}$ for some $p \in M$ then u is constant.*

In particular there are no bounded f-subharmonic functions. Corollary 4.2 follows from Yau's proof that a complete manifold has no positive L^p ($p > 1$) subharmonic functions [44]. The argument only uses integration by parts and a clever choice of test function and so is valid also for the weighted measure and Laplacian.

While Theorem 3.1 has applications when $\lambda > 0$ it is not strong enough to extend results for a general lower bound, for these results we apply Theorem 1.2. It is well known that a lower bound on volume growth for manifolds with $\mathrm{Ric} \geq 0$ can be derived from the volume comparison for annulli, see [46]. Thus Theorem 1.3 follows from (3.11). We give the proof here for completeness and to motivate Theorem 4.3.

Proof of Theorem 1.3: Let M be a manifold with $\mathrm{Ric}_f \geq 0$ for a bounded function f. Let $p \in M$ and let γ be a geodesic ray based at p in M. Then, applying the annulus relative volume comparison (3.11) to annuli centered at $\gamma(t)$, we obtain

$$\frac{\mathrm{Vol}_f(B(\gamma(t), t-1))}{\mathrm{Vol}_f(A(\gamma(t), t-1, t+1))} \geq \frac{(t-1)^{n+4k}}{(t+1)^{n+4k} - (t-1)^{n+4k}} \geq c(n,k)t \qquad \forall\ t \geq 2.$$

But $B(\gamma(0), 1) \subset A(\gamma(t), t-1, t+1)$ so we have

$$\mathrm{Vol}_f(B(p, t-1)) \geq c(n,k)\mathrm{Vol}_f(B(p,1))t \qquad \forall t \geq 2.$$

■

Using the volume comparison (3.10) in place of (3.11) we can also prove a lower bound on the volume growth for certain convex f.

Theorem 4.3 *If $\mathrm{Ric}_f \geq 0$ where f is convex function such that the set of critical points of f is unbounded, then M has at least linear f-volume growth.*

The hypothesis on the critical point set is necessary by Examples 7.1 and 7.2.

Proof: Fix $p \in M$. Since the set of critical points of a convex function is connected, for every t there is $x(t)$, a critical point of f, such that $d(p, x(t)) = t$. But $\nabla f(x(t)) = 0$ and f is convex so $\partial r f \geq 0$ in all the radial directions from $x(t)$, therefore we can apply (3.10) and repeat the arguments in the proof of Theorem 1.3 to prove the result. ■

In [24] Milnor observed that polynomial volume growth on the universal cover of a manifold restricts the structure of its fundamental group. Thus Theorem 1.2 also implies the following extension of Milnor's Theorem.

Theorem 4.4 *Let M be a complete manifold with $Ric_f \geq 0$.*

1. *If f is a convex function that obtains its minimum then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree less than or equal n. In particular, $b_1(M) \leq n$.*

2. *If $|f| \leq k$ then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree less than or equal to $n + 4k$. In particular, $b_1(M) \leq n + 4k$.*

Part 1 follows because at a pre-image of the minimum point in the universal cover, $\partial_r f \geq 0$. Gromov [13] has shown that a finitely generated group has polynomial growth if and only if it is almost nilpotent. Moreover, the work of the first author and Wilking shows that any finitely generated almost nilpotent group is the fundamental group of a manifold with $\mathrm{Ric} \geq 0$ [39, 40]. Therefore, there is a complete classification of the finitely generated groups that can be realized as the fundamental group of a complete manifold with $\mathrm{Ric} \geq 0$. Combining these results with Theorem 4.4 we expand this classification to a larger class of manifolds.

Corollary 4.5 *A finitely generated group G is the fundamental group of some manifold with*

1. $\mathrm{Ric}_f \geq 0$ *for some bounded f or*
2. $\mathrm{Ric}_f \geq 0$ *for some convex f which obtains its minimum*

if and only if G is almost nilpotent.

It would be interesting to know if Corollary 4.5 holds without any assumption on f. Example 7.4 shows that the Milnor argument can not be applied since the f-volume growth of a manifold with $\mathrm{Ric}_f \geq 0$ may be exponential, so a different method of proof would be needed.

In [3] Anderson uses similar covering arguments to show, for example, that if $\mathrm{Ric} \geq 0$ and and M has euclidean volume growth then $\pi_1(M)$ is finite. He also finds interesting relationships between the first betti number, volume growth, and finite generation of fundamental group of manifolds with $\mathrm{Ric} \geq 0$. These relationships also carry over to manifolds satisfying the hypotheses of Theorem 4.4. We leave these statements to the interested reader.

Applying the relative volume comparison Theorem 1.2 we also have the following extensions of theorems of Gromov [14] and Anderson [2].

Theorem 4.6 *For the class of manifolds M^n with $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{diam}_M \leq D$ and $|f| \leq k$ ($|\nabla f| \leq a$), the first Betti number $b_1 \leq C(n,k,HD^2)$ ($C(n,HD^2,aD)$).*

Theorem 4.7 *For the class of manifolds M^n with $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{Vol}_f \geq V$, $\mathrm{diam}_M \leq D$ and $|f| \leq k$ ($|\nabla f| \leq a$) there are only finitely many isomorphism types of $\pi_1(M)$.*

Remark In the case when $|\nabla f|$ is bounded, Ric_f bounded from below implies Ric_f^N is also bounded from below (with different lower bound). Therefore in this case the results can also been proven using the volume comparison in [34, 23, 6] for the Ric_f^N tensor.

5 The Splitting Theorem.

An important application of the mean curvature comparison is the extension of the Cheeger-Gromoll splitting theorem. After writing the original version of this paper, we learned that Lichnerowicz had proven the splitting theorem, see [20, 21].

Theorem 5.1 (Lichnerowicz-Cheeger-Gromoll Splitting Theorem) *If $\mathrm{Ric}_f \geq 0$ for some bounded f and M contains a line, then $M = N^{n-1} \times \mathbb{R}$ and f is constant along the line.*

For completeness we retain our complete proof here.

Remark In [10] Fang, Li, and Zhang show that only an upper bound on f is needed in the above theorem. Example 7.2 shows that the upper bound on f is necessary.

Recall that $m_f = \Delta_f(r)$, the f-Laplacian of the distance function. From (1.1), we get a local Laplacian comparison for distance functions

$$\Delta_f(r) \leq \frac{n+4k-1}{r} \quad \text{for all } x \in M \setminus \{p, C_p\}. \tag{5.1}$$

Where C_p is the cut locus of p. To prove the splitting theorem we apply this estimate to the Busemann functions.

Definition 5.2 *If γ is a ray then Busemann function associated to γ is the function*

$$b^{\gamma}(x) = \lim_{t\to\infty}(t - d(x, \gamma(t))). \tag{5.2}$$

¿From the triangle inequality the Busemann function is Lipschitz continuous with Lipschitz constant 1 and thus is differential almost everywhere. At the points where b_γ is not smooth we interpret the f-laplacian in the sense of barriers.

Definition 5.3 *For a continuous function h on $M, q \in M$, a function h_q defined in a neighborhood U of q, is a lower barrier of h at q if h_q is $C^2(U)$ and*

$$h_q(q) = h(q), \quad h_q(x) \le h(x) \ (x \in U). \tag{5.3}$$

Definition 5.4 *We say that $\Delta_f(h) \ge a$ in the barrier sense if, for every $\varepsilon > 0$, there exists a lower barrier function h_ε such that $\Delta_f(h_\varepsilon) > a - \varepsilon$. An upper bound on Δ_f is defined similarly in terms of upper barriers.*

The local Laplacian comparison is applied to give the following key lemma.

Lemma 5.5 *If M is a complete, noncompact manifold with $\mathrm{Ric}_f \ge 0$ for some bounded function f then $\Delta_f(b^\gamma) \ge 0$ in the barrier sense.*

Remark As in [10], one can use the inequality (2.21) to prove Lemma 5.5 only assuming an upper bound on f.

Proof: For the Busemann function at a point q we have a family of barrier functions defined as follows. Given $t_i \to \infty$, let σ_i be minimal geodesics from q to $\gamma(t_i)$, parametrized by arc length. The sequence $\sigma_i'(0)$ subconverges to some $v_0 \in T_qM$. We call the geodesic $\overline{\gamma}$ such that $\overline{\gamma}'(0) = v_0$ an asymptotic ray to γ.

Define the function $h_t(x) = t - d(x, \overline{\gamma}(t)) + b^\gamma(q)$. Since $\overline{\gamma}$ is a ray, the points $q = \overline{\gamma}(0)$ and $\overline{\gamma}(t)$ are not cut points to each other, therefore the function $d(x, \overline{\gamma}(t))$ is smooth in a neighborhood of q and thus so is h_t. Clearly $h_t(q) = b^\gamma(q)$, thus to show that h_t is a lower barrier for b^γ we only need to show that $h_t(x) \le b^\gamma(x)$. To see this, first note that for any s,

$$-d(x, \overline{\gamma}(t)) \le -d(x, \gamma(s)) + d(\gamma(s), \overline{\gamma}(t)) = s - d(x, \gamma(s)) - s + d(\gamma(s), \overline{\gamma}(t)). \tag{5.4}$$

Taking $s \to \infty$ this gives

$$-d(x, \overline{\gamma}(t)) \le b^\gamma(x) - b^\gamma(\overline{\gamma}(t)). \tag{5.5}$$

Also,

$$\begin{aligned} b^\gamma(q) &= \lim_{i\to\infty}(t_i - d(q, \gamma(t_i))) \\ &= \lim_{i\to\infty}(t_i - d(q, \sigma_i(t)) - d(\sigma_i(t), \gamma(t_i))) \\ &= -d(q, \overline{\gamma}(t)) + \lim_{i\to\infty}(t_i - d(\sigma_i(t), \gamma(t_i))) \\ &= -t + b^\gamma(\overline{\gamma}(t)). \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) gives

$$h_t(x) \le b^\gamma(x), \tag{5.7}$$

so h_t is a lower barrier function for b^γ. By (5.1), we have that

$$\Delta_f(h_t)(x) = \Delta_f(-d(x, \overline{\gamma}(t))) \ge -\frac{n + 4k - 1}{t}. \tag{5.8}$$

Taking $t \to \infty$ proves the lemma. ■

Note that since Δ_f is just a perturbation of Δ by a lower order term, the strong maximum principle and elliptic regularity still hold for Δ_f. Namely if h is a continuous function such that $\Delta_f(h) \geq 0$ in the barrier sense and h has an interior maximum then h is constant and if $\Delta_f(h) = 0$ (i.e ≥ 0 and ≤ 0) in the barrier sense then h is smooth. We now apply the lemma and these two theorems to finish the proof of the splitting theorem.

Proof of Theorem 5.1: Denote by γ_+ and γ_- the two rays which form the line γ and let b^+, b^- denote their Busemann functions.

The function $b^+ + b^-$ has a maximum at $\gamma(0)$ and satisfies $\Delta_f(b^+ + b^-) \geq 0$, thus by the strong maximum principle the function is constant and equal to 0. But then $b^+ = -b^-$ so that $0 \leq \Delta_f(b^+) = -\Delta_f(b^-) \leq 0$ which then implies, by elliptic regularity, that the functions b^+ and b^- are smooth.

Moreover, for any point q we can consider asymptotic rays $\overline{\gamma}_+$ and $\overline{\gamma}_-$ to $\gamma_+$and γ_- and denote their Busemann functions by $\overline{b}^+$ and $\overline{b}^-$. From (5.7) it follows that

$$\overline{b}^+(x) + b^+(q) \leq b^+(x). \tag{5.9}$$

We will show that this inequality is, in fact, an equality when γ_+ extends to a line.

First we show that the two asymptotic rays $\overline{\gamma}_+$ and $\overline{\gamma}_-$ form a line. By the triangle inequality, for any t

$$\begin{aligned} d(\overline{\gamma}_+(s_1), \overline{\gamma}_-(s_2)) &\geq d(\overline{\gamma}_-(s_2), \gamma_+(t)) - d(\gamma_+(t), \overline{\gamma}_+(s_1)) \\ &= t - d(\gamma_+(t), \overline{\gamma}_+(s_1)) - (t - d(\overline{\gamma}_-(s_2), \gamma_+(t)), \end{aligned}$$

so by taking $t \to \infty$ we have

$$\begin{aligned} d(\overline{\gamma}_+(s_1), \overline{\gamma}_-(s_2)) &\geq b^+(\overline{\gamma}^+(s_1)) - b^+(\overline{\gamma}^-(s_2)) \\ &= b^+(\overline{\gamma}^+(s_1)) + b^-(\overline{\gamma}^-(s_2)) \\ &\geq \overline{b}^+(\overline{\gamma}^+(s_1)) + b^+(q) + \overline{b}^-(\overline{\gamma}^-(s_2)) + b^-(q) \\ &= s_1 + s_2. \end{aligned}$$

Thus, any asymptotic ray to γ_+ forms a line with any asymptotic ray to γ_-. Applying the same argument given above for b^+ and b^- we see that $\overline{b}^+ = -\overline{b}^-$. By Applying (5.9) to b^-

$$-\overline{b}^-(x) - b^-(q) \geq -b^-(x).$$

Substituting $b^+ = -b^-$ and $\overline{b}^+ = -\overline{b}^-$ we have

$$\overline{b}^+(x) + b^+(q) \geq b^+(x).$$

This along with (5.9), gives

$$\overline{b}^+(x) + b^+(q) = b^+(x).$$

Thus, $\overline{b}^+$ and b^+ differ only by a constant. Clearly, at q the derivative of $\overline{b}^+$ in the direction of $\overline{\gamma}'_+(0)$ is 1. Since $\overline{b}^+$ has Lipschitz constant 1, this implies that $\nabla b^+(q) = \overline{\gamma}'_+(0)$.

¿From the Bochner formula (2.2) and a direct computation one has the following Bochner formula with measure,

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \tag{5.10}$$

Now apply this to b^+, since $||\nabla b^+|| = 1$, we have

$$0 = ||\text{Hess}\, b^+||^2 + \nabla b^+(\Delta_f(b^+)) + \text{Ric}_f(\nabla b^+, \nabla b^+). \tag{5.11}$$

Since $\Delta_f(b^+) = 0$ and $\text{Ric}_f \geq 0$ we then have that $\text{Hess}\, b^+ = 0$ which, along with the fact that $||\nabla b^+|| = 1$ implies that M splits isometrically in the direction of ∇b^+.

To see that f must be constant in the splitting direction note that from (5.11) we now have $\text{Ric}_f(\nabla b^+, \nabla b^+) = 0$ and ∇b^+ points in the splitting direction so $\text{Ric}(\nabla b^+, \nabla b^+) = 0$. Therefore $\text{Hess} f(\nabla b^+, \nabla b^+) = 0$. Since f is bounded f must be constant in ∇b^+ direction. ■

As Lichnerowicz points out, the clever covering arguments in [9] along with Theorem 5.1 imply the following structure theorem for compact manifolds with $\text{Ric}_f \geq 0$.

Theorem 5.6 *If M is compact and $Ric_f \geq 0$ then M is finitely covered by $N \times T^k$ where N is a compact simply connected manifold and f is constant on the flat torus T^k.*

Theorem 5.6 has the following topological consequences.

Corollary 5.7 *Let M be compact with $Ric_f \geq 0$ then*

1. *$b_1(M) \leq n$.*
2. *$\pi_1(M)$ has a free abelian subgroup of finite index of rank $\leq n$.*
3. *$b_1(M)$ or $\pi_1(M)$ has a free abelian subgroup of rank n if and only if M is a flat torus and f is a constant function.*
4. *$\pi_1(M)$ is finite if $\text{Ric}_f > 0$ at one point.*

We also note that the splitting theorem has been used by Oprea [28] to derive information about the Lusternik-Schnirelmann category of compact manifolds with non-negative Ricci curvature. These arguments also clearly carry over to the Ric_f case.

For noncompact manifolds with positive Ricci curvature the splitting theorem has also been used by Cheeger and Gromoll [9] and Sormani [36] to give some other topological obstructions. These results also hold for Ric_f with f bounded.

Theorem 5.8 *Suppose M is a complete manifold with $\text{Ric}_f > 0$ for some bounded f then*

1. *M has only one end and*
2. *M has the loops to infinity property.*

In particular, if M is simply connected at infinity then M is simply connected.

6 Other applications of the mean curvature comparison.

Theorem 1.1 can be used to prove an excess estimate. Recall that for $p, q \in M$ the excess function is $e_{p,q}(x) = d(p,x) + d(q,x) - d(p,q)$. Let $h(x) = d(x, \gamma)$ where γ is a fixed minimal geodesic from p to q, then (1.3) along with the arguments in [1, Proposition 2.3] imply the following version of the Abresch-Gromoll excess estimate.

Theorem 6.1 *Let $\text{Ric}_f \geq 0$, $|f| \leq k$ and $h(x) < \min\{d(p,x), d(q,x)\}$ then*

$$e_{p,q}(x) \leq 2\left(\frac{n+4k-1}{n+4k-2}\right)\left(\frac{1}{2}Ch^{n+4k}\right)^{\frac{1}{n+4k-1}}$$

where

$$C = 2\left(\frac{n+4k-1}{n+4k}\right)\left(\frac{1}{d(p,x)-h(x)} + \frac{1}{d(q,x)-h(x)}\right)$$

Remark (1.1) also implies an excess estimate for manifolds with $\mathrm{Ric} \geq (n-1)H$ and $|\nabla f| \leq a$, however the constant C will depend on $H \cdot d(p,q)^2$ and e^{ah}. The mean curvature comparison for Ric_f^N discussed in the appendix also implies an excess estimate.

Theorem 6.1 gives extensions of theorems of Abresch-Gromoll [1] and Sormani [35] to the case where $\mathrm{Ric}_f \geq 0$ for a bounded f.

Theorem 6.2 *Let be M a complete noncompact manifold with $Ric_f \geq 0$ for some bounded f.*

1. *If M has bounded diameter growth and sectional curvature bounded below then M is homeomorphic to the interior of a compact manifold with boundary.*

2. *If M has sublinear diameter growth then M has finitely generated fundamental group.*

Remark If we consider $|f| \leq k$, the arguments in [1] and [35] say slightly more. Namely, the diameter growth in the first part can be of order $\leq \frac{1}{n+4k-1}$ and in the second part one can derive a explicity constant $S_{n,k}$ such that the diameter growth only needs to be $\leq S_{n,k} \cdot r$. Also see [43] and [41].

We can also apply the mean curvature comparison to the excess function to prove the Myers' theorem. We note that the excess function was also used to prove a Myers' theorem in [31]. It is interesting that this proof is exactly suited to our situation, since we only have a uniform bound on mean curvature when $r \leq \frac{\pi}{2\sqrt{H}}$, while other arguments do not seem to easily generalize.

Proof of Theorem 1.4 Let p_1, p_2 are two points in M with $d(p_1,p_2) \geq \frac{\pi}{\sqrt{H}}$ and set $B = d(p_1,p_2) - \frac{\pi}{\sqrt{H}}$. Let $r_1(x) = d(p_1,x)$ and $r_2(x) = d(p_2,x)$ and e be the excess function for the points p_1 and p_2. By the triangle inequality, $e(x) \geq 0$ and $e(\gamma(t)) = 0$ where γ is a minimal geodesic from p_1 to p_2. Therefore, $\Delta_f(e)(\gamma(t)) \geq 0$.

Let $y_1 = \gamma\left(\frac{\pi}{2\sqrt{H}}\right)$ and $y_2 = \gamma\left(\frac{\pi}{2\sqrt{H}} + B\right)$. For $i = 1$ and 2, $r_i(y_i) = \frac{\pi}{2\sqrt{H}}$ so, by (2.20), we have

$$\Delta_f(r_i(y_i)) \leq 2k\sqrt{H}. \tag{6.1}$$

(1.2) does not give an estimate for $\Delta_f(r_1(y_2))$ since $r_1(y_2) > \frac{\pi}{2\sqrt{H}}$ but by (2.1) and (6.1) we have

$$\Delta_f(r_1(y_2)) \leq 2k\sqrt{H} - B(n-1)H. \tag{6.2}$$

So

$$0 \leq \Delta_f(e)(y_2) = \Delta_f(r_1)(y_2) + \Delta_f(r_2)(y_2) \leq 4k\sqrt{H} - B(n-1)H \tag{6.3}$$

which implies $B \leq \frac{4k}{(n-1)\sqrt{H}}$ and $d(p_1,p_2) \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}$. ■

As we have seen, there is no bound on the distance between two points in a complete manifold with $\mathrm{Ric}_f \geq (n-1)H > 0$. However, by slightly modifying the argument above one can prove a distance bound between two hypersurfaces that depends on the f-mean curvature of the hypersurfaces, here for a hypersurface N the f-mean curvature at a point $x \in N$ with respect to the normal vector n is

$$H_n^f(x) = H_n(x) + \langle n, \nabla f \rangle(x) \tag{6.4}$$

where H_n is the regular mean curvature. m_f is then the f-mean curvature of the geodesic sphere with respect to the inward pointing normal.

Theorem 6.3 *Let $\mathrm{Ric}_f \geq (n-1)H > 0$ and let N_1 and N_2 be two compact hypersurfaces in M then*

$$d(N_1,N_2) \leq \frac{\max_{p \in N_1} |H^f_{n_1}(p)| + \max_{q \in N_2} |H^f_{n_2}(q)|}{2(n-1)H} \tag{6.5}$$

Proof: Let $e_{N_1,N_2}(x) = r_1(x) + r_2(x) - d(N_1, N_2)$ where $r_i(x) = d(x, N_i)$. Then, by applying the Bochner formula to r_i in the same way we applied it to the distance to a point in Section 2, we have

$$\Delta_f(r_i)(x) \leq \max_{p \in N_i} |H^f_{n_i}(x)| - (n-1)Hd(N_i, x)$$

One now can prove the theorem using a similar argument as in the proof of Theorem 1.4. ■

A similar argument also shows Frankel's Theorem is true for Ric_f.

Theorem 6.4 *Any two compact f-minimal hypersurfaces in a manifold with $\mathrm{Ric}_f > 0$ intersect.*

One also has a rigidity statement when $\mathrm{Ric}_f \geq 0$ and M has two f-minimal hypersurface which do not intersect, see [33] for the statement and proof in the $\mathrm{Ric} \geq 0$ case.

7 Examples and Remarks

The most well known example is the following soliton, often referred to as the Gaussian soliton.

Example 7.1 *Let $M = \mathbb{R}^n$ with Euclidean metric g_0, $f(x) = \frac{\lambda}{2}|x|^2$. Then $\mathrm{Hess} f = \lambda g_0$ and $\mathrm{Ric}_f = \lambda g_0$.*

This example shows that, unlike the case of Ricci curvature uniformly bounded from below by a positive constant, the manifold could be noncompact when $\mathrm{Ric}_f \geq \lambda g$ and $\lambda > 0$.

¿From this we construct the following.

Example 7.2 *Let $M = \mathbb{H}^n$ be the hyperbolic space. Fixed any $p \in M$, let $f(x) = (n-1)r^2 = (n-1)d^2(p,x)$. Now Hess $r^2 = 2|\nabla r|^2 + 2r\mathrm{Hess} r \geq 2I$, therefore $\mathrm{Ric}_f \geq (n-1)$.*

This example shows that the Cheeger-Gromoll splitting theorem and Abresch-Gromoll's excess estimate do not hold for $\mathrm{Ric}_f \geq 0$, in fact they don't even hold for $\mathrm{Ric}_f \geq \lambda > 0$. Note that the only properties of hyperbolic space used are that $\mathrm{Ric} \geq -(n-1)$ and that Hess $r^2 \geq 2I$. But Hess $r^2 \geq 2I$ for any Cartan-Hadamard manifold, therefore any Cartan-Hadamard manifold with Ricci curvature bounded below has a metric with $\mathrm{Ric}_f \geq 0$ on it. On the other hand, in these examples $\mathrm{Ric} < 0$. When $\mathrm{Ric} < 0$ ($\mathrm{Ric} \leq 0$) and $\mathrm{Ric}_f \geq 0$($\mathrm{Ric}_f > 0$), then $\mathrm{Hess} f > 0$ and f is strictly convex. Therefore M has to homeomorphic to $\mathbb{R}^n$.

A large class of examples are given by gradient Ricci solitons. Compact expanding or steady solitons are Einstein (f is constant) [29]. There are nontrivial compact shrinking solitons [16, 7]. These examples also have positive Ricci curvature but in the noncompact case there are Kahler Ricci shrinking solitons that do not have nonnegative Ricci curvature [11]. Clearly there are examples with $\mathrm{Ric}_f \geq 0$ but Ricci curvature is not nonnegative, like Example 7.2. One can also construct example that f is bounded. In fact one can use the following general local perturbation.

Example 7.3 *Let M^n be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$ except in a neighborhood of a point p, U_p. If the sectional curvature is ≤ 1 and $\mathrm{Ric} \geq -1$ on U_p and $U_p \subset B(p, \pi/4)$, then the distance function $r(x) = d(p,x)$ is strictly convex (with a singularity at p) on U_p. On $B(p, \pi/4)$ let $f = r^2$. If $\mathrm{Ric} \geq 1$ on the annulus $A(p, \pi/4, \pi/2)$, then one can extend f smoothly to M so that f is constant outside $B(p, \pi/2)$ and $\mathrm{Ric}_f \geq 0$ on M.*

The following example shows that there are manifolds with $Ric_f \geq 0$ which do not have polynomial f-volume growth.

Example 7.4 *Let $M = \mathbb{R}^n$ with Euclidean metric, $f(x_1, \cdots, x_n) = x_1$. Since Hess $f = 0$, $\mathrm{Ric}_f = \mathrm{Ric} = 0$. On the other hand $\mathrm{Vol}_f(B(0,r))$ is of exponential growth. Along the x_1 direction, $m_f - m_H = -1$ which does not goes to zero.*

In this example $|\nabla f| \leq 1$, so $\mathrm{Ric}_f \geq 0$ and $|\nabla f|$ bounded does not imply polynomial f-volume growth either.

Question 7.5 *If M^n has a complete metric and measure such that $\mathrm{Ric}_f \geq 0$ and f is bounded, does M^n has a metric with $\mathrm{Ric} \geq 0$?*

There is no counterexample even without the f bounded condition.

It is also natural to consider the scalar curvature with measure. As pointed out by Perelman in [29, 1.3] the corresponding scalar curvature equation is $S_f = 2\Delta f - |\nabla f|^2 + S$. Note that this is different than taking the trace of Ric_f which is $\Delta f + S$. However, The Lichnerovicz formula and theorem naturally extend to S_f. But $\mathrm{Ric}_f \geq 0$ doesn't imply $S_f \geq 0$ anymore. So one can ask the following question.

Question 7.6 *If M^n is a compact spin manifold with $\mathrm{Ric}_f > 0$, is the $\hat{A}$-genus zero?*

One could try to see if the $K3$ surface has a metric with $\mathrm{Ric}_f > 0$. If this were true it would give a negative answer to Question 7.5.

A Mean curvature comparison for N-Bakry-Emery Ricci tensor

In [6] the volume comparison theorem and Myers' theorem for the N-Bakry-Emery Ricci tensor are proven using what we have called a mean curvature comparison (actually their work is slightly more general than the cases treated in this paper). In this appendix, for clarity, we state this comparison in the language we have used above, which is standard in Riemannian geometry.

Recall the definition of the N-Bakry-Emery tensor is

$$\mathrm{Ric}_f^N = \mathrm{Ric}_f - \frac{1}{N} df \otimes df \quad \text{for } N > 0.$$

The main idea is that the a Bochner formula holds for Ric_f^N that looks like the Bochner formula for the Ricci tensor of an $n+N$ dimensional manifold . This formula seems to have been Bakry and Emery's original motivation for the definition of the Bakry-Emery Ricci tensor and for their more general curvature dimension inequalities for diffusion operators [4]. See [17, 18] for elementary proofs of the inequality.

$$\frac{1}{2}\Delta_f |\nabla u|^2 \geq \frac{(\Delta_f(u))^2}{N+n} + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f^N(\nabla u, \nabla u)$$

For the distance function, we actually have

$$m_f' \leq -\frac{(m_f)^2}{n+N-1} - \mathrm{Ric}_f^N(\partial_r, \partial_r).$$

Thus, using the standard Sturm-Liouville comparison argument, or an argument similar to the one we give above, one has the mean curvature comparison.

Theorem A.1 (Mean curvature comparison for N-Bakry-Emery) *[6] Suppose that $N > 0$ and $\mathrm{Ric}_f^N \geq (n+N-1)H$, then*

$$m_f(r) \leq m_H^{n+N}(r).$$

This comparison along with the methods used above gives proofs of the comparison theorems for Ric_f^N.

The Bochner formula for Ric_f^N has many other applications to other geometric problems not treated here such as eigenvalue problems and Liouville theorems, see for example [5] and [18] and the references there in.

In [23] Lott shows that if M is compact with $\mathrm{Ric}_f^q \geq \lambda$ for some positive integer $q \geq 2$, then, in fact, there is a family of metrics on $M \times S^q$ with Ricci curvature bounded below by λ. Moreover, the metrics on the sphere collapse so that M is a Gromov-Hausdorff limit of $n+q$ dimensional manifolds with Ricci curvature bounded below by λ. This gives an alternate approach to prove many of the comparison and topological theorems for Ric_f^q.

B ODE proof of mean curvature comparison

Theorem B.1 (Mean Curvature Comparison) *Assume* $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$,

a) if $\partial_r f \geq -a$ $(a \geq 0)$ *along a minimal geodesic segment from* p *(when* $H > 0$ *assume* $r \leq \pi/2\sqrt{H}$*) then*

$$m_f(r) - m_H(r) \leq a \tag{B-1}$$

along that minimal geodesic segment from p*. Equality holds if and only if the radial sectional curvatures are equal to* H *and* $f(t) = f(p) - at$ *for all* $t < r$.
In particular when $a = 0$*, we have*

$$m_f(r) \leq m_H(r) \tag{B-2}$$

and equality holds if and only if all radial sectional curvatures are H *and* f *is constant.*

b) if $|f| \leq k$ *along a minimal geodesic segment from* p *(when* $H > 0$ *assume* $r \leq \pi/2\sqrt{H}$*) then*

$$m_f(r) - m_H \leq (n-1)e^{\frac{4k}{n-1}} \left(\frac{\sqrt{|H|}\mathrm{sn}_H(2r) + 2|H|r}{\mathrm{sn}_H^2(r)} \right) \tag{B-3}$$

along that minimal geodesic segment from p*, where* $\mathrm{sn}_H(r)$ *is the unique function satisfying*

$$\mathrm{sn}_H''(r) + H\mathrm{sn}_H(r) = 0, \quad \mathrm{sn}_H(0) = 0, \quad \mathrm{sn}_H'(0) = 1.$$

In particular when $H = 0$ *we have*

$$m_f(r) - \frac{n-1}{r} \leq 4(n-1)e^{\frac{4k}{n-1}} \frac{1}{r}. \tag{B-4}$$

Proof: We compare m_f to the mean curvature of the model space. Note that the mean curvature of the model space m_H satisfies

$$m_H' = -\frac{m_H^2}{n-1} - (n-1)H. \tag{B-5}$$

Using $\mathrm{Ric}_f \geq (n-1)H$, and subtracting (2.5) by (B-5) gives

$$(m_f - m_H)' \leq -\frac{1}{n-1}\left[(m_f + \partial_r f)^2 - m_H^2\right] \tag{B-6}$$

$$= -\frac{1}{n-1}\left[(m_f - m_H + \partial_r f)(m_f + m_H + \partial_r f)\right]. \tag{B-7}$$

Proof of Part a): Write (B-7) as the following

$$(m_f - m_H - a)' \leq -\frac{1}{n-1}\left[(m_f - m_H - a + a + \partial_r f)(m_f + m_H + \partial_r f)\right]. \tag{B-8}$$

Let us define $\psi_{a,H} = \max\{m_f - m_H - a, 0\} = (m_f - m_H - a)_+$, and declare that $\psi_{a,H} = 0$ whenever it becomes undefined. Since $\partial_r f \geq -a$, $a + \partial_r f \geq 0$. When $\psi_{a,H} \geq 0$, $m_f + m_H + \partial_r f \geq a + \partial_r f + 2m_H \geq 2m_H$ which is ≥ 0 if $m_H \geq 0$. Using this and (B-8) gives

$$\psi'_{a,H} \leq -\frac{1}{n-1}(m_f + m_H + \partial_r f)\psi_{a,H} \leq 0. \tag{B-9}$$

Since $\lim_{r\to 0} \psi_{a,H}(r) = (-\partial f_r(0) - a)_+ = 0$, we have $\psi_{a,H}(r) = 0$ for all $r \geq 0$. This finishes the proof of the inequality.

Now suppose that $m_f = m_H + a$, then from (B-6) we have that $m = m_H$ which implies that $\partial_r f = -a$. So $\partial_r^2 f \equiv 0$ which then implies that $\mathrm{Ric}(\partial_r, \partial_r) = \mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$. Now the rigidity for the regular mean curvature comparison implies that all the sectional curvatures are constant and equal to H.

Proof of Part b): Write (B-7) as

$$\begin{aligned}(m_f - m_H)' &\leq -\frac{1}{n-1}\left[(m_f - m_H + \partial_r f)(m_f - m_H + 2m_H + \partial_r f)\right] \\ &= -\frac{1}{n-1}\left[(m_f - m_H)^2 + 2(m_H + \partial_r f)(m_f - m_H) + \partial_r f(2m_H + \partial_r f)\right]\end{aligned} \tag{B-10}$$

Now define $\psi = \max\{m_f - m_H, 0\} = (m_f - m_H)_+$, the error from the mean curvature comparison, and declare that $\psi = 0$ whenever it becomes undefined. Define

$$\rho = \left[-\frac{1}{n-1}\partial_r f(2m_H + \partial_r f)\right]_+. \tag{B-11}$$

Using this notation and inequality (B-10) we obtain

$$\psi' \leq -\frac{1}{n-1}\psi^2 - \frac{2}{n-1}(m_H + \partial_r f)\psi + \rho. \tag{B-12}$$

We would like to estimate ψ in terms of ρ. It is enough to use the linear differential inequality

$$\psi' + \frac{2}{n-1}(m_H + \partial_r f)\psi \leq \rho. \tag{B-13}$$

When $\partial_r f = 0$ (in the usual case), we have $\rho = 0$ and $\psi = 0$, getting the classical mean curvature comparison. In general, by (B-11), the definition of ρ, when $m_H > 0$

$$\rho > 0 \Longleftrightarrow -2m_H < \partial_r f < 0. \tag{B-14}$$

Also

$$\rho \leq \left(-\frac{2}{n-1}(\partial_r f)m_H\right)_+. \tag{B-15}$$

Therefore we have

$$\rho \leq \frac{4}{n-1}m_H^2.$$

Note that $m_H = (n-1)\frac{\mathrm{sn}'_H(r)}{\mathrm{sn}_H(r)}$. Now (B-13) becomes

$$\psi' + \left(2\frac{\mathrm{sn}'_H(r)}{\mathrm{sn}_H(r)} + \frac{2}{n-1}\partial_r f\right)\psi \leq 4(n-1)\left(\frac{\mathrm{sn}'_H(r)}{\mathrm{sn}_H(r)}\right)^2.$$

Multiply this by the integrating factor $\mathrm{sn}_H^2(r)e^{\frac{2}{n-1}f(r)}$ to obtain

$$\left(\mathrm{sn}_H^2(r)e^{\frac{2}{n-1}f(r)}\psi(r)\right)' \leq 4(n-1)e^{\frac{2}{n-1}f(r)}(\mathrm{sn}_H'(r))^2.$$

Since $\psi(0)$ is bounded, integrate this from 0 to r gives

$$\mathrm{sn}_H^2(r)e^{\frac{2}{n-1}f(r)}\psi(r) \leq 4(n-1)\int_0^r e^{\frac{2}{n-1}f(t)}(\mathrm{sn}_H'(r))^2 dt. \tag{B-16}$$

Since $|f| \leq k$, we have

$$\psi(r) \leq (n-1)e^{\frac{4k}{n-1}}\left(\frac{\sqrt{|H|}\mathrm{sn}_H(2r) + 2|H|r}{\mathrm{sn}_H^2(r)}\right).$$

When $H = 0$, $\mathrm{sn}_H(r) = r$, from (B-16) we get

$$\psi(r) \leq 4(n-1)e^{\frac{4k}{n-1}}\frac{1}{r}.$$

This completes the proof of Part b). ■

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