A CONTINUUM THEORY OF CHIRAL SMECTIC C LIQUID CRYSTALS[*]

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Abstract. We formulate a nonlinear continuum theory of flow of chiral smectic C liquid crystals (C*) involving molecular director, layer order parameter, polarization vector, flow velocity, and hydrostatic pressure fields. In addition to chiral orientational ordering, smectic C* phases also present positional ordering, with molecular centers of mass arranged in one dimensional layers. The nonzero tilt angle of the molecular director with respect to the layer normal together with the chirality is responsible for the ferroelectric nature of the phase. This results in a stronger coupling with applied electric fields than the dielectric nematic. We apply the model to study the molecular reorientation dynamics in homeotropic geometry under the influence of an applied electric field. The switching process between states with opposite polarization is understood by the traveling wave solution of the system. We prove existence and uniqueness of the traveling wave and show that the predicted switching time is smaller than that when the flow effect is neglected. We also obtain bounds on the speed of switching and an optimality condition on the parameters of the problem. Numerical simulations confirm the predictions of the analysis.

1. Introduction. We develop a model of smectic C* liquid crystals accounting for elastic, hydrodynamic, and electrostatic effects. The free energy includes the Oseen–Frank energy of nematic liquid crystals, the smectic C energy of the form proposed by Chen and Lubensky, and the ferroelectric electrostatic energy. We apply the governing equations to study the switching dynamics, in homeotropic geometry, between two states with opposite electric polarization. We apply a variational method to characterize the speed of the switching traveling wave and show that the predicted speed is greater than in the approach that neglects flow. We also obtain an optimality condition of the speed in terms of the parameters of the problem. We perform numerical simulations to illustrate the dynamics of switching. A main feature of our work is the study of the backflow effect due to the spontaneous polarization of the liquid crystal.

Liquid crystal phases form when a material has a degree of positional or orientational ordering yet stays in a liquid state. In the nematic state, molecules tend to align themselves along a preferred direction with no positional order of centers of mass. The unit vector field $\mathbf{n}$, nematic director, represents the average direction of molecular alignment. Moreover, if the liquid crystal is chiral, $\mathbf{n}$ follows a helical pattern with temperature-dependent pitch. Upon lowering the temperature, or increasing concentration, according to whether the liquid crystal is thermotropic or lyotropic, the nematic liquid crystal experiences a transition to the smectic A phase

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with molecules arranged along equally spaced layers. The molecules tend to align themselves along the direction perpendicular to the layers. Upon transition to the lower temperature smectic C phase, a symmetry break occurs, with molecules making a nonzero tilt angle with the layer normal. Values of the tilt angle α are found to be between 20 and 35 degrees and depend on the material and temperature.

We consider chiral smectic C liquid crystals and label them C*, according to conventional notation. One relevant feature of liquid crystal molecules that form smectic C* phases is the presence of a *side chain* giving a transverse electric dipole and therefore yielding the polarization field $\mathbf{P}$ of the theory. Level surfaces of the scalar variable ϕ describe smectic layers. A schematic representation of the fields of a smectic C* phase is given in Figure 1, where the normal vector to the layers corresponds to the axis of a cone of semiangle α, with $\mathbf{n}$ being allowed to rotate on the surface. The polarization vector $\mathbf{P}$ is perpendicular to both the layer normal $\nabla\phi$ and the director field [16]. Additional fields of the hydrodynamic theory are the velocity field $\mathbf{u}$ and the hydrostatic pressure p. The latter is the constraint associated with the assumption of fluid incompressibility.

In chiral configurations, since $\mathbf{P}$ rotates with $\mathbf{n}$, the net polarization of the material is zero. Therefore, ferroelectric states correspond to configurations with constant $\mathbf{n}$. The electrostatic effects due to polarization dominate the dielectric effects of standard nematic liquid crystals. Consequently, faster switching devices are achieved with smectic C* liquid crystals.

In this article, we study the transition between states with opposite polarization and determine lower bounds for the speed of the connecting traveling wave. The latter corresponds to a chiral configuration with periodically varying $\mathbf{n}$ and $\mathbf{P}$. The switching takes place upon reversing the direction of the applied electric field. The stability of the polarized states was studied in [23]. The traveling wave of our problem represents the backflow effect, that is, the flow generated by changes in the applied electric field.

The free energy density of the model consists of nematic, smectic C, and electrostatic contributions. The form of the smectic C free energy, $\mathbf{F}_S$, that we study was introduced by Chen and Lubensky in 1976, based on the Landau–de Gennes model for smectic A [6]. They investigated the nematic to smectic phase transition, and it was later used by Renn and Lubensky to predict the twist grain boundary phase in cholesteric smectic [19]. However, the free energy density $\mathbf{F}_S$ is degenerate in that it lacks second order coercivity in the direction $\mathbf{n}$. In order to avoid the anisotropic quartic order derivatives in the Chen–Lubensky model, Luk'yanchuk proposed a modified model [20]. The new model was later used in [13] to rigorously analyze the temperature phase transition from chiral nematic to chiral smectic liquid crystals. The analysis of the ferroelectric smectic C* phases was carried out in [24], where the energy minimizers are further required to satisfy the electrostatic Maxwell equations. The hydrodynamic theory that we propose combines the approaches by Leslie and Ericksen (for details, see [8], [4], and [29]) for nematic and the work by W. E [9] for smectic A liquid crystals. The latter follows the model by Kleman and Parodi [14] also for smectic A phases, where the concepts of permeation force and molecular field were introduced as forces driving smectic A flow. However, since the layer position completely specifies the director field in smectic A, W. E shows that only the permeation force is responsible for the dynamics of smectic A liquid crystals. This is not the case for the smectic C modeling, where both forces are needed to describe the hydrodynamics. We use a variational approach together with the dissipation inequality to determine the elastic and viscous components, respectively, of

such forces. Furthermore, the Lorenz force associated with the charge density $-\text{div}\mathbf{P}$ enters the equation of balance of linear momentum.

Leslie, Stewart, and Nakagawa also developed a nonlinear continuum theory for smectic C liquid crystals, using the $\mathbf{c}$ director, which is the projection of $\mathbf{n}$ onto the layer, and the unit vector normal to the layers (see [18] and [29]). Their theory is constrained to exclude variations in the layer spacing thickness and changes in tilt with respect to the layer. The nonlinear continuum theory in the present paper is also restricted to the constant tilt angle case, excluding the variation of the tilt angle between the director and the layer normal as in [18]. However, our model allows the variation of the layer spacing thickness.

The second part of the paper is devoted to the study of the switching dynamics of a smectic C* sample confined between parallel bounding planes. We assume that the electric field is applied parallel to the smectic layers. We derive the governing equation of the director and the flow equation in the homeotropic geometry, where the smectic layers are parallel to the bounding plates. When the flow is neglected, the director profile can be understood by the traveling wave solution of the resulting nonlinear reaction diffusion equation (see [7], [21], [25], and [26]). This equation also represents the gradient flow of the energy. One main goal of our work is to study the traveling wave solutions, taking flow and ferroelectric effects into account and estimating the speed of the corresponding traveling wave. The variational characterization of the speed follows the approach in [2] for reaction-diffusion equations. Furthermore, we obtain an optimal lower bound of the speed in terms of the viscous and smectic parameters of the model. Numerical simulations of the problem explore ranges of parameters, from the case in which flow is neglected to cases in which parameters approach the optimal lower bound of the speed. We find a very good agreement with the predictions of the analysis.

Section 2 is devoted to static theory, and dissipation and hydrodynamics are discussed in section 3. The analysis of traveling waves of the switching problem and the corresponding numerical simulations are developed in section 4.

2. Hydrostatic theory.

2.1. Smectic C* free energy. The total free energy density consists of the nematic f_n and smectic f_s parts. The Oseen–Frank energy density for a nematic is given by

$$f_n = \frac{K_1}{2}(\nabla \cdot \mathbf{n})^2 + \frac{K_2}{2}(\mathbf{n} \cdot \nabla \times \mathbf{n} + \tau)^2 + \frac{K_3}{2}|\mathbf{n} \times (\nabla \times \mathbf{n})|^2,$$

where K_1, K_2, and K_3 are the splay, twist, and bend elastic constants, respectively. The parameter τ denotes the cholesteric twist.

In order to associate smectic and nematic structure with a state $(\mathbf{n}, \Psi)$, we write

$$\Psi(\mathbf{x}) = \rho(\mathbf{x})e^{i\varphi(\mathbf{x})}.$$

Then the molecular mass density is defined by

$$\delta(\mathbf{x}) = \rho_0(\mathbf{x}) + \frac{1}{2}(\Psi(\mathbf{x}) + \Psi^*(\mathbf{x})) = \rho_0(\mathbf{x}) + \rho(\mathbf{x})\cos\varphi(\mathbf{x}),$$

where ρ_0 is a locally uniform mass density, $\rho(\mathbf{x})$ is the mass density of the smectic layers, and φ parametrizes the layers so that $\nabla\varphi$ is the direction of the layer normal.

Now the smectic C energy density is given by

$$f_s = \frac{D}{2}|\mathbf{D_n}^2\Psi|^2 - \frac{C_\perp}{2}|\mathbf{D_n}\Psi|^2 + \frac{C_\parallel}{2}|\mathbf{n} \cdot \mathbf{D_n}\Psi|^2,$$

where $\mathbf{D_n} = \nabla - iq\mathbf{n}$, $\mathbf{D_n}^2 = \mathbf{D_n} \cdot \mathbf{D_n}$, and $D, C_\perp, C_\parallel$ are positive constants. The model for smectic C energy was proposed by Chen and Lubensky [6], but we use the modified model, introduced by Luk'yanchuk [20]. Since we investigate smectic structure far from the nematic–smectic transition, we assume that the magnitude of the smectic order parameter is a constant. We may assume that $\Psi = e^{i\varphi}$. Then f_s becomes

$$f_s = \frac{D}{2}|\nabla\varphi - q\mathbf{n}|^4 + \frac{D}{2}(\Delta\varphi - q\nabla \cdot \mathbf{n})^2 - \frac{C_\perp}{2}|\nabla\varphi - q\mathbf{n}|^2 + \frac{C_\parallel}{2}(\mathbf{n} \cdot \nabla\varphi - q)^2$$

$$= \frac{D}{2}\left(|\nabla\varphi - q\mathbf{n}|^2 - \frac{C_\perp}{2D}\right)^2 + \frac{C_\parallel}{2}(\mathbf{n} \cdot \nabla\varphi - q)^2 + \frac{D}{2}(\Delta\varphi - q\nabla \cdot \mathbf{n})^2.$$

If $\mathbf{n}$ is a constant and φ is linear, then we can see that the energy is minimized if and only if $\mathbf{n} \cdot \nabla\varphi = q$ and $|\nabla\varphi - q\mathbf{n}|^2 = \frac{C_\perp}{2D}$. This corresponds to a uniform smectic C state with tilt angle α, between the director and the layer normal, determined by $\tan^2\alpha = C_\perp/(2Dq^2)$ and layer thickness d satisfying $(\frac{2\pi}{d})^2 = q^2 + \frac{C_\perp}{2D}$.

We get the free energy density

$$f_d = f_n + f_s.$$

Note that there are two constraints:

(2.1) $|\mathbf{n}| = 1$ and $\mathbf{n} \cdot \nabla\varphi = \cos\alpha|\nabla\varphi|.$

We consider the total smectic C free energy density

(2.2) $\tilde{f} = f_d + f_l,$

where the last term is present in order to make use of Lagrange multipliers:

$$f_l = \frac{\lambda}{2}(\mathbf{n} \cdot \mathbf{n} - 1) + \beta(\mathbf{n} \cdot \nabla\varphi - \cos\alpha|\nabla\varphi|).$$

2.2. The molecular field and the permeation force. In the nematic, the molecular field can be obtained through the deformation of the director field while the centers of gravity of the molecules are fixed. On the other hand, the directors are parallel to the layer normal in smectic A. As a result, W. E instead discussed the permeation forces, the normal forces acting on layers in [9]. In smectic C, the directors are tilted with respect to the layer normal, and hence both the molecular field and the permeation force need to be discussed. First we obtain the equilibrium conditions in bulk by writing the variation of the total free energy with respect to the director and the layer normal variations, while keeping the material undeformed. Let D be any region inside the liquid crystal. We get

$$\delta\int_D \tilde{f} = \int_D \left(\frac{\partial\tilde{f}}{\partial(\Delta\varphi)}\delta(\Delta\varphi) + \frac{\partial\tilde{f}}{\partial(\partial_i\varphi)}\delta(\partial_i\varphi) + \frac{\partial\tilde{f}}{\partial(\partial_i\mathbf{n}_j)}\delta(\partial_i\mathbf{n}_j) + \frac{\partial\tilde{f}}{\partial\mathbf{n}_i}\delta\mathbf{n}_i\right)d\mathbf{x}$$

$$= \int_D \left(\partial_i \left[\partial_i \left(\frac{\partial \tilde{f}}{\partial(\Delta\varphi)} \right) - \frac{\partial \tilde{f}}{\partial(\partial_i\varphi)} \right] \delta\varphi + \left[\frac{\partial \tilde{f}}{\partial \mathbf{n}_i} - \partial_j \left(\frac{\partial \tilde{f}}{\partial(\partial_j\mathbf{n}_i)} \right) \right] \delta\mathbf{n}_i \right) d\mathbf{x}$$

$$+ \int_{\partial D} \left(\frac{\partial \tilde{f}}{\partial(\Delta\varphi)} \partial_j(\delta\varphi) + \left[\frac{\partial \tilde{f}}{\partial(\partial_j\varphi)} - \partial_j(\frac{\partial \tilde{f}}{\partial(\Delta\varphi)}) \right] \delta\varphi + \frac{\partial \tilde{f}}{\partial(\partial_j\mathbf{n}_i)} \delta\mathbf{n}_i \right) \nu_j \, ds$$

$$=: \int_D (-g\delta\varphi - \mathbf{h}_i\delta\mathbf{n}_i) \, d\mathbf{x} + \int_{\partial D} \left(\frac{\partial \tilde{f}}{\partial(\Delta\varphi)} \partial_j(\delta\varphi) - \tau_j\delta\varphi + \pi_{ij}\delta\mathbf{n}_i \right) \nu_j \, ds,$$

where

$$
\begin{aligned}
g &= -\nabla \cdot \tau \\
&= -\nabla \cdot \left[\nabla \left(\frac{\partial f_d}{\partial(\Delta\varphi)} \right) - \frac{\partial f_d}{\partial(\nabla\varphi)} - \beta \left(\mathbf{n} - \cos\alpha \frac{\nabla\varphi}{|\nabla\varphi|} \right) \right],
\end{aligned}
$$

(2.3)

$$\mathbf{h}_i = -\frac{\partial f_d}{\partial\mathbf{n}_i} + \partial_j\pi_{ij} - \lambda\mathbf{n}_i - \beta\partial_i\varphi,$$

using the notation

(2.4)
$$\pi_{ij} = \frac{\partial f_d}{\partial(\partial_j\mathbf{n}_i)}.$$

2.3. The elastic stress. We now calculate the elastic stress associated with the infinitesimal deformation of the body, while holding the location of the layers and the director field fixed. For this, we let

$$
\begin{aligned}
\mathbf{r}' &= \mathbf{r} + \mathbf{u}(\mathbf{r}), \\
\mathbf{n}'(\mathbf{r}') &= \mathbf{n}'(\mathbf{r} + \mathbf{u}) = \mathbf{n}(\mathbf{r}), \\
\varphi'(r') &= \varphi(\mathbf{r} + \mathbf{u}) = \varphi(\mathbf{r}).
\end{aligned}
$$

Using the relations

$$\frac{\partial r'_i}{\partial r_j} = \delta_{ij} + \frac{\partial u_i}{\partial r_j} \quad \text{and} \quad \frac{\partial r_i}{\partial r'_j} \simeq \delta_{ij} - \frac{\partial(\delta u_i)}{\partial r_j},$$

we get

$$\frac{\partial\varphi'}{\partial r'_j} \simeq \frac{\partial\varphi}{\partial r_j} - \frac{\partial\varphi}{\partial r_k}\frac{\partial u_k}{\partial r_j} \quad \text{and} \quad \frac{\partial\mathbf{n}'_i}{\partial r'_j} \simeq \frac{\partial\mathbf{n}_i}{\partial r_j} - \frac{\partial\mathbf{n}_i}{\partial r_k}\frac{\partial u_k}{\partial r_j}.$$

Hence

$$
\begin{aligned}
\delta(\partial_i\varphi) &\simeq -\frac{\partial\varphi}{\partial r_k}\frac{\partial u_k}{\partial r_i}, \\
\delta(\partial_i\mathbf{n}_j) &\simeq -\frac{\partial\mathbf{n}_j}{\partial r_k}\frac{\partial u_k}{\partial r_i}, \\
\delta(\Delta\varphi) &\simeq -2(\partial_{ik}\varphi)(\partial_i u_k) - (\partial_k\varphi)\Delta u_k.
\end{aligned}
$$

Taking these approximations into account, we now calculate the corresponding variation of the energy of a subdomain D in Ω. For this, we use integration by parts. This

gives

$$\delta \int_D \tilde{f} = \int_D \left(\frac{\partial \tilde{f}}{\partial(\Delta\varphi)} \delta(\Delta\varphi) + \frac{\partial \tilde{f}}{\partial(\partial_i\varphi)} \delta(\partial_i\varphi) + \frac{\partial \tilde{f}}{\partial(\partial_j \mathbf{n}_i)} \delta(\partial_j \mathbf{n}_i) \right) d\mathbf{x}$$

$$= \int_D \left(\frac{\partial f_d}{\partial(\Delta\varphi)} \left[-2(\partial_{jk}\varphi)(\partial_j \mathbf{u}_k) - (\partial_k\varphi)(\partial_j^2 \mathbf{u}_k) \right] \right) d\mathbf{x}$$

$$+ \int_D \left(\left[\frac{\partial f_d}{\partial(\partial_j\varphi)} + \beta \left(\mathbf{n}_j - \cos\alpha \frac{\partial_j\varphi}{|\nabla\varphi|} \right) \right] \left(-(\partial_k\varphi)(\partial_j \mathbf{u}_k) \right) \right) d\mathbf{x}$$

$$+ \int_D \left(\frac{\partial f_d}{\partial(\partial_i \mathbf{n}_j)} \left(-(\partial_k \mathbf{n}_i)(\partial_j \mathbf{u}_k) \right) \right) d\mathbf{x} - \int_{\partial D} \frac{\partial \tilde{f}}{\partial(\Delta\varphi)} (\partial_k\varphi)(\partial_j \mathbf{u}_k)\nu_j \, ds$$

$$=: \int_D \left(\sigma_{kj}^d (\partial_j \mathbf{u}_k) \right) d\mathbf{x} - \int_{\partial D} \frac{\partial \tilde{f}}{\partial(\Delta\varphi)} (\partial_k\varphi)(\partial_j \mathbf{u}_k)\nu_j \, ds,$$

where

(2.5)
$$\sigma_{kj}^d = \left[-\frac{\partial f_d}{\partial(\Delta\varphi)} \partial_{jk}\varphi + \tau_j(\partial_k\varphi) - \pi_{ij}(\partial_k \mathbf{n}_i) \right]$$

is the deviatoric part of the stress tensor. To take this incompressibility constraint into account, we modify the previous calculations to include the corresponding Lagrange multiplier term. For this, let us consider the free energy density

$$f = \tilde{f} - p\nabla \cdot \mathbf{u},$$

where p is a Lagrange multiplier. This leads to a modified elastic stress

(2.6)
$$\sigma_{kj}^e = \sigma_{kj}^d - p\delta_{kj}.$$

2.4. The equilibrium equations. By combining all variations from the previous sections, we have the total variation of $\tilde{f}$:

(2.7)
$$\delta \int_D \tilde{f} = \int_D \left(\sigma_{kj}^e (\partial_j \mathbf{u}_k) - \mathbf{h}_k \delta \mathbf{n}_k - g\delta\varphi \right) d\mathbf{x}$$
$$+ \int_{\partial D} \left(\frac{\partial \tilde{f}}{\partial(\Delta\varphi)} \partial_j(\delta\varphi) - \tau_j \delta\varphi + \pi_{kj} \delta \mathbf{n}_k - \frac{\partial \tilde{f}}{\partial(\Delta\varphi)} (\partial_k\varphi)(\partial_j \mathbf{u}_k) \right) \nu_j \, ds.$$

By integration by parts, it becomes

$$\delta \int_D \tilde{f} = \int_D \left(-\partial_j(\sigma_{kj}^e)\mathbf{u}_k - \mathbf{h}_k \delta \mathbf{n}_k - g\delta\varphi \right) d\mathbf{x}$$
$$+ \int_{\partial D} \left(\frac{\partial \tilde{f}}{\partial(\Delta\varphi)} \partial_j(\delta\varphi) - \tau_j \delta\varphi + \pi_{kj} \delta \mathbf{n}_k \right) \nu_j \, ds$$
$$+ \int_{\partial D} \left(\sigma_{kj}^e \mathbf{u}_k - \frac{\partial \tilde{f}}{\partial(\Delta\varphi)} (\partial_k\varphi)(\partial_j \mathbf{u}_k) \right) \nu_j \, ds.$$

Then the hydrostatic equilibrium condition is

$$\delta \int_D \tilde{f} = 0$$

for all $D \subseteq \Omega$ and for admissible variations. Taking variations such that $\delta \mathbf{n} = 0$, $\delta \varphi = 0$, and $\mathbf{u}$ and its gradient vanish at the boundary gives the system of partial differential equations

(2.8) $\mathbf{h} = 0, \qquad g = 0,$

and

(2.9) $\partial_j \sigma_{kj} = 0.$

These, together with two constraint relations (2.1), give the equilibrium equations for smectic C without the external fields. In three dimensions, the system (2.1) and (2.8) consists of six scalar equations for the six unknowns $\mathbf{n}$, φ, λ, and β. Well posedness of this system, in its variational form, was studied in [13] and [24]. Moreover, in [13], the authors performed an extensive phase transition and stability analysis of equilibrium states. In [24], the role of permanent polarization was a main focus of the work.

Notice that the combined system (2.1), (2.8), and (2.9) is overdetermined. This issue in nematic liquid crystals was addressed by Ericksen in [10]. He argues that artificial body forces have to be included in the equations for the system to be closed.

2.5. The balance of torques. We integrate by parts (2.7) to obtain

(2.10)
$$\delta \int_D \tilde{f} = \int_D \left(\sigma_{kj}^e (\partial_j \mathbf{u}_k) - \mathbf{h}_k \delta \mathbf{n}_k - \tau_k \partial_k (\delta \varphi) \right) d\mathbf{r}$$
$$+ \int_{\partial D} \left(\frac{\partial \tilde{f}}{\partial (\Delta \varphi)} \partial_j (\delta \varphi) + \pi_{kj} \delta \mathbf{n}_k - \frac{\partial \tilde{f}}{\partial (\Delta \varphi)} (\partial_k \varphi)(\partial_j \mathbf{u}_k) \right) \nu_j \, ds.$$

Notice that $\tilde{f}$ is invariant under the rotation of the centers of gravity, the directors, and the layers by the same angle ω. Now the energy is unchanged under the following replacement:

$$\mathbf{u}(\mathbf{r}) = \omega \times \mathbf{r},$$
$$\delta \mathbf{n}(\mathbf{r}) = \omega \times \mathbf{n},$$
$$\delta \varphi = 0,$$

where ω is the rotation vector. Then we have

$$\partial_j \mathbf{u}_k = \varepsilon_{kpj} \omega_p.$$

Therefore, we have, from (2.10),

(2.11)
$$\delta \int_D f_d = \varepsilon_{kpj} \omega_p \int_D \left[\sigma_{kj}^d - \mathbf{h}_k \mathbf{n}_j + \beta \left(\mathbf{n}_j - \cos \alpha \frac{\partial_j \varphi}{|\nabla \varphi|} \right) \partial_k \varphi - \lambda \mathbf{n}_k \mathbf{n}_j - \beta \partial_k \varphi \mathbf{n}_j \right] d\mathbf{r}$$
$$+ \omega_p \int_{\partial D} \left(\varepsilon_{kpq} \pi_{kj} \mathbf{n}_q \nu_j - \frac{\partial f}{\partial (\Delta \varphi)} \varepsilon_{kpj} (\partial_k \varphi) \nu_j \right) ds$$
$$= \varepsilon_{kpj} \omega_p \int_D \left[\sigma_{kj}^d - \mathbf{h}_k \mathbf{n}_j \right] d\mathbf{r}$$
$$+ \omega_p \int_{\partial D} \left(\varepsilon_{kpq} \pi_{kj} \mathbf{n}_q \nu_j - \frac{\partial f}{\partial (\Delta \varphi)} \varepsilon_{kpj} (\partial_k \varphi) \nu_j \right) ds = 0.$$

From the equilibrium condition (2.8), we obtain

$$\int_D \varepsilon_{kpj}(\sigma_{kj}^d)d\mathbf{r} + \int_{\partial D} \varepsilon_{kpq}\left(\mathbf{s}_k\mathbf{n}_q + (\partial_q\varphi)\,\mathbf{m}_k\right)ds = 0,$$

where $\mathbf{s}_i = \pi_{ij}\nu_j$ and $\mathbf{m}_i = \frac{\partial f}{\partial(\Delta\varphi)}\nu_i$. Using the notation Γ for the antisymmetric part of the elastic stress, we have

(2.12)
$$\varepsilon_{pji}\sigma_{ij}^d = \Gamma_p(\sigma^d),$$

and using the fact $\Gamma_p(\sigma^e) = \Gamma_p(\sigma^d)$, we have

(2.13)
$$\int_D \Gamma_p(\sigma^e)\,d\mathbf{r} + \int_{\partial D} (\mathbf{n}\times\mathbf{s} + \nabla\varphi\times\mathbf{m}) = 0.$$

From (2.9), we have

$$0 = \int_D \varepsilon_{jpq}\left(\partial_i(\sigma_{ji}^e)\right)\mathbf{r}_q$$

$$= \varepsilon_{jpq}\left(-\int_D (\sigma_{ji}^e + \phi_{ji})\partial_i\mathbf{r}_q + \int_{\partial D} (\sigma_{ji}^e)\,\mathbf{r}_q\,\nu_i\right)$$

$$= -\int_D \varepsilon_{jpi}(\sigma_{ji}^e) + \varepsilon_{jpq}\int_{\partial D} (\sigma_{ji}^e)\,\mathbf{r}_q\,\nu_i.$$

Therefore,

$$\int_D \Gamma_p(\sigma^e)\,d\mathbf{r} = \int_{\partial D} \mathbf{r}\times\mathbf{t}\,d\mathbf{x},$$

where $\mathbf{t}_j = (\sigma_{ji}^e)\,\nu_i$. Inserting this equation into (2.13), we finally obtain

(2.14)
$$\int_{\partial D} (\mathbf{r}\times\mathbf{t} + \mathbf{n}\times\mathbf{s} + \nabla\varphi\times\mathbf{m})\,ds = 0.$$

This indicates that there are three contributions to these surface torques: mechanical torque (due to the stress tensor), director torque, and layer torque. This is the analogue of the balance of torques in nematic liquid crystals given by equation (3.115) of [8].

3. Hydrodynamic theory. In this section, we derive the hydrodynamic equations for smectic C liquid crystals following previous work by Ericksen and Leslie (see [8], [4], [29], and [17]) for nematics and work by W. E [9] for smectic A. As we mentioned in the introduction, both the director and the layer functions are hydrodynamic variables.

3.1. Balance laws. The equations of balance of mass, linear momentum, energy, and angular momentum are given by

(3.1)
$$\frac{d}{dt}\int_D \rho\,d\mathbf{x} = 0,$$

(3.2)
$$\frac{d}{dt}\int_D \rho\mathbf{v}_i\,d\mathbf{x} = \int_{\partial D} \sigma_{ij}ds_j,$$

(3.3)
$$\frac{d}{dt}\int_D E\,d\mathbf{x} = \int_{\partial D} (\sigma\mathbf{v} + \dot\varphi\tau + \dot{\mathbf{n}}\pi)\cdot ds - \int_{\partial D} \mathbf{q}\cdot ds,$$

(3.4)
$$\frac{d}{dt}\int_D (\mathbf{r}\times\rho\mathbf{v})\,d\mathbf{r} = \int_{\partial D} (\mathbf{r}\times\mathbf{t} + \mathbf{n}\times\mathbf{s} + \nabla\varphi\times\mathbf{m})\,ds,$$

where $D \subseteq \Omega$. We neglect body forces for simplicity, but they can easily be included as needed. Here, σ, τ, π, $\mathbf{t}$, $\mathbf{s}$, and $\mathbf{m}$ consist of the equilibrium components obtained in the previous sections and the dissipative components to be calculated next. The energy density in (3.3) is given by $E = \frac{1}{2}|\mathbf{v}|^2 + e$, where e denotes the internal energy per unit mass. The terms on the right-hand side of (3.3) represent the work done by the stress, the layer permeation force, and the director force, respectively, on the material. The vector field $\mathbf{q}$ denotes the heat flux.

Since the above balance laws are valid for any $D \subseteq \Omega$, by the Reynolds transport theorem, (3.1), (3.2), and (3.3) yield

$$(3.5) \qquad \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{or} \quad \dot{\rho} = -\rho \nabla \cdot \mathbf{v},$$

$$(3.6) \qquad \rho \dot{\mathbf{v}} = \nabla \cdot \sigma \quad \text{or} \quad \rho \dot{v}_i = \partial_j \sigma_{ij},$$

$$(3.7) \qquad \rho \left(\frac{1}{2} \mathbf{v}^2 + e \right)^{\cdot} = \nabla \cdot (\sigma \mathbf{v} + \dot{\varphi} \tau + \dot{\mathbf{n}} \pi) - \nabla \cdot \mathbf{q},$$

where $\dot{f} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$ is a material derivative. The local form of (3.4) becomes a symmetry relation on constitutive equations, analogous to (2.28) of [10] for the nematic liquid crystal in the static case. It is guaranteed to hold for fields satisfying the balance of linear momentum, provided that the constitutive equations are invariant under rigid body rotations. It will be used later in the connection with the entropy inequality. In this paper, we assume incompressibility of the flow; that is, $\nabla \cdot \mathbf{v} = 0$ holds, and consequently, ρ is constant.

3.2. The entropy inequality. We assume that the second law of thermodynamics in the form of the Clausius–Duhem inequality

$$(3.8) \qquad \rho \dot{S} + \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) \geq 0$$

holds for all processes. Here field S denotes the entropy of the system per unit mass. We use this inequality to determine the forms of the dissipative contribution to stresses and forces [17]. Taking (3.6) into account, we rewrite the balance of energy (3.7) as follows:

$$\rho \dot{e} = -\nabla \cdot \mathbf{q} + Tr(\sigma \nabla \mathbf{v}) + \nabla \cdot (\dot{\varphi} \tau + \dot{\mathbf{n}} \pi).$$

We let

$$(3.9) \qquad H = e - TS$$

denote the Helmholtz free energy density. Substituting (3.9) into inequality (3.8), using the balance of energy, and omitting the pure divergence terms, we obtain

$$\rho \dot{H} = \rho(\dot{e} - T\dot{S} - S\dot{T})$$
$$\leq Tr(\sigma \nabla \mathbf{v}) - \nabla \cdot \mathbf{q} + \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) T - \rho S \dot{T}$$
$$= Tr(\sigma \nabla \mathbf{v}) - \rho S \dot{T} - \frac{\mathbf{q} \cdot \nabla T}{T}.$$

Since $H = H(\rho, \mathbf{n}, \nabla \mathbf{n}, \nabla \varphi, \Delta \varphi, T)$,

$$\dot{H} = \frac{\partial H}{\partial T} \dot{T} + \frac{\partial H}{\partial \mathbf{n}} \dot{\mathbf{n}} + \frac{\partial H}{\partial (\nabla \mathbf{n})} (\nabla \mathbf{n})^{\cdot} + \frac{\partial H}{\partial (\nabla \varphi)} (\nabla \varphi)^{\cdot} + \frac{\partial H}{\partial (\Delta \varphi)} (\Delta \varphi)^{\cdot} + \frac{\partial H}{\partial \rho} \dot{\rho}.$$

A direct computation gives

$$\nabla\dot\varphi = (\nabla\varphi)\dot{} + \nabla\mathbf{v}\nabla\varphi,$$
$$\nabla\dot{\mathbf{n}} = (\nabla\mathbf{n})\dot{} + \nabla\mathbf{v}\nabla\mathbf{n},$$
$$\Delta\dot\varphi = (\Delta\varphi)\dot{} + 2Tr(\nabla^2\varphi\nabla\mathbf{v}) + \Delta\mathbf{v}\cdot\nabla\varphi.$$

So, we get

$$Tr(\sigma\nabla\mathbf{v}) - \frac{1}{T}\mathbf{q}\cdot\nabla T - \rho\dot{T}\left(S + \frac{\partial H}{\partial T}\right)$$
$$-\rho\frac{\partial H}{\partial\mathbf{n}}\dot{\mathbf{n}} - \rho\frac{\partial H}{\partial(\nabla\mathbf{n})}\{\nabla\dot{\mathbf{n}} - \nabla\mathbf{v}\nabla\mathbf{n}\} - \rho\frac{\partial H}{\partial(\nabla\varphi)}\{\nabla\dot\varphi - (\nabla\mathbf{v})(\nabla\varphi)\}$$
$$-\rho\frac{\partial H}{\partial(\Delta\varphi)}\left\{\Delta\dot\varphi - 2Tr(\nabla^2\varphi\nabla\mathbf{v}) - \Delta\mathbf{v}\cdot\nabla\varphi - \rho\frac{\partial H}{\partial\rho}\dot\rho\right\} \geq 0.$$

Since, in particular, such an inequality holds for all possible choices of $\dot{T}$, we find that $S = -\frac{\partial H}{\partial T}$. For smectic C liquid crystals, we take $\tilde{f} = \rho H$, as in (2.2). Moreover, since the density is constant, the previous inequality becomes

$$Tr\left[\left(\sigma + \frac{\partial\tilde{f}}{\partial(\nabla\mathbf{n})}\nabla\mathbf{n} + \frac{\partial\tilde{f}}{\partial(\Delta\varphi)}\nabla^2\varphi - \nabla\left\{\frac{\partial\tilde{f}}{\partial(\Delta\varphi)}\nabla\varphi\right\}\right)\nabla\mathbf{v}\right] - \frac{1}{T}\mathbf{q}\cdot\nabla T$$
$$-\left(\frac{\partial\tilde{f}}{\partial\mathbf{n}} - \nabla\frac{\partial\tilde{f}}{\partial(\nabla\mathbf{n})}\right)\dot{\mathbf{n}} - \left(-\nabla\cdot\frac{\partial\tilde{f}}{\partial(\nabla\varphi)} + \Delta\frac{\partial\tilde{f}}{\partial(\Delta\varphi)}\right)\dot\varphi \geq 0.$$

Assuming that the stress consists of elastic and dissipative parts, σ^e (equation (2.6)) and σ^v, respectively, we write

$$\sigma = \sigma^e + \sigma^v.$$

This, together with substituting (2.3), (2.4), and (2.5) into the inequality, gives

$$Tr(\sigma^v\nabla\mathbf{v}) - \frac{1}{T}\mathbf{q}\cdot\nabla T + \mathbf{h}\cdot\dot{\mathbf{n}} + g\dot\varphi \geq 0.$$

Let us introduce the notation

$$2\sigma^{sym} = \sigma^v + (\sigma^v)^T,$$
$$2D = \nabla\mathbf{v} + (\nabla\mathbf{v})^T, \qquad 2\mathbf{w} = \nabla\times\mathbf{v}.$$

We denote $D = (d_{ij})$. From (2.12), we have

(3.10)
$$\Gamma(\sigma) = (\sigma_{32} - \sigma_{23}, \sigma_{13} - \sigma_{31}, \sigma_{21} - \sigma_{12}).$$

Therefore,

(3.11)
$$Tr(\sigma^{sym}D) + \Gamma(\sigma^v)\cdot\mathbf{w} + \mathbf{h}\cdot\dot{\mathbf{n}} + g\dot\varphi - \frac{1}{T}\mathbf{q}\cdot\nabla T \geq 0.$$

Following de Gennes [8], in order to characterize $\Gamma(\sigma^v)$, we need to consider the balance of angular momentum. Using integration by parts, (3.4) becomes

$$\int_D \mathbf{r}\times\rho\dot{\mathbf{v}}\,d\mathbf{r} = \int_D (\Gamma(\sigma) + \mathbf{r}\times(\nabla\cdot\sigma))\,d\mathbf{r} + \int_{\partial D}(\mathbf{n}\times\mathbf{s} + \nabla\varphi\times\mathbf{m})\,ds.$$

By substituting (3.6) into this, we get

$$\int_D \Gamma(\sigma)\,d\mathbf{x} + \int_{\partial D} (\mathbf{n} \times \mathbf{s} + \nabla\varphi \times \mathbf{m})\,ds = 0.$$

Moreover, using (2.11), we now get

$$(3.12) \qquad \int_D \Gamma(\sigma^v)\,d\mathbf{r} = \int_D (-\mathbf{n} \times \mathbf{h})\,d\mathbf{r},$$

which, substituted into the inequality, gives

$$(3.13) \qquad Tr(\sigma^{sym}D) + \mathbf{h} \cdot \mathbf{N} + g\dot{\varphi} - \frac{1}{T}\mathbf{q} \cdot \nabla T \geq 0,$$

where $\mathbf{N} = \dot{\mathbf{n}} - \mathbf{w} \times \mathbf{n}$. Observe that $\mathbf{n} \cdot \mathbf{N} = 0$ holds as a result of the constraint $|\mathbf{n}| = 1$.

We conclude this subsection by writing the skew part of the viscous stress. Since (3.12) holds for all $D \subseteq \Omega$, we have

$$(3.14) \qquad \sigma_{ij}^{skw} = \frac{1}{2}\varepsilon_{kji}\Gamma_k = -\frac{1}{2}\varepsilon_{kji}\mathbf{n}_j h_i.$$

From now on, we will denote $\mathbf{l} = \nabla\varphi$.

3.3. Coefficients of viscosity. We consider linear dependence of the dissipative forces on their fluxes. We require $\mathbf{q}$ and σ^{sym} to be invariant under the simultaneous transformations $\mathbf{n} \to -\mathbf{n}$ and $\nabla\varphi \to -\nabla\varphi$. We impose that $\mathbf{h}$ and $\dot{\varphi}$ change into $-\mathbf{h}$ and $-\dot{\varphi}$, respectively, under the same transformation. Hence the most general form of the equations is

$$(3.15) \qquad \sigma_{ij}^{sym} = A_{ijk}^1 \mathbf{N}_k + A_{ijkm}^2 d_{km},$$

$$(3.16) \qquad \mathbf{h}_i = B_{ij}^1 \mathbf{N}_j + B_{ijk}^2 d_{jk},$$

$$(3.17) \qquad \mathbf{q}_i = C_{ij}^3 \frac{1}{T}\frac{\partial T}{\partial x_j} + C_i^4 g,$$

$$(3.18) \qquad \dot{\varphi} = D_i^3 \frac{1}{T}\frac{\partial T}{\partial x_i} + D^4 g,$$

where A, B, C, and D are functions of $\mathbf{n}_i$ and $\partial_i\varphi$. The most general form of $\mathbf{h}$ that meets the invariance requirement is

$$(3.19) \qquad \begin{aligned} \mathbf{h} = {} & \beta_1 \mathbf{N} + \beta_2 (\mathbf{N} \cdot \mathbf{l})\mathbf{l} + \beta_3 D\mathbf{n} + \beta_4 D\mathbf{l} + \beta_5 (\mathbf{n} \cdot D\mathbf{l})\mathbf{l} \\ & + \beta_6 (\mathbf{n} \cdot D\mathbf{n})\mathbf{l} + \beta_7 (\mathbf{l} \cdot D\mathbf{l})\mathbf{l}. \end{aligned}$$

The skew-symmetric part of the viscous stress follows from (3.14) together with (3.19). Writing the symmetric part of the stress tensor explicitly from (3.15) and adding to it the skew part, gives

$$(3.20) \qquad \begin{aligned} \sigma^v = {} & \alpha_1 D + \alpha_2 D\mathbf{n} \otimes \mathbf{n} + \alpha_3 \mathbf{n} \otimes D\mathbf{n} + \alpha_4 (D\mathbf{l} \otimes \mathbf{l} + \mathbf{l} \otimes D\mathbf{l}) + \alpha_5 D\mathbf{l} \otimes \mathbf{n} \\ & + \alpha_6 \mathbf{n} \otimes D\mathbf{l} + \alpha_7 (D\mathbf{n} \otimes \mathbf{l} + \mathbf{l} \otimes D\mathbf{n}) + \alpha_8 (\mathbf{l} \cdot D\mathbf{l})\mathbf{l} \otimes \mathbf{n} + \alpha_9 (\mathbf{l} \cdot D\mathbf{l})\mathbf{n} \otimes \mathbf{l} \\ & + \alpha_{10} (\mathbf{n} \cdot D\mathbf{l})\mathbf{l} \otimes \mathbf{l} + \alpha_{11} (\mathbf{l} \cdot D\mathbf{l})\mathbf{n} \otimes \mathbf{n} + \alpha_{12} (\mathbf{n} \cdot D\mathbf{l})\mathbf{l} \otimes \mathbf{n} \\ & + \alpha_{13} (\mathbf{n} \cdot D\mathbf{l})\mathbf{n} \otimes \mathbf{l} + \alpha_{14} (\mathbf{n} \cdot D\mathbf{n})\mathbf{l} \otimes \mathbf{l} + \alpha_{15} (\mathbf{n} \cdot D\mathbf{l})\mathbf{n} \otimes \mathbf{n} \\ & + \alpha_{16} (\mathbf{n} \cdot D\mathbf{n})\mathbf{l} \otimes \mathbf{n} + \alpha_{17} (\mathbf{n} \cdot D\mathbf{n})\mathbf{n} \otimes \mathbf{l} + \alpha_{18} (\mathbf{n} \cdot D\mathbf{n})\mathbf{n} \otimes \mathbf{n} \\ & + \alpha_{19} (\mathbf{l} \cdot D\mathbf{l})\mathbf{l} \otimes \mathbf{l} + \alpha_{20} (\mathbf{l} \otimes \mathbf{N} + \mathbf{N} \otimes \mathbf{l}) + \alpha_{21} \mathbf{N} \otimes \mathbf{n} + \alpha_{22} \mathbf{n} \otimes \mathbf{N} \\ & + \alpha_{23} (\mathbf{l} \cdot \mathbf{N})\mathbf{l} \otimes \mathbf{l} + \alpha_{24} (\mathbf{l} \cdot \mathbf{N})\mathbf{l} \otimes \mathbf{n} + \alpha_{25} (\mathbf{l} \cdot \mathbf{N})\mathbf{n} \otimes \mathbf{l} + \alpha_{26} (\mathbf{l} \cdot \mathbf{N})\mathbf{n} \otimes \mathbf{n}. \end{aligned}$$

We also get

$$\mathbf{q} = \frac{1}{T}(\mu_1 \partial_i T + \mu_2 \mathbf{n}_i \mathbf{n}_j \partial_j T + \mu_3 \varphi_i \varphi_j \partial_j T + \mu_4 \mathbf{n}_i \varphi_j \partial_j T + \mu_5 \mathbf{n}_j \varphi_i \partial_j T)$$

(3.21)
$$+ (\gamma_1 \mathbf{n}_i + \gamma_2 \varphi_i)g,$$

$$\dot{\varphi} = \frac{1}{T}(\gamma_1' \mathbf{n}_i \partial_i T + \gamma_2' \varphi \partial_i T) + \gamma_3' g.$$

The viscosity coefficients α, β, γ, μ, and γ' cannot be arbitrarily chosen: they satisfy inequalities that follow from (3.13). Further restrictions result from Onsager's reciprocal relations (see, for instance, [8], [4], and [22]):

(3.22)
$$\beta_1 = \alpha_{22} - \alpha_{21}, \qquad \beta_2 = \alpha_{25} - \alpha_{24}, \qquad \beta_3 = \alpha_3 - \alpha_2 = \alpha_{21} + \alpha_{22},$$
$$\beta_4 = \alpha_6 - \alpha_5 = 2\alpha_{20}, \qquad \beta_5 = \alpha_{13} - \alpha_{12} = \alpha_{24} + \alpha_{25},$$
$$\beta_6 = \alpha_{17} - \alpha_{16} = \alpha_{26}, \qquad \beta_7 = \alpha_9 - \alpha_8 = \alpha_{23},$$
$$\gamma_1' = \gamma_1, \qquad \gamma_2' = \gamma_2.$$

We end this section by summarizing the governing equations of the hydrodynamic of smectic C. It consists of 11 equations and 11 unknowns. The latter are the pressure p, the velocity field $\mathbf{v}$, the director $\mathbf{n}$, the layer φ, the temperature T, and Lagrange multipliers λ and β. The equations are as follows:

- balance of linear momentum equation (3.6):

(3.23)
$$\rho\dot{\mathbf{v}} = \nabla \cdot (-pI + \sigma^d + \sigma^v) + \mathbf{f},$$

where the elastic stress σ^d is given from (2.5), the viscous part σ^v is from (3.20), and $\mathbf{f}$ is an external force;

- molecular field equation:

(3.24)
$$\frac{\partial f_d}{\partial \mathbf{n}_i} - \partial_j \pi_{ij} + \lambda \mathbf{n}_i + \beta \partial_i \varphi + \mathbf{h}_i = 0,$$

where $\mathbf{h}$ is given from (3.19);

- permeation force equation:

$$\dot{\varphi} = \frac{1}{T}(\gamma_1' \mathbf{n}_i \partial_i T + \gamma_2' \varphi \partial_i T) + \gamma_3' g,$$

where g is defined in (2.3);

- balance of energy equation:

$$E_t + \nabla \cdot (E\mathbf{v} + \mathbf{q} - \sigma\mathbf{v} - \dot{\varphi}\tau - \dot{\mathbf{n}}\pi) = 0,$$

where $\mathbf{q}$ is given from (3.21);

- incompressibility condition:

$$\nabla \cdot \mathbf{v} = 0;$$

- two constraints:

$$|\mathbf{n}| = 1 \qquad \text{and} \qquad \mathbf{n} \cdot \nabla\varphi = \cos\alpha |\nabla\varphi|.$$

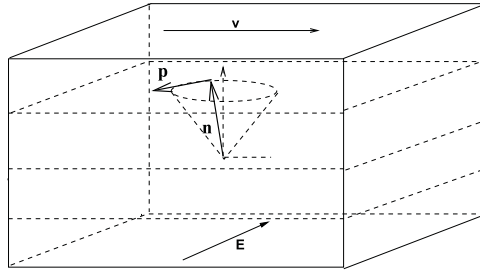In the isothermal case, this reduces to 10 equations and 10 unknowns.

FIG. 1. *Smectic C* liquid crystals in the homeotropic geometry; the polarization is perpendicular to both the layer normal and the director.*

4. Switching dynamics.

4.1. Electrostatic energy. In order to investigate the electric effect, we consider the electric energy [8], [29],

$$(4.1) \qquad f_e = -\int_\Omega \mathbf{D} \cdot d\mathbf{E} = -\int_\Omega (\varepsilon_\perp \mathbf{E} + \varepsilon_a (\mathbf{n} \cdot \mathbf{E})\mathbf{n} + \mathbf{P}) \cdot d\mathbf{E},$$

where $\mathbf{E}$ denotes the electric field, $\mathbf{P}$ denotes the ferroelectric polarization, and ε_a represents the dielectric anisotropy. Since chiral smectic C liquid crystals are known to be ferroelectric, they possess a spontaneous polarization $\mathbf{P}$. Dropping the constant term in the electric energy, (4.1) reduces to

$$(4.2) \qquad f_e = -\frac{1}{2}\int_\Omega \varepsilon_a (\mathbf{n} \cdot \mathbf{E})^2 \, d\mathbf{x} - \int_\Omega \mathbf{P} \cdot \mathbf{E} \, d\mathbf{x}.$$

Since the magnitude of the polarization is small in smectic C* liquid crystals, we assume that it is constant, P_0. Furthermore, since the chiral molecules create a spontaneous polarization within each layer and the polarization is perpendicular to the director (see Figure 1), we write

$$(4.3) \qquad \mathbf{P} = P_0 \frac{\nabla \varphi \times \mathbf{n}}{|\nabla \varphi \times \mathbf{n}|}.$$

The polarization $\mathbf{P}$ also gives an electrostatic charge density, $-\nabla \cdot \mathbf{P}$. The electrostatic effects of chiral smectic C liquid crystals were studied in [24]. As a result of the electric field, the Lorentz force $\mathbf{f} = (-\nabla \cdot \mathbf{P})\mathbf{E}$ has to be included in (3.23).

4.2. The model. We consider the homeotropic geometry where the liquid crystal is confined between two parallel plates with the smectic layers parallel to the plates (Figure 1). Let

$$(4.4) \qquad \begin{aligned} \mathbf{n} &= (\cos\phi \sin\alpha, \sin\phi \sin\alpha, \cos\alpha), \\ \mathbf{v} &= (v, 0, 0), \\ \nabla\varphi &= (0, 0, k), \end{aligned}$$

with $\phi = \phi(z)$, $v = v(z)$, $p = p(x, y, z)$, and α and k constant. We consider the switching dynamics between states with opposite polarization when a uniform electric field is applied in a direction parallel to the layer, i.e., $\mathbf{E} = E_0(0, 1, 0)$ in (4.2). We also restrict our attention to the case when $\varepsilon_a < 0$, which applies to many smectic C

liquid crystals. This tends to align the director and polarization fields along directions perpendicular and parallel to the applied field, respectively. Note that, since $\nabla \cdot \mathbf{P} = 0$, $\mathbf{f}$ in (3.23) vanishes.

The balance of linear momentum (3.23) yields

$$
\rho v_t = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}(\sigma_{13}^d + \sigma_{13}^v),
$$

(4.5)
$$
0 = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}(\sigma_{23}^d + \sigma_{23}^v),
$$

$$
0 = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}(\sigma_{33}^d + \sigma_{33}^v).
$$

The first two equations imply that p is linear on x and y. So, p is of the form

$$
p(x, y, z, t) = k_0(t) + k_1(t)x + k_2(t)y + \sigma_{33}^d + \sigma_{33}^v.
$$

Hence, (4.5) reduces to

$$
\rho v_t = -k_1(t) + \frac{\partial}{\partial z}(\sigma_{13}^d + \sigma_{13}^v).
$$

Also, as a result of (4.4), equation (3.24) reduces to a single equation for ϕ. Hence the system of governing equations is

(4.6)
$$
\rho v_t = \frac{\partial}{\partial z}\left[g(\phi)v_z - \eta_3(\sin \phi)\phi_t\right] - k_1(t),
$$
$$
2\beta_1 \sin^2 \alpha \phi_t = \lambda_1 \sin \phi v_z + 2K\phi_{zz} - 2P_0 E_0 \sin \phi - |\varepsilon_a|E_0^2 \sin^2 \alpha \sin 2\phi,
$$

where

$$
\lambda_1 = \sin \alpha((-\beta_1 + \beta_3)\cos \alpha + \beta_4 k),
$$
$$
K = \sin^2 \alpha(K_2 \sin^2 \alpha + K_3 \cos^2 \alpha),
$$
$$
g(\phi) = \frac{1}{2}(\eta_1 + \eta_2 \cos^2 \phi),
$$
$$
\eta_1 = \alpha_1 + \alpha_4 k^2(\alpha_2 - \alpha_{21})\cos^2 \alpha + (\alpha_5 + \alpha_7 - \alpha_{20})k \cos \alpha,
$$
$$
\eta_2 = \sin^2 \alpha\Big(\alpha_3 + \alpha_{22} + k^2(\alpha_{13} + \alpha_{25}) + (\alpha_{15} + 2\alpha_{17} + \alpha_{26})k \cos \alpha
$$
$$
+ 2\alpha_{18}\cos^2 \alpha\Big),
$$
$$
\eta_3 = \sin \alpha(\alpha_{20} k + \alpha_{21}\cos \alpha),
$$
$$
\frac{\partial p}{\partial x} = k_1(t).
$$

Onsager reciprocal relation (3.22) now gives

(4.7) $$\lambda_1 = 2\sin \alpha (\alpha_{20}k + \alpha_{21}\cos \alpha) = 2\eta_3.$$

Note that we may derive the following inequalities from the dissipation inequality (3.13):

(4.8) $$\beta_1 > 0, \quad g(\phi) > 0, \quad \text{and} \quad \beta_1 g(\phi)\sin^2 \alpha - \eta_3^2 \sin^2 \phi > 0.$$

The first inequality can be obtained from the shear flow alignment.

Let $V = v + K(t)/\rho$, where $K(t)$ is an antiderivative of $k_1(t)$. Using (4.7) and new variables

$$\bar{z} = \left(\frac{P_0 E_c}{K}\right)^{\frac{1}{2}} z, \quad \bar{t} = \frac{P_0 E_c}{\beta_1 \sin^2 \alpha} t, \quad u = \frac{\beta_1}{\sqrt{K P_0 E_c}} V, \quad \text{and} \quad E_c = \frac{2P_0}{|\varepsilon_a| \sin^2 \alpha},$$

the system (4.6) becomes

(4.9)
$$\varepsilon u_{\bar{t}} = \frac{\partial}{\partial \bar{z}} \left(\frac{\sin^2 \alpha}{\eta_1} g(\phi) u_{\bar{z}} - \frac{\eta_3}{\eta_1}(\sin \phi)\phi_{\bar{t}}\right),$$

$$\phi_{\bar{t}} = \frac{\eta_3}{\beta_1} \sin \phi \, u_{\bar{z}} + \phi_{\bar{z}\bar{z}} - e \sin \phi - e^2 \sin 2\phi,$$

where

$$\varepsilon = \frac{\rho K}{\beta_1 \eta_1} \quad \text{and} \quad e = E_0/E_c.$$

We may assume that the dimensionless parameter $\varepsilon \ll 1$ since the viscous coefficients are much bigger than the elastic coefficients.

4.3. Traveling wave solution. In this section, we study the traveling wave solutions of system (4.9) to understand the director profile when the electric field is applied. For this, we look for a solution of (4.9) in the form $w(\zeta) = u(\bar{z}, \bar{t})$ and $\theta(\zeta) = \phi(\bar{z}, \bar{t})$, where $\zeta = \bar{z} - c\bar{t}$, such that θ connects two bistable states, $\theta = 0$ and $\theta = \pi$. Then the traveling wave solution is $(w(\zeta), \theta(\zeta)) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ and $c \in \mathbb{R}$ satisfying

(4.10)
$$-c\varepsilon \, w' = \left[\frac{\sin^2 \alpha}{\eta_1} g(\theta)w' + c\frac{\eta_3}{\eta_1} \sin \theta \, \theta'\right]',$$

(4.11)
$$-c \, \theta' = \frac{\eta_3}{\beta_1} \sin \theta w' + \theta'' - e \sin \theta - e^2 \sin 2\theta,$$

with $w(-\infty) = w'(-\infty) = 0$, $\theta(-\infty) = 0$, and $\theta(\infty) = \pi$. This is consistent with the system approaching an equilibrium state. Here the $'$ denotes the derivative with respect to ζ. Integrating (4.10) and using the relations $w(-\infty) = w'(-\infty) = \theta(-\infty) = 0$, equation (4.10) reduces to

(4.12)
$$-c\varepsilon \, w = \frac{\sin^2 \alpha}{\eta_1} g(\theta)w' + c\frac{\eta_3}{\eta_1} \sin \theta \, \theta'.$$

Substituting this into (4.11), we have

(4.13)
$$\theta'' + ch(\theta)\theta' - e \sin \theta - e^2 \sin 2\theta - c\varepsilon\frac{\eta_3 \sin \theta}{\beta_1 \sin^2 \alpha \, g(\theta)} w = 0,$$

where

(4.14)
$$0 < h(\theta) := 1 - \frac{\eta_3^2}{\beta_1 \sin^2 \alpha} \frac{\sin^2 \theta}{g(\theta)} \leq 1.$$

Notice that the inequalities follow from (4.8). Introducing the rescaled variable $v = \varepsilon w$, we rewrite the system (4.12) and (4.13) as follows:

(4.15)
$$v' + c \, \varepsilon b(\theta)v + c \, \varepsilon \beta_1 d(\theta)\theta' = 0,$$

$$\theta'' + ch(\theta)\theta' + E(\theta) - c \, d(\theta)v = 0,$$

where

$$b(\theta) = \frac{\eta_1}{\sin^2 \alpha \, g(\theta)},$$

$$E(\theta) = -e \sin \theta - e^2 \sin 2\theta,$$

$$d(\theta) = \frac{\eta_3}{\beta_1 \sin^2 \alpha} \frac{\sin \theta}{g(\theta)}.$$

If $\theta(\zeta)$ and $v(\zeta)$ are solutions of the system, then so are $\theta(\zeta+\zeta_0)$ and $v(\zeta+\zeta_0)$ for any constant ζ_0. Hence we impose a normalized condition, $\theta(0) = \frac{1}{2}[\theta(-\infty) + \theta(\infty)] = \frac{\pi}{2}$. Using the condition $v(-\infty) = 0$, we solve the first equation for v,

$$v(\varepsilon, c, \theta, \zeta) = -c\varepsilon\beta_1 e^{-c\varepsilon\beta(\zeta)} \int_{-\infty}^{\zeta} d(\theta(s))\theta'(s)e^{c\varepsilon\beta(s)} \, ds,$$

where

$$\beta(\zeta) = \int b(\theta(s)) \, ds.$$

Substituting this expression into the second equation of (4.15), we have

(4.16) $$\qquad \theta'' + ch(\theta)\theta' + E(\theta) - c \, d(\theta)v(\varepsilon, c, \theta, \zeta) = 0.$$

For $\varepsilon = 0$, equation (4.16) becomes

(4.17) $$\qquad\qquad \theta'' + ch(\theta)\theta' + E(\theta) = 0.$$

From now on, we will restrict our attention to the case $e > \frac{1}{2}$ so that the term $E(\theta)$ is cubic-like. In fact, if $e > \frac{1}{2}$, the term $E(\theta) = -e \sin\theta(1 + 2e\cos\theta)$ has an intermediate zero. In this case, (4.17) with a bistable nonlinearity has an increasing traveling wave solution (c_0, θ_0) with $c_0 > 0$ that $\theta_0 \to 0$ as $\zeta \to -\infty$ and $\theta_0 \to \pi$ as $\zeta \to \infty$, thanks to the condition $h(\theta) > 0$ for any θ [15]. Furthermore, we can easily see that [11], [1], from the phase plane analysis, θ_0 satisfies

$$|\theta_0(\zeta) - \pi| \leq K e^{-\mu_1 \zeta}, \qquad |\theta_0'(\zeta)| \leq K e^{-\mu_1 \zeta}$$

for $\zeta \geq 0$ and for some constant $\mu_1 > 0$, and

$$|\theta_0(\zeta)| \leq K e^{\mu_2 \zeta}, \qquad |\theta_0'(\zeta)| \leq K e^{\mu_2 \zeta}$$

for $\zeta \leq 0$ and for some constant $\mu_2 > 0$.

Motivated by the work in [12], we look for solutions of (4.16) of the form

$$\theta = \theta_0 + s(\zeta, c, \varepsilon),$$

$$c = c_0 + \sigma.$$

Substituting these into (4.16) and letting $r = (s, \sigma)$, we define the operator F:

$$F(r; \varepsilon) = \theta_0'' + s'' + (c_0 + \sigma)h(\theta_0 + s)(\theta_0' + s') + E(\theta_0 + s)$$
$$-(c_0 + \sigma)d(\theta_0 + s)v(\varepsilon, c_0 + \sigma, \theta_0 + s).$$

Note that r satisfies the boundary conditions

(4.18) $$\qquad\qquad r(-\infty; \varepsilon) = r(\infty; \varepsilon) = 0.$$

For a fixed constant μ satisfying $0 < \mu < \min\{\mu_1, \mu_2\}$, we define the function spaces

$$B_\mu^n(\mathbb{R}) = \left\{ u \in C^n(\mathbb{R}) : \|u\|_{B_\mu^n(\mathbb{R})} \equiv \sum_{i=0}^n \sup_{x \in \mathbb{R}} \left| e^{\mu|x|} \left(\frac{d}{dx} \right)^i u(x) \right| < \infty \right\},$$

$$\dot{B}_\mu^n(\mathbb{R}) = \{ u \in B_\mu^n(\mathbb{R}) : u(0) = 0 \}.$$

Note that the mapping F is differentiable from X into Y, where

$$X = \dot{B}_\mu^2 \times \mathbb{R},$$
$$Y = B_\mu^0.$$

The next lemma establishes that F meets the hypotheses of the implicit function theorem.

LEMMA 4.1.
   (i) F is a continuous mapping, and $\|F(r; \varepsilon) - F(r; 0)\|_Y \to 0$ as $\varepsilon \to 0$.
   (ii) F is continuously Fréchet differentiable with respect to r, and

$$\|F_r(r; \varepsilon)[\tilde{r}] - F_r(r; 0)[\tilde{r}]\|_Y \to 0 \text{ as } \varepsilon \to 0.$$

   (iii) $F_r(0; 0)$ has a bounded inverse.
   Proof.
   (i) We have

$$|e^{\mu|z|}(F(r; \varepsilon) - F(r; 0))|$$
$$= e^{\mu|z|} \left| (c_0 + \sigma)^2 d(\theta_0 + s)\varepsilon\beta_1 g_-(z) \int_{-\infty}^z d(\theta_0 + s)(\theta_0' + s')g_+(t)e^{\mu|t|}e^{-\mu|t|} \, dt \right|$$
$$\leq C\varepsilon(c_0 + \sigma)\|\theta_0' + s'\|_{B_\mu^0} e^{\mu|z|} \left| \int_{-\infty}^z e^{-\mu|t|} dt \right|$$
$$\leq C\varepsilon(c_0 + \sigma)\|\theta_0 + s\|_{B_\mu^2},$$

where $g_\pm(z) = \exp(\pm\varepsilon(c_0 + \sigma)\beta(z))$ and $\beta'(z) = b(\theta_0 + s)(z)$. Notice that we used the fact that $\beta(z)$ is increasing.
   (ii) Note that

$$e^{\mu|z|}|F_r(r; \varepsilon)\tilde{r} - F_r(r; 0)\tilde{r}|$$
$$= e^{\mu|z|}|\tilde{\sigma}d(\theta_0 + s)v_\varepsilon(z) - (c_0 + \sigma)d'(\theta_0 + s)v_\varepsilon(z)\tilde{s} + (c_0 + \sigma)d(\theta_0 + s)v_\varepsilon'(z)\tilde{s}|,$$

where $v_\varepsilon(z) = v(\varepsilon, c_0 + \sigma, \theta_0 + s, z)$. Since we have

$$v_\varepsilon'(z) = -(c_0 + \sigma)\varepsilon b(\theta_0 + s)v_\varepsilon(z) - (c_0 + \sigma)\varepsilon\beta_1 d(\theta_0 + s)(\theta_0' + s'),$$

the rest of the proof follows as in part (i).
   (iii) It suffices to show that for any $g \in Y$, the linear problem

$$F_r(0; 0)\tilde{r} = g$$

has a unique solution $\tilde{r} \in \dot{B}_\mu^2$ such that

$$\|\tilde{r}\|_{\dot{B}_\mu^2} \leq C\|g\|_Y.$$

The above linear problem can be explicitly written as

$$(4.19) \qquad \tilde{s}'' + c_0 h(\theta_0)\tilde{s}' + (c_0 h(\theta_0)\theta_0' + E'(\theta_0))\tilde{s} = G,$$

where

$$G = g - h(\theta_0)\theta_0'\tilde{\sigma}.$$

The proof of existence and uniqueness of solution of (4.19) satisfying boundary condition (4.18) follows as that of Lemma 3 in [15].   □

The proof of the following theorem uses the implicit function theorem together with Lemma 4.1.

THEOREM 4.2. *For $\varepsilon > 0$ sufficiently small, there exists a unique (up to translation in ζ) $(c_\varepsilon, \theta_\varepsilon, v_\varepsilon)$ satisfying* (4.15) *such that*

$$\|\theta_\varepsilon - \theta_0\|_{B_\mu^2} + \|v_\varepsilon\|_{B_\mu^1} + |c_\varepsilon - c_0| \longrightarrow 0 \qquad as \ \varepsilon \to 0.$$

4.4. Speed of the traveling wave. In this section, we study the speed of the traveling front of (4.16). We follow the variational approach in [2] for reaction-diffusion equations.

We first consider front propagation for the reaction-diffusion equation

$$(4.20) \qquad \theta'' + c\theta' = H(\theta),$$

where $H(\theta) = e \sin\theta + e^2 \sin 2\theta$. This is the traveling wave equation for a switching problem for smectic C liquid crystals, when flow effects are neglected (see [25], [26]). It is known that there exists a unique heteroclinic solution (c_s, θ_s) of (4.20) such that $\theta(-\infty) = 0$, $\theta(\infty) = \pi$, and $\theta'(\zeta) > 0$ for $|\zeta| < \infty$. This equation can also be obtained from our model by neglecting the flow. The authors of [7] found the explicit wave front solution (θ_s, c_s) of (4.20), given by

$$\theta_s(\zeta) = 2\arctan(e^{\sqrt{2}e\zeta}), \quad c_s = \frac{1}{\sqrt{2}}.$$

Following the work by Benguria and Depassier in [2], we obtain the variational expression of the speed for (4.20):

$$(4.21) \qquad c_s^2 = \max \frac{2 \int_0^\pi Hf d\theta}{\int_0^\pi \frac{f^2}{f'} d\theta},$$

where the maximum is taken over all positive increasing functions f in $(0, \pi)$. We denote the maximizing function by $\hat{f}$.

With the help of Theorem 4.2, we will investigate c_0, the speed of the traveling wave solution of (4.17). We let θ be a solution of (4.17). The same proof as in [2] leads to the variational principle for the speed of the traveling wave of (4.17). Noting that $0 < h(\theta) \leq 1$,

$$(4.22) \qquad c_0^2 = \max \frac{2 \int_0^\pi Hf d\theta}{\int_0^\pi \frac{h^2 f^2}{f'} d\theta},$$

where the maximum is taken over all positive increasing functions f in $(0, \pi)$ for which the integrals exist.

Now we compare the speeds c_0 and c_s using (4.21) and (4.22). From (4.22), we get

$$c_0^2 \geq \frac{2 \int_0^\pi H \hat{f} d\theta}{\int_0^\pi \frac{h^2 \hat{f}^2}{\hat{f}'} d\theta}.$$

Since $0 < h(\theta) \leq 1$, we have

$$h^2(\theta) \leq h(\theta) = 1 - \frac{2\eta_3^2 \sin^2 \theta}{\beta_1 \sin^2 \alpha \, (\eta_1 + \eta_2 \cos^2 \theta)} \leq 1 - A \sin^2 \theta,$$

where

(4.23) $$A := \frac{2\eta_3^2}{\beta_1 \sin^2 \alpha \, (\eta_1 + \max\{\eta_2, 0\})} \geq 0.$$

It follows from (4.14) that $0 \leq A < 1$. In fact, if $\max\{\eta_2, 0\} = 0$, then

$$A = \frac{2\eta_3^2}{\beta_1 \eta_1 \sin^2 \alpha} = 1 - h\left(\frac{\pi}{2}\right) < 1.$$

Also, if $\max\{\eta_2, 0\} = \eta_2$, then

$$A = \frac{2\eta_3^2}{\beta_1 \sin^2 \alpha (\eta_1 + \eta_2)} \leq 1 - h\left(\frac{\pi}{2}\right) < 1.$$

From (4.21) and (4.22), we have

(4.24) $$c_0^2 \geq \frac{2 \int_0^\pi H \hat{f} d\theta}{\int_0^\pi \frac{\hat{f}^2}{\hat{f}'} d\theta - A \int_0^\pi \frac{\sin^2 \theta \, \hat{f}^2}{\hat{f}'} d\theta} = \frac{2 \int_0^\pi H \hat{f} d\theta}{\int_0^\pi \frac{\hat{f}^2}{\hat{f}'} d\theta (1 - A \cdot M)} = \frac{c_s^2}{1 - A \cdot M},$$

where M is a fixed number given by

$$M = \frac{\int_0^\pi \frac{\sin^2 \theta \, \hat{f}^2}{\hat{f}'} d\theta}{\int_0^\pi \frac{\hat{f}^2}{\hat{f}'} d\theta}.$$

Notice that $0 < M < 1$ is independent of viscosity coefficients, since $\hat{f}$ is the maximizing function for c_s. We rewrite (4.24) as

(4.25) $$\left(\frac{c_0}{c_s}\right)^2 \geq \frac{1}{1 - A \cdot M}.$$

The inequality (4.25) shows that the switching is faster when the flow is taken into consideration. The value A in (4.23) is the control parameter; i.e., A is the quantity which measures flow effects. If A is close to 1, then flow effects are expected to be strong, and if A is close to 0, then flow effects are weak. In particular, we see that the ratio of c_0 to c_s increases as A approaches 1. In view of this control parameter, the optimal switching time is obtained when $A = 1$. The sufficient condition for this is

$$\eta_2 \leq 0 \qquad \text{and} \qquad 2\eta_3^2 = \beta_1 \eta_1 \sin^2 \alpha.$$

This condition depends only on the viscosity, the tilt angle, and the layer thickness of the material. In [5], the control parameter was also found. Our control parameter A is analogous to that given by Carlsson, Clark, and Zou in [5].
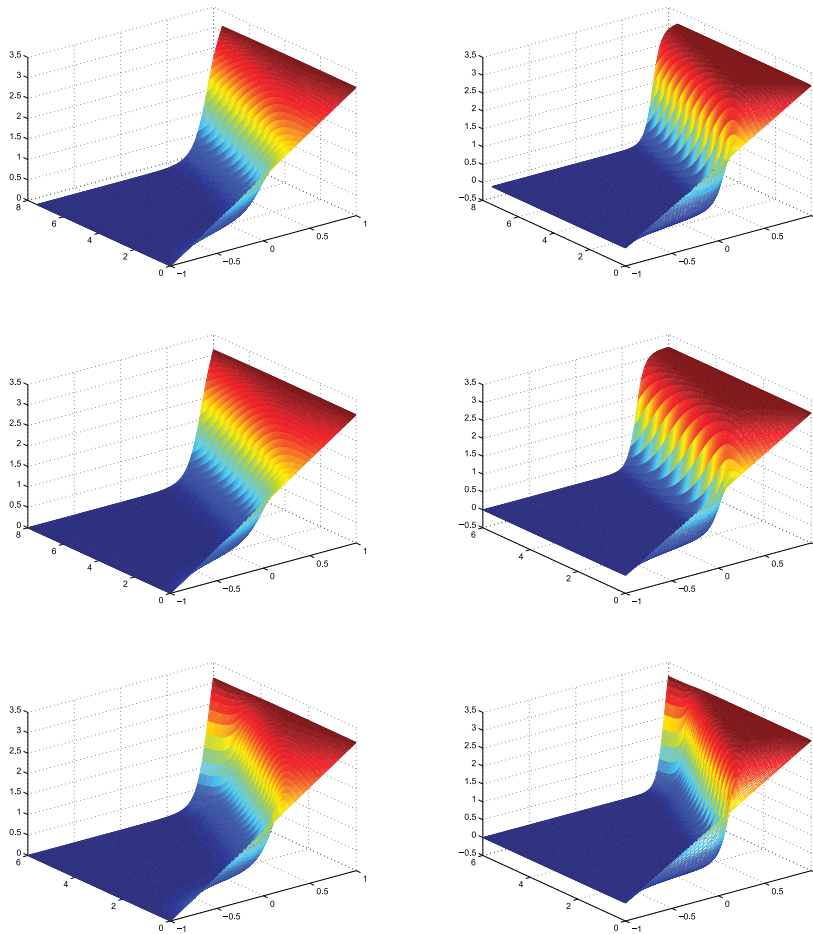
Fig. 2. *Director configuration when the initial condition is the linear function connecting* 0 *and* π. *The applied electric field corresponds to* $e = 0.75$ *on the left column and* $e = 1.3$ *on the right column. The upper row depicts the director profile when flow is neglected, while in the second and third rows the flow effects are included. For simulations in the second row, the control parameter* A *is close to* 0, *while for the third row,* A *is close to* 1.

4.5. Numerical simulation. In order to solve the system (4.9) numerically, we use a second order semi-implicit scheme for time discretization. This scheme requires us to solve two Helmholtz equations at each time step, which we do by means of a spectral Galerkin method (see [27] and [28]). We impose the homogeneous boundary and initial conditions on u and assume strong anchoring conditions for ϕ, i.e., $\phi(0, t) = 0$ and $\phi(L, t) = \pi$, where L is the domain size. From (4.3) and (4.4) we see that the director configurations $\phi = 0$ and $\phi = \pi$ correspond to the polarization pointing in the same and opposite directions as the applied electric field, respectively.

For ε, we simply take $\varepsilon = 10^{-6}$. For the tilt angle and viscosity coefficients appearing in the system, we use $\eta_1 = 3.8$, $\eta_2 = -0.2$, $\beta_1 = 40.9706$, and $\alpha = \pi/8$. This set of parameters, employed in [3], gives a value of the control parameter A in (4.23) of approximately 0.5. In [3], the authors study the macroscopic equations of
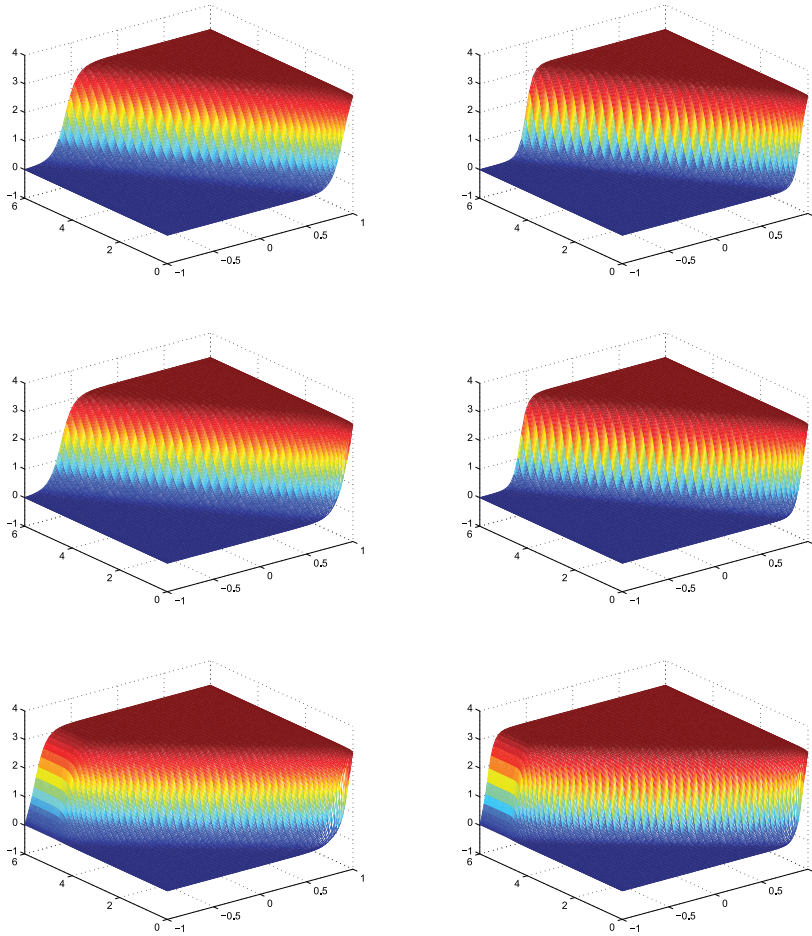
FIG. 3. *Director configuration when the sign of the electric field is reversed at $t = 0$. The arrangement of the simulations in rows and columns follows the analogous criteria to those in Figure 2.*

smectic C* liquid crystals [18] in a homeotropic geometry to investigate the backflow effect upon the removal of a strong electric field. Their approach is based on linear analysis, replacing the nonlinear functions by their initial values. In our simulations, we vary the parameter η_3 in order for A to span the interval $(0, 1)$.

We consider two types of initial conditions for ϕ corresponding to the simulations shown in Figures 2 and 3, respectively. In Figure 2, the initial value ϕ_0 is the linear function connecting two bistable states, 0 and π. When a positive electric field is applied, the molecules start to switch so that the polarization is parallel to the applied field in most of the cell except near the top plate where the strong anchoring condition, $\phi(L, t) = \pi$, is imposed. Figure 2 depicts the director configuration with $e = 0.75$ in the left and $e = 1.3$ in the right columns, respectively. The flow effect is neglected in simulations in the first row, and it is included in the second and third rows. The control parameter A is close to 0 in the middle, while A is close to 1 in the third row. As we may expect from (4.25), the simulations in first and second rows depict almost

the same switching time, while the third row describes faster switching dynamics.

In Figure 3, we numerically investigate the switching behavior when the sign of the electric field is alternating, proceeding as follows: we first obtain the director profile of the equilibrium state in a positive electric field and then impose it as an initial condition of the problem with an applied negative electric field. In [21], the authors investigated the switching time for the static model when alternating fields are applied.

The simulations show that the predicted switching process is faster when the flow is taken into consideration. We note that the switching is already faster even with a very small, but nonzero, value of A.

Zou, Clark, and Carlsson in [30] also performed numerical simulations for reorientation dynamics with various boundary conditions, based on the model proposed by Leslie, Stewart, and Nakagawa [18]. In bookshelf geometry, they showed that the switching process is generally faster when backflow is present. They also numerically confirmed that the control parameter found in [5] is a measure of the contribution of the backflow effects. The control parameter in [30] is defined as the average of $1 - h(\theta)$ over θ. In the previous section, we also identified an analogous control parameter, which is dependent only on parameters of the problem, but we obtained bounds on it that rigorously allow us to quantify the backflow effects in the switching time. In particular, the upper bound on A yields an optimality condition on the parameters.

5. Conclusion. In this paper, we presented a nonlinear continuum theory of smectic C* liquid crystals. Since the smectic C liquid crystals have molecules tilted with respect to the layers, we use both the director and the layer functions as variables in the hydrodynamic theory. For the general framework, we employed the approach by Ericksen and Leslie for the hydrodynamic theory of the nematic liquid crystals. Also, motivated by the work of W. E on the continuum theory of the smectic A liquid crystals, we obtained the dynamic equations for the director $\mathbf{n}$ and the layer variable φ.

We applied the model to study the switching dynamics between two states with opposite polarization in the homeotropic geometry. Even though there are 22 viscosity coefficients in our hydrodynamic theory, the system of equations reduces to two equations with only four viscosity constants, η_1, η_2, η_3, and β_1. These constants are further constrained by the entropy inequality. We understand the molecular reorientation via the propagation of a traveling wave. We proved the existence and uniqueness of the traveling wave solution and further analyzed the speed of the front. We showed that the flow generally makes the switching faster and that there is a control parameter that determines the importance of the flow effect. This analysis was confirmed by the numerical simulations.

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