ODE Math 119B, Spring 2010

Quiz 2 May 24th, 2010

Name: ____________________

Perm number: ____________________

Announcement : Thursday's office hours (May 27th) are moved, from 9:00-10:30am to 1:00-2:30pm.

1. Consider the system

$$\begin{aligned} x' &= -y + x(1-x^2-y^2)(4-x^2-y^2) \\ y' &= x + y(1-x^2-y^2)(4-x^2-y^2) \end{aligned}$$

   (a) Write the system in polar coordinates.
   (b) Find periodic orbits.
   (c) Draw the phase portrait for this system.
   (d) Determine the stability of each periodic orbit by using the Poincare map.

(a). $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} x' = r'\cos\theta - r\cdot\sin\theta\,\theta' \\ y' = r'\sin\theta + r'\cos\theta\,\theta' \end{cases} \Rightarrow \begin{cases} r' = \cos\theta\cdot x' + \sin\theta\, y' \\ \theta' = \frac{1}{r}(-\sin\theta\, x' + \cos\theta\, y') \end{cases}$

$r' = \cos\theta\cdot(-r\cancel{\sin\theta} + r\cos\theta\cdot(1-r^2)(4-r^2)) + \sin\theta\,(r\cancel{\cos\theta} + r\sin\theta(1-r^2)(4-r^2))$

$= r(1-r^2)(4-r^2)$

$\theta' = \frac{1}{r}\left(-\sin\theta\cdot(-r\sin\theta + r\cos\theta\cancel{(1-r^2)(4-r^2)}) + \cos\theta\cdot(r\cos\theta + r\sin\theta\cancel{(1-r^2)(4-r^2)})\right)$

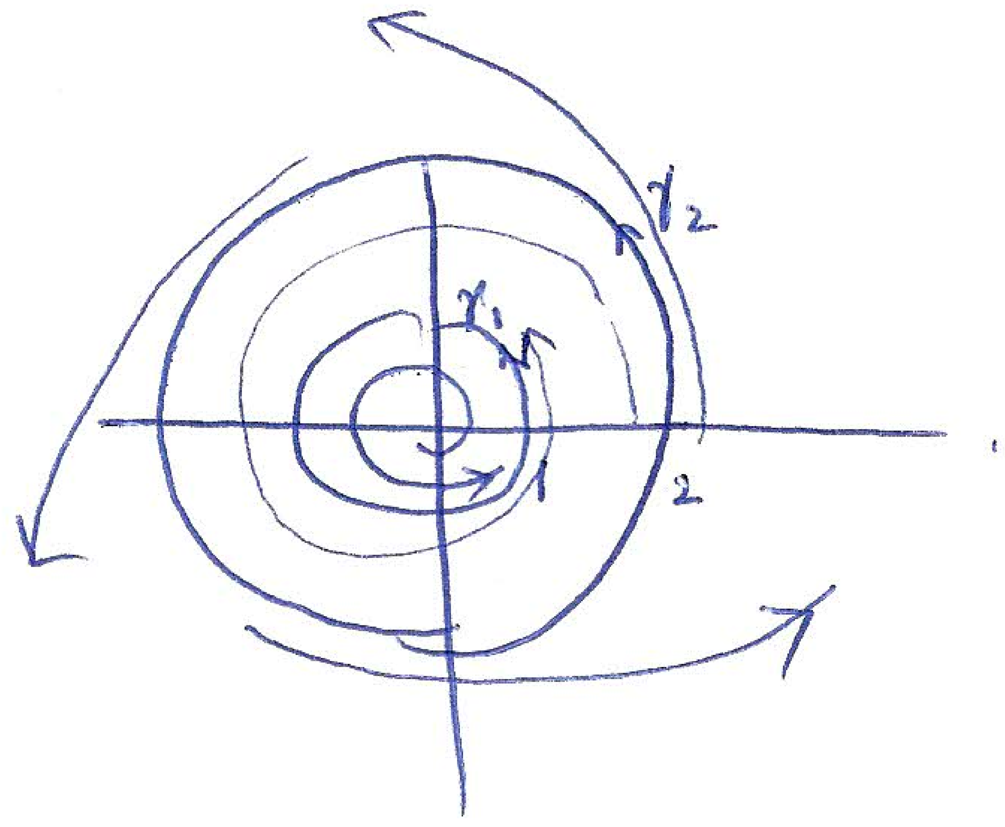
$= \frac{1}{r}\cdot r = 1.$

i.e. $\begin{cases} r' = r(1-r^2)(4-r^2) \\ \theta' = 1. \end{cases}$

(b). $r=1$ & $r=2$. From $\theta = \theta_0 + C$, there are two periodic orbits;

$\gamma_1(t)$; $(\cos t, \sin t)^T$

$\gamma_2(t)$; $(2\cos t, 2\sin t)^T$



(c). $r' > 0$ for $0 < r < 1$ & $r > 2$.

$r' < 0$ for $1 < r < 2$.

End of exam.

(d). $\nabla\cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$

$= (1-r^2)(4-r^2) + x\cdot(-2x)\cdot(4-r^2) + x\cdot(1-r^2)(-2x)$

$+ (1-r^2)(4-r^2) + y\cdot(-2y)(4-r^2) + y(1-r^2)(-2y)$

$= 2(1-r^2)(4-r^2) - 2r^2(4-r^2) - 2r^2(1-r^2)$

where $r^2 = x^2+y^2$.

$\Rightarrow (\nabla\cdot f)(\gamma_1(t)) = -2\cdot(4-1) = -6$

$\Rightarrow \int_0^{2\pi} \nabla\cdot f(\gamma_1(t))\,dt < 0 \Rightarrow \gamma_1(t)$; stable.

$(\nabla\cdot f)(\gamma_2(t)) = -2\cdot 4\cdot(1-4) = 24 > 0$

$\Rightarrow \int_0^{2\pi} (\nabla\cdot f)(\gamma_2(t))\,dt > 0 \Rightarrow \gamma_2(t)$; unstable.