Hw 3 solution

Note Title 2010-05-03

§2.12.

#2. $\begin{cases} x' = y \\ y' = -y + \alpha x^2 + xy \end{cases}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}}_{\substack{\parallel \\ A}} \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \alpha x^2 + xy \end{pmatrix}}_{\substack{\parallel \\ F(x,y)}}$$

$\text{tr}\,A = -1$ & $\det A = 0$

$p(\lambda) = \lambda^2 + \lambda = \lambda(\lambda+1) = 0$; $\lambda = 0, -1$

$A \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: eigenvector ass. w/ 0

$A + I = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; eigenvector ass. w/ -1.

Then $A = SBS^{-1}$

where $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ & $S = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

Let $\vec{w} = \begin{pmatrix} u \\ v \end{pmatrix} = S^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$. $\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = S\vec{\omega}$

$S\vec{\omega}\,' = AS\vec{\omega} + F(S\vec{\omega})$

Taking S^{-1},

$\vec{\omega}\,' = B\vec{\omega} + G(\vec{\omega})$, where $G(\vec{\omega}) = S^{-1}F(S\vec{\omega})$

$G(\omega) = S^{-1}F(u+v, -v)$

$$= -1 \cdot \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \alpha(u+v)^2 + (u+v)(-v) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \alpha u^2 + (2\alpha-1)uv + (\alpha-1)v^2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha u^2 + (2\alpha-1)uv + (\alpha-1)v^2 \\ -\alpha u^2 - (2\alpha-1)uv - (\alpha-1)v^2 \end{pmatrix} \,//$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow C = [0], \quad P = [-1]$$

Local center manifold; $v = h(u)$

& $h(0) = h'(0) = 0$

Use approximation; $h(u) = au^2 + O(u^3)$

$\Rightarrow h'(u) = 2au + O(u^2)$

Since the center manifold is invariant under the flow,

$v(t) = h(u(t)) \quad \forall t.$

$\Rightarrow v'(t) = h'(u) \cdot u'(t)$

From $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \alpha u^2 + (2\alpha-1)uv + (\alpha-1)v^2 \\ -v - \alpha u^2 - (2\alpha-1)uv - (\alpha-1)v^2 \end{pmatrix}$,

& $v = h(u) = au^2 + O(u^3)$,

$$-au^2 - \alpha u^2 - (2\alpha-1)u \cdot au^2 - (\alpha-1) \cdot a^2u^4$$
$$= 2au \cdot (\alpha u^2 + (2\alpha-1)u \cdot au^2 + (\alpha-1) \cdot a^2u^4) + O(u^3)$$

Collecting terms,

$(-a-\alpha)u^2 = 0$

$\Longrightarrow a = -\alpha \,//$

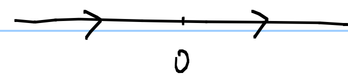
$$\Rightarrow h(u) = -\alpha u^2 + O(u^3).$$

Reduced flow:

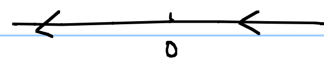
$$u' = \alpha u^2 + (2\alpha-1)\cdot u \cdot (-\alpha u^2) + (\alpha-1)\cdot \alpha^2 u^4 + O(u^3)$$

$$\Rightarrow \quad u' \approx \alpha u^2$$

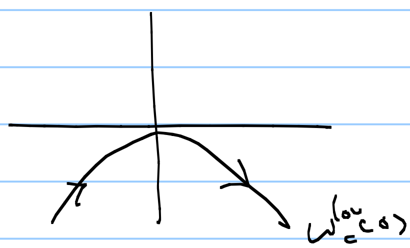
$\alpha > 0$



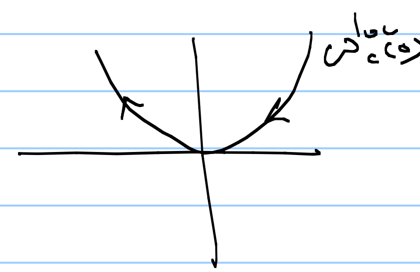
$\alpha < 0$



$$\Rightarrow \quad W_c^{lc}(0) = \{ (u, v) : v \approx -\alpha u^2 \}$$



$\alpha > 0$

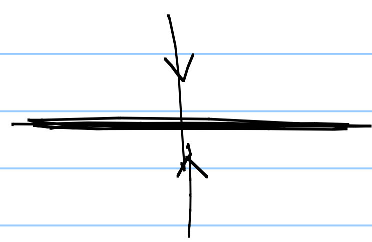


$\alpha < 0$

$\alpha = 0$; $\quad h(u) = 0$; $\quad u' = 0$ (reduced flow)

$\Rightarrow$ 0 : stable

(Every pt is an equil. pt)



$W_c^{loc}(0)$.

#5. (a) $\begin{cases} x_1' = -x_2 + x_1 y \\ x_2' = x_1 + x_2 y \\ y' = -y - x_1^2 - x_2^2 + y^2 \end{cases}$

$$\begin{pmatrix} x_1' \\ x_2' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} + \begin{pmatrix} x_1 y \\ x_2 y \\ -x_1^2 - x_2^2 + y^2 \end{pmatrix}.$$

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad P = [-1]$$

Local center manif ;

$$y = h(x_1, x_2)$$
$$= ax_1^2 + bx_1x_2 + cx_2^2 + O(|x|^2)$$

$$y' = \nabla h \cdot x'$$

$$\Rightarrow -y - x_1^2 - x_2^2 + y^2 = (2ax_1 + bx_2,\ bx_1 + 2cx_2) \cdot \begin{pmatrix} -x_2 + x_1 y \\ x_1 + x_2 y \end{pmatrix}$$

$$\Rightarrow -ax_1^2 - bx_1x_2 - cx_2^2 - x_1^2 - x_2^2$$
$$= -2ax_1x_2 - bx_2^2 + bx_1^2 + 2cx_1x_2 + O(|x|^3)$$

Collecting terms,

$O(x_1^2)$: $-a - 1 = b$

$O(x_1x_2)$: $-b = -2a + 2c$

$O(x_2^2)$: $-c - 1 = -b$

$\Longrightarrow b = 0, \quad a = -1, \quad c = -1$

$\Rightarrow h(x) \approx -x_1^2 - x_2^2.$

Reduced flow:

$$(*) \begin{cases} x_1' \approx -x_2 + x_1(-x_1^2 - x_2^2) \\ x_2' \approx x_1 + x_2(-x_1^2 - x_2^2) \end{cases}$$

Let $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$

Then $\dot V(x(t)) = (x_1, x_2) \cdot \begin{pmatrix} -x_2 - x_1^3 - x_1 x_2^2 \\ x_1 - x_2 x_1^2 - x_2^3 \end{pmatrix}$

$$= -x_1x_2 - x_1^4 - x_1^2 x_2^2 + x_1 x_2 - x_2^2 x_1^2 - x_2^4$$

$$= -(x_1^4 + 2x_1^2 x_2^2 + x_2^4) < 0$$

, if $(x_1, x_2) \neq (0,0)$.

$\Rightarrow (0,0)$; asymptotically stable on the center manifold.

Polar coordinate :

$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta.$

$$\Rightarrow x_1' = r'\cos\theta + r\cdot(-\sin\theta)\theta' \quad — ①$$

$$x_2' = r'\sin\theta + r\cdot\cos\theta\cdot\theta' \quad — ②$$

① × cos θ + ② × sin θ gives

$$x_1'\cos\theta + x_2'\sin\theta = r'$$

From (*),

$$r' = [-r\sin\theta + r\cos\theta\cdot(-r^2)]\cos\theta$$
$$+ [r\cos\theta + r\sin\theta(-r^2)]\sin\theta$$

$$\Rightarrow r' = -r^3$$

Also, $-①\times\sin\theta + ②\times\cos\theta$ gives

$$-x_1'\sin\theta + x_2'\cos\theta = r\theta'.$$

$$\Rightarrow r\theta' = -(-r\sin\theta - r^3\cos\theta)\sin\theta + (r\cos\theta - r^3\sin\theta)\cdot\cos\theta$$

$$= r$$

$$\Rightarrow \theta' = 1.$$

Since $r(t)\to 0$ as $t\to\infty$ & $\theta = \theta_0 + t$, the solution curve spirals counterclockwise to the origin. Also, note that there is no unstable manifold. Thus, the origin is asymptotically stable for the given nonlinear problem.