Midterm solution

Note Title 2010-03-08

1. (a) $e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$

(b) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = D + N.$

$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ & $N^3 = 0.$, $D = 2I.$

$$e^{At} = e^{Dt+Nt} = e^{Dt} \cdot e^{Nt} \quad (\text{since } DN = ND)$$

$$= e^{2t} \cdot I \cdot \left(I + Nt + \frac{N^2t^2}{2!} + \frac{N^3t^3}{3!} + \cdots \right)$$

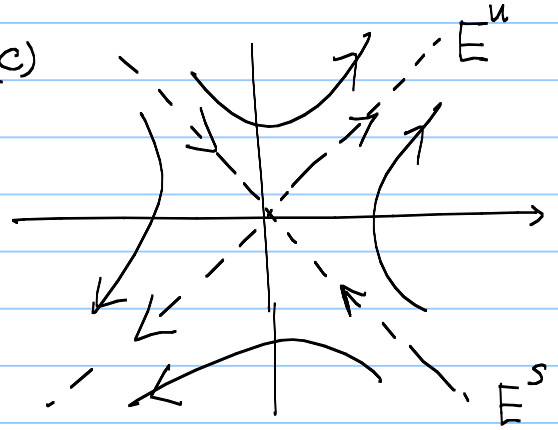
$$= e^{2t} \cdot \left(I + Nt + \frac{N^2t^2}{2!} \right)$$

$$= e^{2t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \quad //$$

2. (a) $x = e^{At} \cdot x_0 = e^{At} \cdot 4 v_1 = 4 e^{\lambda_1 t} v_1 = 4 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) $E^s = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$ & $E^u = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$

(c)



(d) $x(t) = e^{At}\cdot x_0 + e^{At}\cdot\int_0^t e^{-As}\begin{pmatrix}1\\-1\end{pmatrix} ds$

$$= e^{At}\cdot(v_1+v_2) + e^{At}\int_0^t e^{-As}\cdot v_2\, ds$$

$$= e^{At}(v_1+v_2) + e^{At}\cdot\int_0^t e^{-\lambda_2 s}\, v_2\, ds$$

$$= e^{\lambda_1 t} v_1 + e^{\lambda_2 t} v_2 + e^{At}\cdot\int_0^t e^{5s}\, v_2\, ds$$

$$= e^{3t} v_1 + e^{-5t} v_2 + e^{At}\cdot\frac{1}{5} e^{5s} v_2\Big|_0^t$$

$$= e^{3t} v_1 + e^{-5t} v_2 + \frac{1}{5} e^{\lambda_2 t} v_2\,(e^{5t}-1)$$

$$= e^{3t} v_1 + e^{-5t} v_2 + \frac{1}{5} v_2\cdot(1-e^{-5t})$$

$$= e^{3t}\begin{pmatrix}1\\1\end{pmatrix} + e^{-5t}\cdot\begin{pmatrix}1\\-1\end{pmatrix}\cdot\left(1-\frac{1}{5}\right) + \frac{1}{5}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= e^{3t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{4}{5} e^{-5t}\begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{5}\begin{pmatrix}1\\-1\end{pmatrix}$$

<u>Check</u> $x(0) = \begin{pmatrix}1\\1\end{pmatrix} + \frac{4}{5}\begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{5}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}.$

$$x' = 3e^{3t}\begin{pmatrix}1\\1\end{pmatrix} - 4e^{-5t}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$A = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\cdot\begin{pmatrix}3 & \\ & -5\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}^{-1}$$

$$= \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}3 & 0\\0 & -5\end{pmatrix}\left(-\frac{1}{2}\right)\cdot\begin{pmatrix}-1 & -1\\-1 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}3 & 0\\0 & -5\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}3 & 3\\-5 & 5\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-2 & 8\\8 & -2\end{pmatrix} = \begin{pmatrix}-1 & 4\\4 & -1\end{pmatrix}$$

$$Ax + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} e^{3t} + \frac{4}{5} e^{-5t} + \frac{1}{5} \\ e^{3t} - \frac{4}{5} e^{-5t} - \frac{1}{5} \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{3t} - 4e^{-5t} - 1 + 1 \\ 3e^{3t} + 4e^{-5t} + 1 - 1 \end{pmatrix} = \begin{pmatrix} 3e^{3t} - 4e^{-5t} \\ 3e^{3t} + 4e^{-5t} \end{pmatrix}$$

(e) If λ: e. value of A & v is the corresponding eigenvector, then

$$Av = \lambda v.$$

This implies $A^2 v = \lambda^2 v$.

i.e. A^2 has an eigenvalue λ^2 with eigenvector v.

$\Rightarrow$ A^2 has e. values $3^2 = 9$ & $(-5)^2 = 25$ with the same e. vector, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\Rightarrow$ The solution $x(t)$ is given by

$$x(t) = c_1 e^{9t} v_1 + c_2 e^{25t} \cdot v_2.$$