Homework 6

§2.1.

#4. $\frac{dx}{x^3} = dt. \Rightarrow -\frac{1}{2}x^{-2} = t + C$

$\Rightarrow 2x^2 = \frac{-1}{t+C}$

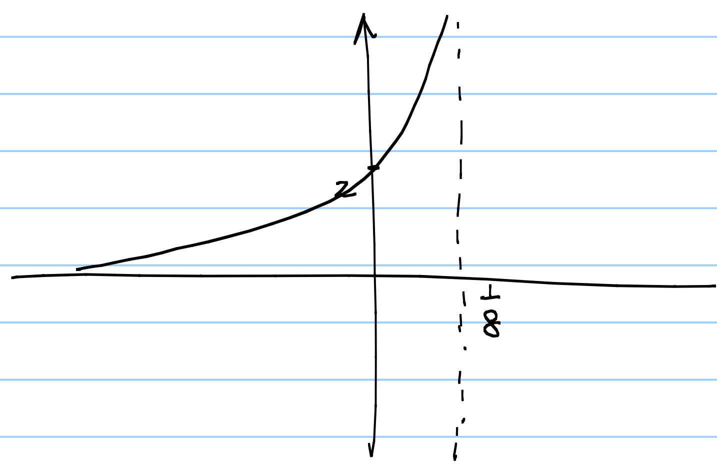
$2 \cdot 4 = -\frac{1}{C} \Rightarrow C = -\frac{1}{8}$

$\Rightarrow x^2 = \frac{1}{2\left(\frac{1}{8} - t\right)} = \frac{1}{\frac{1}{4} - 2t}$

$\Rightarrow x = \frac{1}{\sqrt{\frac{1}{4} - 2t}}$

$\Rightarrow$ IVP has a solution for $t \in (-\infty, \frac{1}{8})$

$\lim_{t \nearrow \frac{1}{8}} x(t) = \lim_{t \nearrow \frac{1}{8}} \frac{1}{\sqrt{\frac{1}{4} - 2t}} = +\infty.$



§2.3

#1. $u(t, y) = e^{At} \cdot y$; linear in y

$\Rightarrow \frac{\partial u}{\partial y}(t, y) = e^{At}.$

i.e. $\phi(t) = e^{At}.$

Since $\Phi' = (e^{At})' = Ae^{At}$ by Lemma in §1.4

$= A \cdot \Phi$

& $\Phi(0) = e^{A \cdot 0} = I$. //

§2.4

1 (a) $\frac{dx}{x^2} = dt$

$\Rightarrow -\frac{1}{x} = t + C \Rightarrow x = \frac{-1}{t+C}$

$\Rightarrow x_0 = -\frac{1}{C} \Rightarrow C = -\frac{1}{x_0}$

$\Rightarrow x = \frac{1}{\frac{1}{x_0} - t} = \frac{x_0}{1 - tx_0}$

Let $I(x_0)$ be the max interval of existence.

If $x_0 > 0$, $I(x_0) = (-\infty, \frac{1}{x_0})$

If $x_0 < 0$, $I(x_0) = (\frac{1}{x_0}, +\infty)$

If $x_0 = 0$, $I(x_0) = (-\infty, +\infty)$

For $x_0 > 0$, $\lim_{t \nearrow 1/x_0} x(t) = +\infty$

For $x_0 < 0$, $\lim_{t \searrow 1/x_0} x(t) = -\infty$

(d) From #4 in §2.1,

$x(t) = \frac{1}{\sqrt{\frac{1}{x_0^2} - 2t}}$ & $I(x_0) = (-\infty, \frac{1}{2x_0^2})$

$$\lim_{t \nearrow \frac{1}{2x_0^2}} x(t) = +\infty$$

#2 (a) From #1(a), $x_1(t) = \dfrac{1}{1-t}$

$$\Rightarrow x_2' - x_2 = 1 - t$$

By integrating factor method,

$$(e^{-t} \cdot x_2)' = (1-t)e^{-t}$$

$u = 1-t$ $dv = e^{-t}$

$du = -dt$ $v = -e^{-t}$

$$\Rightarrow e^{-t} \cdot x_2 = \int (1-t)e^{-t}\,dt + C$$

$$= (t-1)e^{-t} - \int e^{-t}\,dt + C$$

$$= te^{-t} + C$$

$$\Rightarrow x_2(t) = t + ce^{t}$$

$$\Rightarrow x_2(0) = C = 1$$

i.e $x_2(t) = t + e^{t}$.

$$\Rightarrow \quad x(t) = \begin{pmatrix} \frac{1}{1-t} \\ t + e^{t} \end{pmatrix}$$

$I(x_0) = (-\infty, 1)$; max interval

$$\lim_{t \nearrow 1} x(t) = \begin{pmatrix} \infty \\ 1+e \end{pmatrix} \quad \text{DNE}_{\text{//}}$$

(b) $2x_1 \, dx_1 = dt$

$x_1^2 = t + c. \quad \Rightarrow \quad 1 = c$

$\Rightarrow \quad x_1 = \sqrt{t+1}.$

From 2(a), $x_2(t) = \dfrac{1}{1-t}$

i.e. $x(t) = \begin{pmatrix} \sqrt{t+1} \\ \dfrac{1}{1-t} \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} t+1 \geq 0 \\ t \neq 1 \end{matrix}$

$\Rightarrow \quad I(x_0) = (-1, 1)$

$\lim_{t \nearrow 1} x(t) = \begin{pmatrix} \sqrt{2} \\ \infty \end{pmatrix}$ DNE

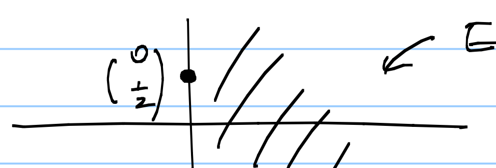
$\lim_{t \searrow -1} x(t) = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$

Since $f(x_1, x_2) = \begin{pmatrix} \dfrac{1}{2x_1} \\ x_2^2 \end{pmatrix}$ & $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$E = (0, \infty) \times \mathbb{R}$ is the set where

f is C^1 & $x_0 \in E$.

i.e.



$\Rightarrow \quad \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \in \partial E$ (boundary of E) //