Proof. If $\lambda = a + ib$, it follows by induction that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k = \begin{bmatrix} \mathrm{Re}(\lambda^k) & -\mathrm{Im}(\lambda^k) \\ \mathrm{Im}(\lambda^k) & \mathrm{Re}(\lambda^k) \end{bmatrix}$$

where Re and Im denote the real and imaginary parts of the complex number λ respectively. Thus,

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \begin{bmatrix} \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) & -\mathrm{Im}\left(\frac{\lambda^k}{k!}\right) \\ \mathrm{Im}\left(\frac{\lambda^k}{k!}\right) & \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Re}(e^\lambda) & -\mathrm{Im}(e^\lambda) \\ \mathrm{Im}(e^\lambda) & \mathrm{Re}(e^\lambda) \end{bmatrix} \\ &= e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}. \end{aligned}$$

Note that if $a = 0$ in Corollary 3, then e^A is simply a rotation through b radians.

Corollary 4. *If*

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

then

$$e^A = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

Proof. Write $A = aI + B$ where

$$B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

Then aI commutes with B and by Proposition 2,

$$e^A = e^{aI}e^B = e^a e^B.$$

And from the definition

$$e^B = I + B + B^2/2! + \cdots = I + B$$

since by direct computation $B^2 = B^3 = \cdots = 0$.

We can now compute the matrix e^{At} for any 2×2 matrix A. In Section 1.8 of this chapter it is shown that there is an invertible 2×2 matrix P (whose columns consist of generalized eigenvectors of A) such that the matrix

$$B = P^{-1}AP$$

has one of the following forms

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

It then follows from the above corollaries and Definition 2 that

$$e^{Bt} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}, \quad e^{Bt} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad e^{Bt} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

respectively. And by Proposition 1, the matrix e^{At} is then given by

$$e^{At} = Pe^{Bt}P^{-1}.$$

As we shall see in Section 1.4, finding the matrix e^{At} is equivalent to solving the linear system (1) in Section 1.1.

Problem Set 3

1. Compute the operator norm of the linear transformation defined by the following matrices:

   (a) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$.

   Hint: In (c) maximize $|A\mathbf{x}|^2 = 26x_1^2 + 10x_1x_2 + x_2^2$ subject to the constraint $x_1^2 + x_2^2 = 1$ and use the result of Problem 2; or use the fact that $\|A\| = [\text{Max eigenvalue of } A^TA]^{1/2}$. Follow this same hint for (b).

2. Show that the operator norm of a linear transformation T on $\mathbf{R}^n$ satisfies

$$\|T\| = \max_{|\mathbf{x}|=1} |T(\mathbf{x})| = \sup_{\mathbf{x}\neq 0} \frac{|T(\mathbf{x})|}{|\mathbf{x}|}.$$

3. Use the lemma in this section to show that if T is an invertible linear transformation then $\|T\| > 0$ and

$$\|T^{-1}\| \geq \frac{1}{\|T\|}.$$

4. If T is a linear transformation on $\mathbf{R}^n$ with $\|T - I\| < 1$, prove that T is invertible and that the series $\sum_{k=0}^{\infty}(I - T)^k$ converges absolutely to T^{-1}.

   Hint: Use the geometric series.

5. Compute the exponentials of the following matrices:

   (a) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

6. (a) For each matrix in Problem 5 find the eigenvalues of e^A.

   (b) Show that if $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of e^A corresponding to the eigenvalue e^λ.

   (c) If $A = P\,\mathrm{diag}[\lambda_j]P^{-1}$, use Corollary 1 to show that
   $$\det e^A = e^{\mathrm{trace}A}.$$
   Also, using the results in the last paragraph of this section, show that this formula holds for any 2×2 matrix A.

7. Compute the exponentials of the following matrices:
   $$\text{(a)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{(b)}\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{(c)}\ \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$
   Hint: Write the matrices in (b) and (c) as a diagonal matrix S plus a matrix N. Show that S and N commute and compute e^S as in part (a) and e^N by using the definition.

8. Find 2×2 matrices A and B such that $e^{A+B} \neq e^A e^B$.

9. Let T be a linear operator on $\mathbf{R}^n$ that leaves a subspace $E \subset \mathbf{R}^n$ invariant; i.e., for all $\mathbf{x} \in E$, $T(\mathbf{x}) \in E$. Show that e^T also leaves E invariant.

1.4 The Fundamental Theorem for Linear Systems

Let A be an $n \times n$ matrix. In this section we establish the fundamental fact that for $\mathbf{x}_0 \in \mathbf{R}^n$ the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{1}$$

has a unique solution for all $t \in \mathbf{R}$ which is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0. \tag{2}$$

Notice the similarity in the form of the solution (2) and the solution $x(t) = e^{at}x_0$ of the elementary first-order differential equation $\dot{x} = ax$ and initial condition $x(0) = x_0$.

In order to prove this theorem, we first compute the derivative of the exponential function e^{At} using the basic fact from analysis that two convergent limit processes can be interchanged if one of them converges uniformly. This is referred to as Moore's Theorem; cf. Graves [G], p. 100 or Rudin [R], p. 149.

Lemma. *Let A be a square matrix, then*

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Proof. Since A commutes with itself, it follows from Proposition 2 and Definition 2 in Section 3 that

$$\begin{aligned} \frac{d}{dt}e^{At} &= \lim_{h\to 0}\frac{e^{A(t+h)} - e^{At}}{h} \\ &= \lim_{h\to 0} e^{At}\frac{(e^{Ah} - I)}{h} \\ &= e^{At}\lim_{h\to 0}\lim_{k\to\infty}\left(A + \frac{A^2h}{2!} + \cdots + \frac{A^kh^{k-1}}{k!}\right) \\ &= Ae^{At}. \end{aligned}$$

The last equality follows since by the theorem in Section 1.3 the series defining e^{Ah} converges uniformly for $|h| \le 1$ and we can therefore interchange the two limits.

Theorem (The Fundamental Theorem for Linear Systems). *Let A be an $n \times n$ matrix. Then for a given $\mathbf{x}_0 \in \mathbf{R}^n$, the initial value problem*

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{1}$$

has a unique solution given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0. \tag{2}$$

Proof. By the preceding lemma, if $\mathbf{x}(t) = e^{At}\mathbf{x}_0$, then

$$\mathbf{x}'(t) = \frac{d}{dt}e^{At}\mathbf{x}_0 = Ae^{At}\mathbf{x}_0 = A\mathbf{x}(t)$$

for all $t \in \mathbf{R}$. Also, $\mathbf{x}(0) = I\mathbf{x}_0 = \mathbf{x}_0$. Thus $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ is a solution. To see that this is the only solution, let $\mathbf{x}(t)$ be any solution of the initial value problem (1) and set

$$\mathbf{y}(t) = e^{-At}\mathbf{x}(t).$$

Then from the above lemma and the fact that $\mathbf{x}(t)$ is a solution of (1)

$$\begin{aligned}\mathbf{y}'(t) &= -Ae^{-At}\mathbf{x}(t) + e^{-At}\mathbf{x}'(t)\\ &= -Ae^{-At}\mathbf{x}(t) + e^{-At}A\mathbf{x}(t)\\ &= \mathbf{0}\end{aligned}$$

for all $t \in \mathbf{R}$ since e^{-At} and A commute. Thus, $\mathbf{y}(t)$ is a constant. Setting $t = 0$ shows that $\mathbf{y}(t) = \mathbf{x}_0$ and therefore any solution of the initial value problem (1) is given by $\mathbf{x}(t) = e^{At}\mathbf{y}(t) = e^{At}\mathbf{x}_0$. This completes the proof of the theorem.

Example. Solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}$$
$$\mathbf{x}(0) = \begin{bmatrix}1\\0\end{bmatrix}$$

for

$$A = \begin{bmatrix}-2 & -1\\ 1 & -2\end{bmatrix}$$

and sketch the solution curve in the phase plane $\mathbf{R}^2$. By the above theorem and Corollary 3 of the last section, the solution is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{-2t}\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = e^{-2t}\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}.$$

It follows that $|\mathbf{x}(t)| = e^{-2t}$ and that the angle $\theta(t) = \tan^{-1}x_2(t)/x_1(t) = t$. The solution curve therefore spirals into the origin as shown in Figure 1 below.

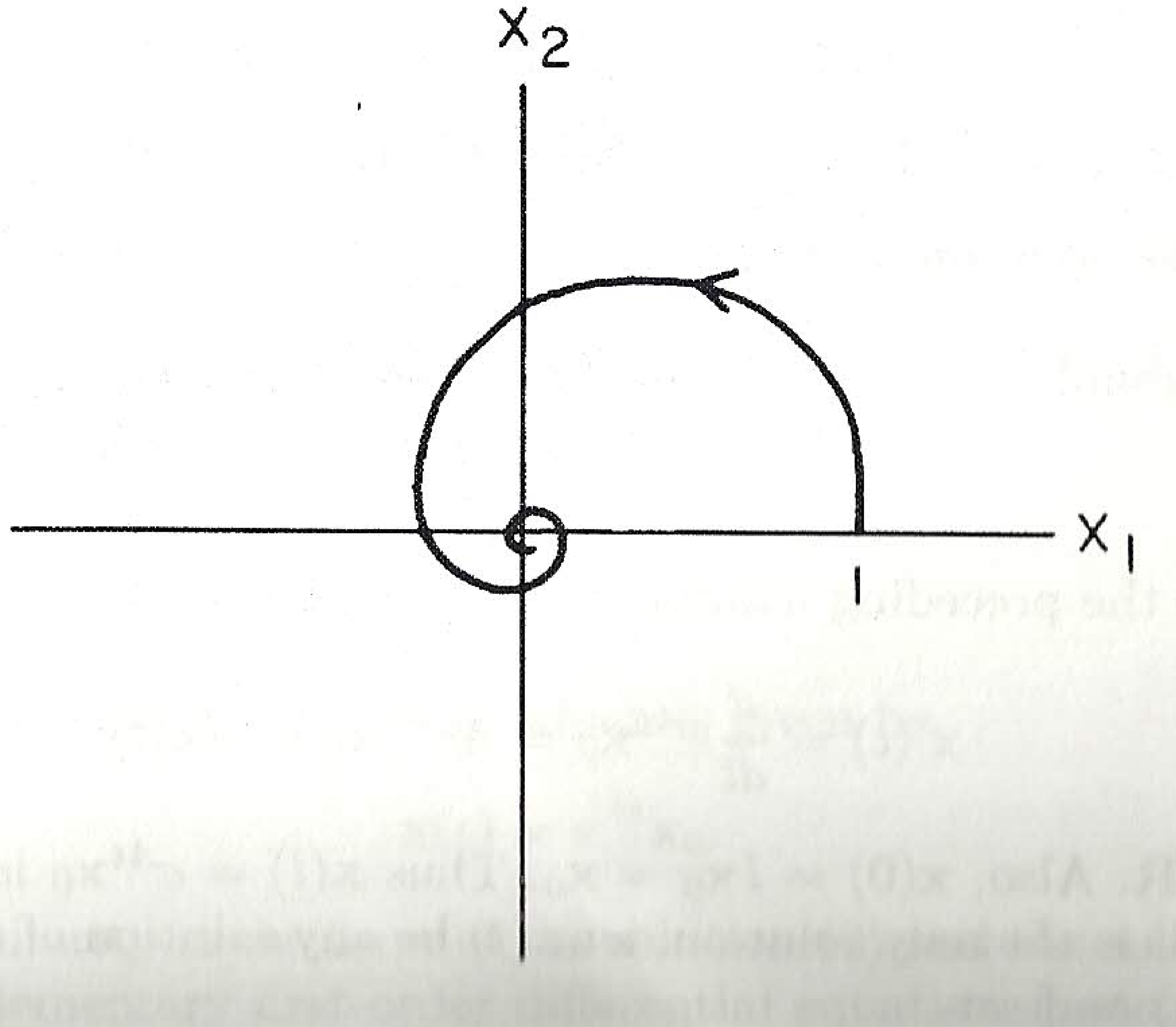

Figure 1

Problem Set 4

1. Use the forms of the matrix e^{Bt} computed in Section 1.3 and the theorem in this section to solve the linear system $\dot{\mathbf{x}} = B\mathbf{x}$ for

   (a) $B = \begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}$

   (b) $B = \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$

   (c) $B = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$.

2. Solve the following linear system and sketch its phase portrait

$$\dot{\mathbf{x}} = \begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}\mathbf{x}.$$

   The origin is called a stable focus for this system.

3. Find e^{At} and solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ for

   (a) $A = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$

   (b) $A = \begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}$.

   Cf. Problem 1 in Problem Set 2.

4. Given

$$A = \begin{bmatrix}1 & 0 & 0\\ 1 & 2 & 0\\ 1 & 0 & -1\end{bmatrix}.$$

   Compute the 3×3 matrix e^{At} and solve $\dot{\mathbf{x}} = A\mathbf{x}$. Cf. Problem 2 in Problem Set 2.

5. Find the solution of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ where

   (a) $A = \begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}$

   (b) $A = \begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$

   (c) $A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

   (d) $A = \begin{bmatrix}-2 & 0 & 0\\ 1 & -2 & 0\\ 0 & 1 & -2\end{bmatrix}$.