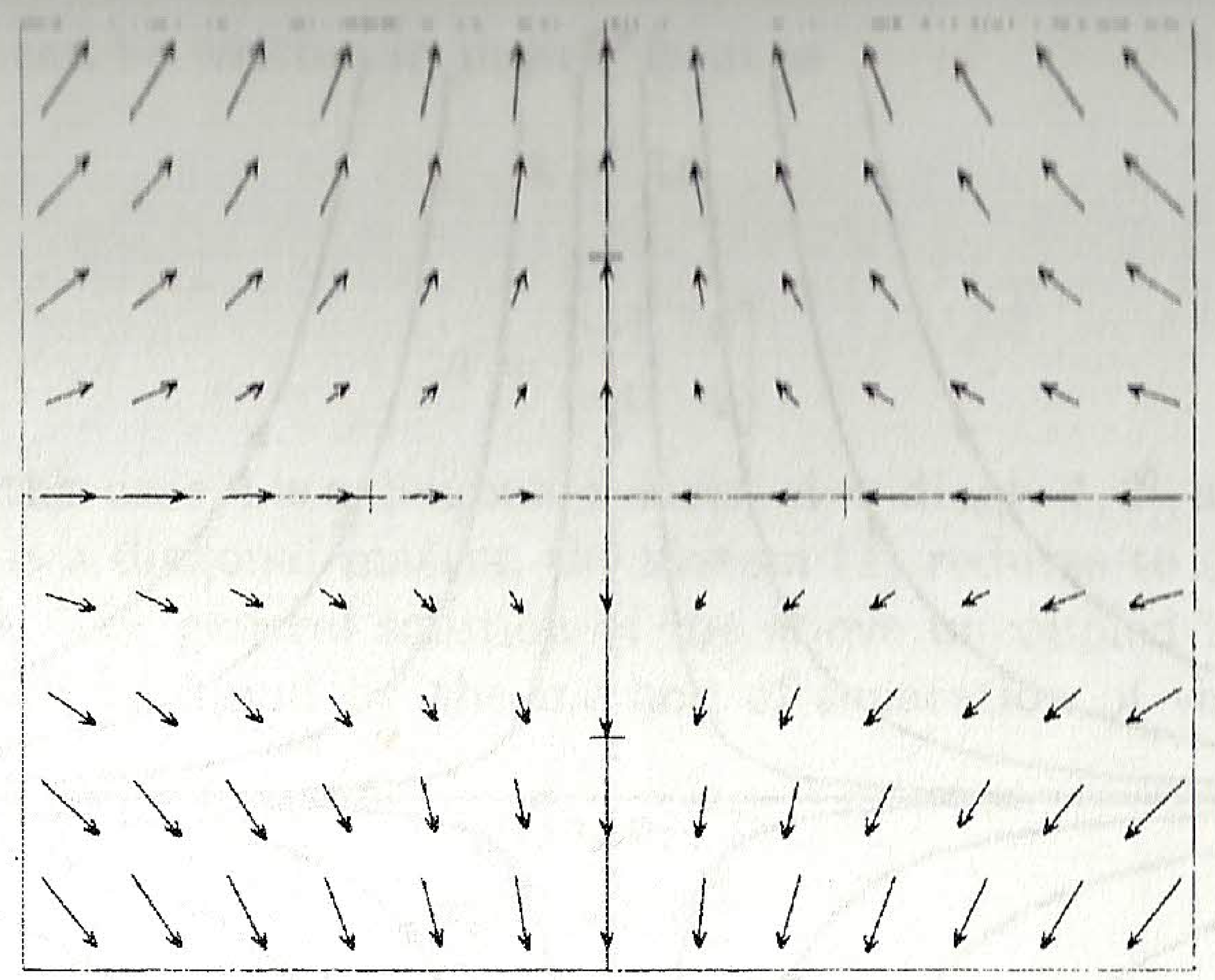

Figure 2

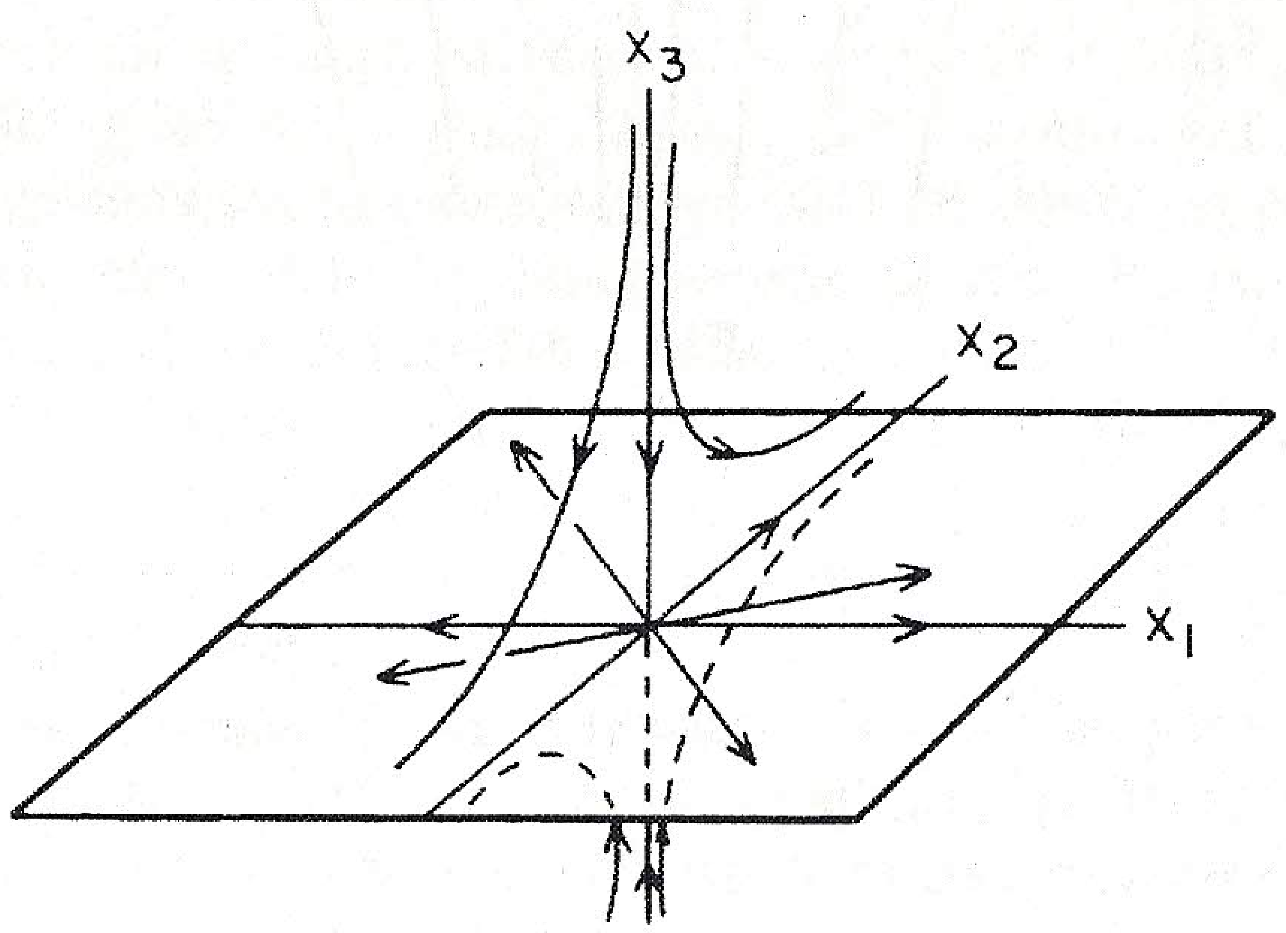

Figure 3

The general solution is given by

$$x_1(t) = c_1 e^t$$
$$x_2(t) = c_2 e^t$$
$$x_3(t) = c_3 e^{-t}.$$

And the phase portrait for this system is shown in Figure 3 above. The x_1, x_2 plane is referred to as the *unstable subspace* of the system (3) and

the x_3 axis is called the *stable subspace* of the system (3). Precise definitions of the stable and unstable subspaces of a linear system will be given in the next section.

Problem Set 1

1. Find the general solution and draw the phase portrait for the following linear systems:

(a) $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$

(b) $\dot{x}_1 = x_1$, $\dot{x}_2 = 2x_2$

(c) $\dot{x}_1 = x_1$, $\dot{x}_2 = 3x_2$

(d) $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1$

(e) $\dot{x}_1 = -x_1 + x_2$, $\dot{x}_2 = -x_2$

Hint: Write (d) as a second-order linear differential equation with constant coefficients, solve it by standard methods, and note that $x_1^2 + x_2^2 =$ constant on the solution curves. In (e), find $x_2(t) = c_2 e^{-t}$ and then the x_1-equation becomes a first order linear differential equation.

2. Find the general solution and draw the phase portraits for the following three-dimensional linear systems:

(a) $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$, $\dot{x}_3 = x_3$

(b) $\dot{x}_1 = -x_1$, $\dot{x}_2 = -x_2$, $\dot{x}_3 = x_3$

(c) $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1$, $\dot{x}_3 = -x_3$

Hint: In (c), show that the solution curves lie on right circular cylinders perpendicular to the x_1, x_2 plane. Identify the stable and unstable subspaces in (a) and (b). The x_3-axis is the stable subspace in (c) and the x_1, x_2 plane is called the center subspace in (c); cf. Section 1.9.

3. Find the general solution of the linear system

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = ax_2$$

where a is a constant. Sketch the phase portraits for $a = -1, a = 0$ and $a = 1$ and notice that the qualitative structure of the phase portrait is the same for all $a < 0$ as well as for all $a > 0$, but that it changes at the parameter value $a = 0$ called a bifurcation value.

4. Find the general solution of the linear system (1) when A is the $n \times n$ diagonal matrix $A = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. What condition on the eigenvalues $\lambda_1, \ldots, \lambda_n$ will guarantee that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ for all solutions $\mathbf{x}(t)$ of (1)?

5. What is the relationship between the vector fields defined by

$$\dot{\mathbf{x}} = A\mathbf{x}$$

and

$$\dot{\mathbf{x}} = kA\mathbf{x}$$

where k is a non-zero constant? (Describe this relationship both for k positive and k negative.)

6. (a) If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions of the linear system (1), prove that for any constants a and b, $\mathbf{w}(t) = a\mathbf{u}(t) + b\mathbf{v}(t)$ is a solution.

(b) For

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

find solutions $\mathbf{u}(t)$ and $\mathbf{v}(t)$ of $\dot{\mathbf{x}} = A\mathbf{x}$ such that every solution is a linear combination of $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

1.2 Diagonalization

The algebraic technique of diagonalizing a square matrix A can be used to reduce the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

to an uncoupled linear system. We first consider the case when A has real, distinct eigenvalues. The following theorem from linear algebra then allows us to solve the linear system (1).

Theorem. *If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A are real and distinct, then any set of corresponding eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ forms a basis for $\mathbf{R}^n$, the matrix $P = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$ is invertible and*

$$P^{-1}AP = \mathrm{diag}[\lambda_1, \ldots, \lambda_n].$$

This theorem says that if a linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is represented by the $n \times n$ matrix A with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for $\mathbf{R}^n$, then with respect to any basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, T is represented by the diagonal matrix of eigenvalues, $\mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. A proof of this theorem can be found, for example, in Lowenthal [Lo].

In order to reduce the system (1) to an uncoupled linear system using the above theorem, define the linear transformation of coordinates

$$\mathbf{y} = P^{-1}\mathbf{x}$$

where P is the invertible matrix defined in the theorem. Then

$$\mathbf{x} = P\mathbf{y},$$
$$\dot{\mathbf{y}} = P^{-1}\dot{\mathbf{x}} = P^{-1}A\mathbf{x} = P^{-1}AP\mathbf{y}$$

and, according to the above theorem, we obtain the uncoupled linear system

$$\dot{\mathbf{y}} = \mathrm{diag}[\lambda_1, \ldots, \lambda_n]\mathbf{y}.$$

This uncoupled linear system has the solution

$$\mathbf{y}(t) = \mathrm{diag}[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}]\mathbf{y}(0).$$

(Cf. problem 4 in Problem Set 1.) And then since $\mathbf{y}(0) = P^{-1}\mathbf{x}(0)$ and $\mathbf{x}(t) = P\mathbf{y}(t)$, it follows that (1) has the solution

$$\mathbf{x}(t) = PE(t)P^{-1}\mathbf{x}(0). \tag{2}$$

where $E(t)$ is the diagonal matrix

$$E(t) = \mathrm{diag}[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}].$$

Corollary. *Under the hypotheses of the above theorem, the solution of the linear system (1) is given by the function $\mathbf{x}(t)$ defined by (2).*

Example. Consider the linear system

$$\dot{x}_1 = -x_1 - 3x_2$$
$$\dot{x}_2 = 2x_2$$

which can be written in the form (1) with the matrix

$$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$. A pair of corresponding eigenvectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The matrix P and its inverse are then given by

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The student should verify that

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then under the coordinate transformation $\mathbf{y} = P^{-1}\mathbf{x}$, we obtain the uncoupled linear system

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= 2y_2 \end{aligned}$$

which has the general solution $y_1(t) = c_1 e^{-t}$, $y_2(t) = c_2 e^{2t}$. The phase portrait for this system is given in Figure 1 in Section 1.1 which is reproduced below. And according to the above corollary, the general solution to the original linear system of this example is given by

$$\mathbf{x}(t) = P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1}\mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$, or equivalently by

$$\begin{aligned} x_1(t) &= c_1 e^{-t} + c_2(e^{-t} - e^{2t}) \\ x_2(t) &= c_2 e^{2t}. \end{aligned} \tag{3}$$

The phase portrait for the linear system of this example can be found by sketching the solution curves defined by (3). It is shown in Figure 2. The

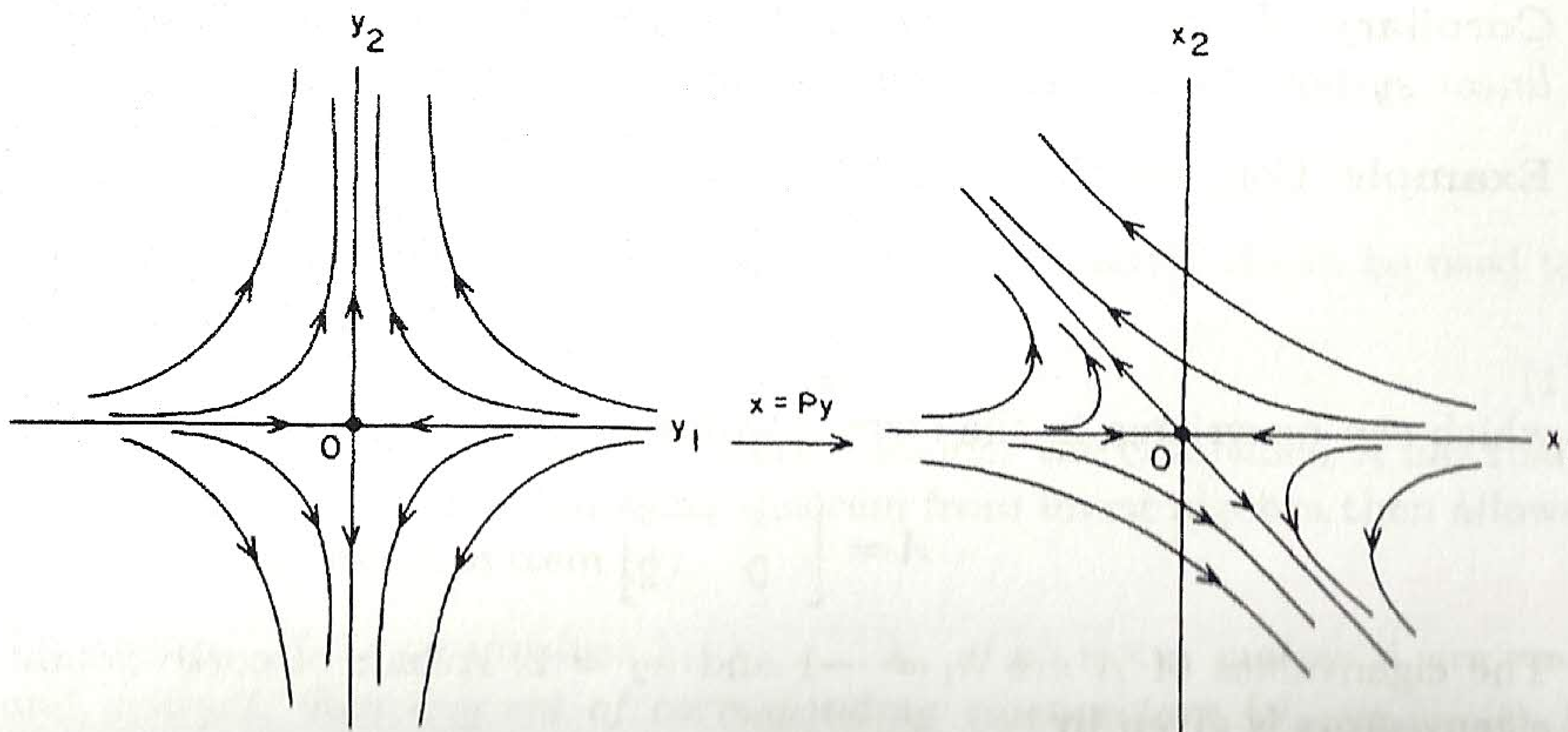

Figure 1 **Figure 2**

phase portrait in Figure 2 can also be obtained from the phase portrait in Figure 1 by applying the linear transformation of coordinates $\mathbf{x} = P\mathbf{y}$. Note that the subspaces spanned by the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the matrix A determine the stable and unstable subspaces of the linear system (1) according to the following definition:

Suppose that the $n \times n$ matrix A has k negative eigenvalues $\lambda_1, \ldots, \lambda_k$ and $n-k$ positive eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ and that these eigenvalues are distinct. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a corresponding set of eigenvectors. Then the *stable and unstable subspaces of the linear system* (1), E^s *and* E^u, are the linear subspaces spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ respectively; i.e.,

$$\begin{aligned} E^s &= \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \\ E^u &= \operatorname{Span}\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}. \end{aligned}$$

If the matrix A has pure imaginary eigenvalues, then there is also a center subspace E^c; cf. Problem 2(c) in Section 1.1. The stable, unstable and center subspaces are defined for the general case in Section 1.9.

Problem Set 2

1. Find the eigenvalues and eigenvectors of the matrix A and show that $B = P^{-1}AP$ is a diagonal matrix. Solve the linear system $\dot{\mathbf{y}} = B\mathbf{y}$ and then solve $\dot{\mathbf{x}} = A\mathbf{x}$ using the above corollary. And then sketch the phase portraits in both the $\mathbf{x}$ plane and $\mathbf{y}$ plane.

   (a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

2. Find the eigenvalues and eigenvectors for the matrix A, solve the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, determine the stable and unstable subspaces for the linear system, and sketch the phase portrait for

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{x}.$$

3. Write the following linear differential equations with constant coefficients in the form of the linear system (1) and solve:

   (a) $\ddot{x} + \dot{x} - 2x = 0$

   (b) $\ddot{x} + x = 0$