Banach SSD spaces and classes of monotone sets

S. Simons

Department of Mathematics, University of California
Santa Barbara, CA 93106-3080, U.S.A.
simons@math.ucsb.edu

In this paper, we unify the theory of SSD spaces and the theory of strongly representable sets, and we apply our results to the theory of the various classes of maximally monotone sets. In particular, we prove that type (ED), dense type, type (D), type (NI) and strongly representable are equivalent concepts and, consequently, that the known properties of strongly representable sets follow from known properties of sets of type (ED).

1. Introduction

In Sections 2–4, we give a more complete version of the algebraic theory of SSD spaces, as introduced in [18], and further developed in [20]. Apart from the fact that we write "$\mathcal{P}_q$" instead of "pos", we use the notation of the latter of these references. What distinguishes the three sections is the number of bilinear forms considered: one in Section 2, two in Section 3, and three in Section 4. The concepts of *SSD space*, *q–positive set* and the convex function, Φ_A, associated with a q–positive set, A, are introduced in Definition 2.2. The functions Φ are the generalizations to SSD spaces of the Fitzpatrick functions of monotone sets. In Definition 2.8, we introduce the q–positive set, $\mathcal{P}_q(f)$, associated with an appropriate convex function, f, and in Definition 2.10, we introduce the *intrinsic conjugate*, $f^{@}$, of a convex function, f. The main result in this section is Theorem 2.14, though the results marked "Lemma" will be used throughout the paper. In Section 3, we consider the situation of an SSD space linked by a linear map to an external vector space. We then define another convex function, Ψ_A, associated with a q–positive set, A, using a convex function, Θ_A, on the external space. The basic properties of these functions are collected together in Lemma 3.2. In Theorem 3.3 and Corollaries 3.4 and 3.5, we discuss a maximal property of the Ψ functions, which complements a minimal property of the Φ functions in certain circumstances. While the material of Section 2 is essentially algebraic $\big($apart from the disguised differentiability argument of Lemma 2.12(a)$\big)$, Theorem 3.3 uses the Fenchel–Moreau theorem from convex analysis, for a (possibly nonhausdorff) locally convex space. Since we have not seen this result in the literature, we give a proof of it for the convenience of the reader in Section 10. In Section 4, we consider the special situation where the external space is also an SSD space, and the allied concept of *SSD–homomorphism*. These are introduced in Definition 4.1. These ideas allow us to apply the analysis of Sections 2 and 3, with the SSD space replaced by the external space. This enables us to generalize (in Definition 4.4 and Theorem 4.5) to SSD spaces some concepts due to Gossez for maximally monotone multifunctions.

In Sections 5 and 6, we specialize to the situation in which the SSD spaces have a Banach space structure also. In Definition 5.1, we introduce the concept of a *Banach SSD space*, which is an SSD space with a Banach space structure satisfying the compatibility condition (26), from which it follows that the Banach space dual can be considered as a linked external space (see Remark 5.9). Section 5 is inspired by Voisei–Zălinescu, [22]. In Definition 5.4, we introduce the concept of *VZ function* on a Banach SSD space. Our main result on VZ functions, established in Theorems 5.6(d) and 5.8(b), is that *if* f *is a lower semicontinuous VZ function then* $\mathcal{P}_q(f)$ *is* maximally q*–positive,* $f^{@}$ *is also a VZ function, and* $\mathcal{P}_q\big(f^{@}\big) = \mathcal{P}_q(f)$. The argument in Theorem 5.6 uses completeness heavily, as well as the fact that a proper, convex, lower semicontinuous function on a Banach space dominates a continuous affine function. Theorem 5.7 contains a characterization of VZ functions in terms of the concept of p*–density* introduced in Definition 5.5. We show in Theorem 5.11 that if f is a lower semicontinuous VZ function on a Banach SSD space and $A = \mathcal{P}_q(f)$ then f lies between Ψ_A and Φ_A and, further, if h is a proper convex function on B that lies between Ψ_A and Φ_A then h and $h^{@}$ are VZ functions. The rest of Section 5 is devoted to some counterexamples. Section 5 uses the material of Sections 2 and 3, but not Section 4. By contrast, Section 6 depends heavily on Section 4. Here we consider the situation where the dual of a Banach SSD space can be made into a Banach SSD space in its own right, satisfying the compatibility condition (49), and we write $\widetilde{q}$ for the function on the dual corresponding to the function q previously defined on the original Banach space. In Definition 6.11, we work towards defining an analog for SSD spaces of the concept of *strongly representable multifunction*, as expounded by Voisei and Zălinescu in [22] and Marques Alves and Svaiter in [5], [6] and [7]. In order to to this, we introduce the concept of *MAS function* in Definition 6.11. The first main result of Section 6 is Theorem 6.12 (which leads to Theorem 9.7), in which we establish that, under a certain mild side condition, the concepts of MAS function and VZ function are identical. The main tools here are Rockafellar's formula for the conjugate of the sum of convex functions, and the fact that the conjugate of the function $\frac{1}{2}\|\cdot\|^2$ on a Banach space is the function $\frac{1}{2}\|\cdot\|^2$ on the dual space, which are both used in Lemma 6.10. The other main result of Section 6 is Theorem 6.15 (which leads to Theorem 9.5), which depends on the concept of a *compatible* topology on B^* introduced in Definition 6.13, and describes the relationship between compatible topologies and the Gossez extension of a maximally q–positive set introduced in Definition 4.4.

Sections 7 and 8 are devoted to a discussion of certain esoteric topologies on the bidual of a Banach space and the Banach SSD dual of a Banach SSD space. Here is some background for the problem. Suppose that E is a nonzero Banach space, and consider the function $\widetilde{q}$: $(x^*, x^{**}) \mapsto \langle x^*, x^{**}\rangle$ from $(E \times E^*)^* = E^* \times E^{**}$ into $\mathbb{R}$. While it is true that the norm topology on $E^* \times E^{**}$ makes $\widetilde{q}$ continuous, it has been known since the work of Gossez on maximally monotone multifunctions that the norm topology is too large to be of any practical use. The reason for this can be traced to the fact that it is not generally compatible in the sense of Definition 6.13. (See Remark 6.14.) In Section 8, we introduce the topologies $\mathcal{T}_{\mathcal{D}}$ on Banach SSD duals, which have the properties that they are sufficiently small that they are compatible and sufficiently large that Theorem 6.15 leads to significant results. The topologies $\mathcal{T}_{\mathcal{D}}$ are based on the topologies $\mathcal{T}_{\mathcal{CLB}}$ on the bidual of a Banach space that have been previously studied, the properties of which are stated in Section 7.

So far, we have been describing general theories, but we have not discussed any particular examples. In Example 2.3, we give three examples of SSD spaces, of which (c) is probably the most interesting. As we observe in Remark 5.2, Example 2.3(a,b,c) are actually Banach SSD spaces. Example 2.4 is the example that leads to results on monotonicity — it is shown in Example 5.3 how to norm this example so that it becomes a Banach SSD space, and in Example 6.5 how to make its dual a Banach SSD dual. We then show in Theorem 8.3 that $\widetilde{q}$ is continuous with respect to the topology $\mathcal{T}_{\mathcal{D}}$, and so we are now in the position that we can apply Theorems 5.6, 6.12 and 6.15 to this example. This leads to the results on monotonicity that we give in Section 9.

We start Section 9 with a brief history of various classes of maximally monotone sets. We define *type (D)*, *dense type*, *type (NI)*, *type (WD)*, *type (ED)* and *strongly representable* in Definitions 9.1, 9.2, 9.3 and 9.6. The easy implications are that, for maximally monotone sets, type (ED) $\Longrightarrow$ dense type $\Longrightarrow$ type (D) $\Longrightarrow$ type (WD) $\Longrightarrow$ type (NI). Marques Alves and Svaiter proved recently in [7] that type (NI) $\Longrightarrow$ type (D). In Theorem 9.5, we show how the techniques discussed in this paper lead to the stronger conclusion that type (NI) $\Longrightarrow$ type (ED). The obvious significance of this is that Theorem 9.5 leads to solutions of several problems that have been open for some time. These issues are discussed in the paragraph preceding Theorem 9.5. However, Theorem 9.5 is significant for another reason. Strongly representable sets were initially introduced in [5] and [22], and it was proved in [6] that a set is strongly representable $\iff$ it is maximally monotone of type (NI). In Theorem 9.7, we show how the techniques discussed in this paper lead to a proof of this equivalence. If we now combine together the results discussed above, we see that a set is strongly representable $\iff$ it is maximally monotone of type (ED). This enables us to use the properties known for maximally monotone sets of type (ED) to obtain results about strongly representable sets, which frequently improve on the results already known. We give these results in Theorems 9.9 and 9.10, with references to what is currently in the literature.

In the Appendix, Section 10, we give a proof of the Fenchel–Moreau theorem for a (possibly nonhausdorff) locally convex space, which we used in Theorem 3.3.

The author would like to thank Constantin Zălinescu for making him aware of the preprints [5] and [22], and Maicon Marques Alves and Benar Svaiter for making him aware of the preprints [6] and [7]. He would also like to thank Radu Ioan Boţ and Constantin Zălinescu for some very perceptive comments on earlier versions of this paper. The author has learned that the acronym "SSD" has also been used to stand for "strongly subdifferentiable". He hopes that this will not cause any confusion. Finally, he would like to thank the anonymous referees, whose insightful comments resulted in great improvements.

2. SSD spaces

Definition 2.1. If X is a nonzero real vector space and $f\colon\ X \to\,]-\infty,\infty]$, we write $\operatorname{dom} f$ for the set $\big\{x \in X\colon\ f(x) \in \mathbb{R}\big\}$. $\operatorname{dom} f$ is the *effective domain* of f. We say that f is *proper* if $\operatorname{dom} f \ne \emptyset$. We write $\mathcal{PC}(X)$ for the set of all proper convex functions from X into $]-\infty,\infty]$. If X is a nonzero real Banach space, we write X^* for the dual space of X $\big($with the pairing $\langle\cdot,\cdot\rangle\colon X \times X^* \to \mathbb{R}\big)$.

Definition 2.2. We will say that $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ is a *symmetrically self–dual space (SSD space)* if B is a nonzero real vector space and $\lfloor\cdot,\cdot\rfloor\colon B \times B \to \mathbb{R}$ is a symmetric bilinear form. In this case, we will *always* write $q(b) := \frac{1}{2}\lfloor b, b\rfloor \quad (b \in B)$. ("$q$" stands for "quadratic".) We do not assume that $\lfloor\cdot,\cdot\rfloor$ separates the points of B, as was done in [18] and [20]. With this caveat, the definitions and many of the results of his section appear in [18] and [20].

Now let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ be an SSD space and $A \subset B$. We say that A is *q–positive* if $A \ne \emptyset$ and

$$b, c \in A \Longrightarrow q(b-c) \ge 0.$$

In this case, since $q(0) = 0$,

$$b \in A \Longrightarrow \inf q(A-b) = 0. \tag{1}$$

We then define $\Phi_A\colon\ B \to \,]{-\infty}, \infty]$ by

$$\Phi_A(b) := \sup\nolimits_A \big[\lfloor\cdot, b\rfloor - q\big] \quad (b \in B). \tag{2}$$

Φ_A is a generalization to SSD spaces of the "Fiztpatrick function" of a monotone set, which was originally introduced in [2] in 1988, but lay dormant until it was rediscovered by Martínez-Legaz and Théra in [9] in 2001. We note then that, for all $b \in B$,

$$\left.\begin{aligned}\Phi_A(b) &= q(b) - \inf\nolimits_{a\in A}\big[q(a) - \lfloor a, b\rfloor + q(b)\big]\\ &= q(b) - \inf\nolimits_{a\in A} q(a-b) = q(b) - \inf q(A-b).\end{aligned}\right\} \tag{3}$$

From (1),

$$\Phi_A = q \text{ on } A. \tag{4}$$

Thus $\Phi_A \in \mathcal{PC}(B)$. We say that A is *maximally q–positive* if A is q–positive and A is not properly contained in any other q–positive set. In this case, if $b \in B$ and $\inf q(A-b) \ge 0$ then clearly $b \in A$. In other words, $\big(b \in B \setminus A \Longrightarrow \inf q(A-b) < 0\big)$. From (1), $\inf q(A-b) \le 0$ and $\big(\inf q(A-b) = 0 \iff b \in A\big)$. Thus, from (3)

$$\Phi_A \ge q \text{ on } B \quad\text{and}\quad \big(\Phi_A(b) = q(b) \iff b \in A\big). \tag{5}$$

We make the elementary observation that if $b \in B$ and $q(b) \ge 0$ then the linear span $\mathbb{R}b$ of $\{b\}$ is q–positive.

We now give some examples of SSD spaces and their associated q–positive sets. These examples are taken from [20, pp. 79–80].

Example 2.3. Let B be a Hilbert space with inner product $(b, c) \mapsto \langle b, c\rangle$ and $T\colon B \to B$ be a self–adjoint linear operator. Then $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ is an SSD space with $\lfloor b, c\rfloor := \langle b, Tc\rangle$ and $q(b) = \frac{1}{2}\langle Tb, b\rangle$. Here are three special cases of this example:

(a) If, for all $b \in B$, $Tb = b$ then $\lfloor b, c\rfloor := \langle b, c\rangle$, $q(b) = \frac{1}{2}\|b\|^2$ and every subset of B is q–positive

(b) If, for all $b \in B$, $Tb = -b$ then $\lfloor b, c\rfloor := -\langle b, c\rangle$, $q(b) = -\frac{1}{2}\|b\|^2$ and the q–positive sets are the singletons.

(c) If $B = \mathbb{R}^3$ and $T(b_1, b_2, b_3) = (b_2, b_1, b_3)$ then

$$\big\lfloor (b_1, b_2, b_3), (c_1, c_2, c_3)\big\rfloor := b_1c_2 + b_2c_1 + b_3c_3,$$

and $q(b_1, b_2, b_3) = b_1b_2 + \frac{1}{2}b_3^2$. Here, if M is any nonempty monotone subset of $\mathbb{R}\times\mathbb{R}$ (in the obvious sense) then $M \times \mathbb{R}$ is a q–positive subset of B. The set $\mathbb{R}(1, -1, 2)$ is a q–positive subset of B which is not contained in a set $M \times \mathbb{R}$ for any monotone subset of $\mathbb{R}\times\mathbb{R}$. The helix $\big\{(\cos\theta, \sin\theta, \theta)\colon \theta \in \mathbb{R}\big\}$ is a q–positive subset of B, but if $0 < \lambda < 1$ then the helix $\big\{(\cos\theta, \sin\theta, \lambda\theta)\colon \theta \in \mathbb{R}\big\}$ is not.

Example 2.4. Let E be a nonzero Banach space and $B := E \times E^*$. For all (x, x^*) and $(y, y^*) \in B$, we set $\lfloor (x, x^*), (y, y^*) \rfloor := \langle x, y^* \rangle + \langle y, x^* \rangle$. Then $\big(B, \lfloor \cdot, \cdot \rfloor\big)$ is an SSD space with $q(x, x^*) = \frac{1}{2}\big[\langle x, x^* \rangle + \langle x, x^* \rangle\big] = \langle x, x^* \rangle$. Consequently, if $(x, x^*), (y, y^*) \in B$ then $\langle x - y, x^* - y^* \rangle = q(x - y, x^* - y^*) = q\big((x, x^*) - (y, y^*)\big)$. Thus if $A \subset B$ then A is q–positive exactly when A is a nonempty monotone subset of B in the usual sense, and A is maximally q–positive exactly when A is a maximally monotone subset of B in the usual sense. We point out that any finite dimensional SSD space of the form described here must have *even* dimension. Thus cases of Example 2.3 with finite odd dimension cannot be of this special form.

Example 2.5. $\big(\mathbb{R}^3, \lfloor \cdot, \cdot \rfloor\big)$ is *not* an SSD space with

$$\lfloor (b_1, b_2, b_3), (c_1, c_2, c_3) \rfloor := b_1 c_2 + b_2 c_3 + b_3 c_1.$$

(The bilinear form $\lfloor \cdot, \cdot \rfloor$ is not symmetric.)

Lemma 2.6. *Let $\big(B, \lfloor \cdot, \cdot \rfloor\big)$ be an SSD space, $f \in \mathcal{PC}(B)$, $f \geq q$ on B and $b, c \in B$. Then*

$$-q(b - c) \leq \Big[\sqrt{(f - q)(b)} + \sqrt{(f - q)(c)}\Big]^2.$$

Proof. We can and will suppose that $0 \leq (f - q)(b) < \infty$ and $0 \leq (f - q)(c) < \infty$. Let $\sqrt{(f - q)(b)} < \beta < \infty$ and $\sqrt{(f - q)(c)} < \gamma < \infty$, so that $\beta^2 + q(b) > f(b)$ and $\gamma^2 + q(c) > f(c)$. Then, writing $\alpha := \beta + \gamma$,

$$\begin{aligned}\beta\gamma + (\gamma q(b) + \beta q(c))/\alpha &= \gamma\big(\beta^2 + q(b)\big)/\alpha + \beta\big(\gamma^2 + q(c)\big)/\alpha \\ &> \gamma f(b)/\alpha + \beta f(c)/\alpha \geq f\big((\gamma b + \beta c)/\alpha\big) \\ &\geq q\big((\gamma b + \beta c)/\alpha\big) = \big(\gamma^2 q(b) + \gamma\beta\lfloor b, c \rfloor + \beta^2 q(c)\big)/\alpha^2.\end{aligned}$$

Clearing of fractions, we obtain

$$\alpha^2\beta\gamma + \alpha\big(\gamma q(b) + \beta q(c)\big) > \gamma^2 q(b) + \gamma\beta\lfloor b, c \rfloor + \beta^2 q(c),$$

from which $\alpha^2\beta\gamma > -\beta\gamma q(b) + \beta\gamma\lfloor b, c \rfloor - \beta\gamma q(c) = -\beta\gamma q(b - c)$. If we now divide by $\beta\gamma$, we obtain $\alpha^2 > -q(b - c)$, and the result follows by letting $\beta \to \sqrt{(f - q)(b)}$ and $\gamma \to \sqrt{(f - q)(c)}$. □

Remark 2.7. It follows from Lemma 2.6 and the Cauchy–Schwarz inequality that

$$-q(b - c) \leq 2(f - q)(b) + 2(f - q)(c).$$

In the situation of Example 2.4, we recover [22, Proposition 1].

Definition 2.8. If $\big(B, \lfloor \cdot, \cdot \rfloor\big)$ is an SSD space, $f \in \mathcal{PC}(B)$ and $f \geq q$ on B, we write

$$\mathcal{P}_q(f) := \big\{b \in B\colon\ f(b) = q(b)\big\}.$$

We note then that (5) implies that

$$\textit{if } A \textit{ is maximally } q\textit{–positive then } A = \mathcal{P}_q(\Phi_A). \tag{6}$$

The following result is suggested by Burachik–Svaiter, [1, Theorem 3.1, pp. 2381–2382] and Penot, [10, Proposition 4(h)$\Longrightarrow$(a), pp. 860–861].

Lemma 2.9. *Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ be an SSD space, $f \in \mathcal{PC}(B)$, $f \ge q$ on B and $\mathcal{P}_q(f) \ne \emptyset$. Then $\mathcal{P}_q(f)$ is a q–positive subset of B.*

Proof. This is immediate from Lemma 2.6. □

We now introduce a concept of conjugate that is intrinsic to an SSD space without any topological conditions.

Definition 2.10. If $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ is an SSD space and $f \in \mathcal{PC}(B)$, we write $f^{@}$ for the Fenchel conjugate of f with respect to the pairing $\lfloor\cdot,\cdot\rfloor$, that is to say,

$$\text{for all } c \in B, \qquad f^{@}(c) := \sup\nolimits_B \big[\lfloor\cdot,c\rfloor - f\big]. \tag{7}$$

The next result gives some basic properties of ${\Phi_A}^{@}$ and ${\Phi_A}^{@@}$. It will be used in Theorem 2.14, Lemma 4.2(e) and Lemma 4.3(c). The proof of Lemma 2.11(c) below is due to Radu Ioan Boţ.

Lemma 2.11. *Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ be an SSD space and A be a nonempty q–positive subset of B. Then:*

(a) ${\Phi_A}^{@} \le q$ *on* A.

(b) ${\Phi_A}^{@} \ge \Phi_A \vee q$ *on* B.

(c) ${\Phi_A}^{@@} = \Phi_A$ *on* B.

Proof. Let $a \in A$ and $b \in B$. From (2), $\lfloor a,b\rfloor - q(a) \le \Phi_A(b)$. Thus $\lfloor b,a\rfloor - \Phi_A(b) \le q(a)$, and we obtain (a) by taking the supremum over $b \in B$. Let $c \in B$. Then, from (4),

$$\begin{aligned}{\Phi_A}^{@}(c) &= \sup\nolimits_B \big[\lfloor c,\cdot\rfloor - \Phi_A\big] \ge \big[\lfloor c,c\rfloor - \Phi_A(c)\big] \vee \sup\nolimits_A \big[\lfloor c,\cdot\rfloor - \Phi_A\big]\\ &= \big[2q(c) - \Phi_A(c)\big] \vee \sup\nolimits_A \big[\lfloor c,\cdot\rfloor - q\big] = \big[2q(c) - \Phi_A(c)\big] \vee \Phi_A(c).\end{aligned}$$

Now if $\Phi_A(c) = \infty$ then obviously $\big[2q(c) - \Phi_A(c)\big] \vee \Phi_A(c) \ge q(c)$, while if $\Phi_A(c) \in \mathbb{R}$ then $\big[2q(c) - \Phi_A(c)\big] \vee \Phi_A(c) \ge \frac{1}{2}\big[2q(c) - \Phi_A(c)\big] + \frac{1}{2}\Phi_A(c) = q(c)$. Thus ${\Phi_A}^{@}(c) \ge \Phi(c) \vee q(c)$. This completes the proof of (b). From (a), for all $b \in B$, ${\Phi_A}^{@@}(b) = \sup_B \big[\lfloor\cdot,b\rfloor - {\Phi_A}^{@}\big] \ge \sup_A \big[\lfloor b,\cdot\rfloor - {\Phi_A}^{@}\big] \ge \sup_A \big[\lfloor b,\cdot\rfloor - q\big] = \Phi_A(b)$. Thus ${\Phi_A}^{@@} \ge \Phi_A$ on B. However, it is obvious that ${\Phi_A}^{@@} \le \Phi_A$ on B, which completes the proof of (c). □

Our next result represents an improvement of the result proved in [20, Lemma 19.12, p. 82], and uses a disguised differentiability argument. Lemma 2.12 will be used in Theorem 2.14, Corollary 3.4, Lemma 4.2 and Theorem 5.8(b). See Remark 2.13 below for another proof of Lemma 2.12(a), due to Constantin Zălinescu.

Lemma 2.12. *Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ be an SSD space, $f \in \mathcal{PC}(B)$ and $f \ge q$ on B. Then:*

(a) $$a \in \mathcal{P}_q(f) \text{ and } b \in B \quad\Longrightarrow\quad \lfloor b,a\rfloor \le q(a) + f(b).$$

(b) $$f^{@} = q \text{ on } \mathcal{P}_q(f).$$

(c) $$\text{If } \mathcal{P}_q(f) \ne \emptyset \quad \text{then} \quad f \ge \Phi_{\mathcal{P}_q(f)} \text{ on } B.$$

Proof. Let $a \in \mathcal{P}_q(f)$ and $b \in B$. Let $\lambda \in]0,1[$. For simplicity in writing, let $\mu := 1-\lambda \in]0,1[$. Then

$$\lambda^2 q(b) + \lambda\mu\lfloor b,a\rfloor + \mu^2 q(a) = q\big(\lambda b + \mu a\big) \le f(\lambda b + \mu a)$$
$$\le \lambda f(b) + \mu f(a) = \lambda f(b) + \mu q(a).$$

Thus $\lambda^2 q(b) + \lambda\mu\lfloor b,a\rfloor \le \lambda f(b) + \lambda\mu q(a)$. We now obtain (a) by dividing by λ and letting $\lambda \to 0$. Now let $a \in \mathcal{P}_q(f)$. From (a), $b \in B \Longrightarrow \lfloor a,b\rfloor - f(b) \le q(a)$, and it follows by taking the supremum over $b \in B$ that $f^@(a) \le q(a)$. On the other hand, $f^@(a) \ge \lfloor a,a\rfloor - f(a) = 2q(a) - q(a) = q(a)$, completing the proof of (b). Finally, let $b \in B$ and $a \in \mathcal{P}_q(f)$. Then, from (a), $f(b) \ge \lfloor b,a\rfloor - q(a)$. Taking the supremum over $a \in \mathcal{P}_q(f)$ and using (2), $f(b) \ge \Phi_{\mathcal{P}_q(f)}(b)$. Thus $f \ge \Phi_{\mathcal{P}_q(f)}$ on B, giving (c). $\square$

Remark 2.13. The author is grateful to Constantin Zălinescu for pointing out to him the following alternative proof of Lemma 2.12(a). From Lemma 2.6, with c replaced by a, $-q(b) + \lfloor b,a\rfloor - q(a) = -q(b-a) \le (f-q)(b)$. Thus $\lfloor b,a\rfloor - q(a) \le f(b)$, as required.

Theorem 2.14. *Let $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ be an SSD space and A be a maximally q–positive subset of B. Then $\Phi_A{}^@ \ge \Phi_A \ge q$ on B and $\mathcal{P}_q\big(\Phi_A{}^@\big) = \mathcal{P}_q\big(\Phi_A\big) = A$.*

Proof. The first assertion follows from Lemma 2.11(b) and (5). It is clear from this and (6) that $\mathcal{P}_q\big(\Phi_A{}^@\big) \subset \mathcal{P}_q\big(\Phi_A\big) = A$. On the other hand, we can apply Lemma 2.12(b) to $f := \Phi_A$ and obtain $\mathcal{P}_q(\Phi_A) \subset \mathcal{P}_q(\Phi_A{}^@)$, which gives the second assertion. $\square$

3. SSD spaces with a linked external space

A word is in order about the *conjugate* of a convex function. If a vector space is paired with itself by a bilinear form, we use the notation $^@$ to denote the conjugate with respect to this pairing. We have already seen an example of this in (7), and we will see another one in (23). If a vector space X is paired with another vector space Y by a bilinear form $\langle\cdot,\cdot\rangle$: $X \times Y \to \mathbb{R}$ and $f \in \mathcal{PC}(X)$, we write f^* for the conjugate of f with respect to this pairing, that is to say, for all $y \in Y$, $f^*(y) := \sup_X \big[\langle\cdot,y\rangle - f\big]$. We will have an example of this in (9): we will come to another case in the first paragraph of Section 7.

We now introduce an important situation, in which an SSD space $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ supports a second duality other than that defined by $\lfloor\cdot,\cdot\rfloor$.

Definition 3.1. Let $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ be an SSD space. We say that $\big(D,\iota,\langle\cdot,\cdot\rangle\big)$ is a *linked external space* if D is a nonzero real vector space, ι: $B \to D$ is a linear map and $\langle\cdot,\cdot\rangle$: $B \times D \to \mathbb{R}$ is a bilinear form such that

$$\text{for all } b,c \in B, \quad \big\langle b,\iota(c)\big\rangle = \lfloor b,c\rfloor. \tag{8}$$

If $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ is an SSD space and ι is the identity map on B then $\big(B,\iota,\lfloor\cdot,\cdot\rfloor\big)$ is a linked external space. We will discuss more interesting examples of totally differing characters in Definition 4.1 and Remark 5.9.

Let $\big(D,\iota,\langle\cdot,\cdot\rangle\big)$ be a linked external space. If $f \in \mathcal{PC}(B)$ and $d \in D$ then we have

$$f^*(d) := \sup_B \big[\langle\cdot,d\rangle - f\big]. \tag{9}$$

It is clear from (9), (8) and (7) that if $f \in \mathcal{PC}(B)$ then

$$f^* \circ \iota = f^{@}. \tag{10}$$

Now let A be a nonempty q–positive subset of B. We then define the function $\Theta_A\colon\ D \to\ ]-\infty, \infty]$ by,

$$\text{for all } d \in D, \quad \Theta_A(d) := \sup\nolimits_A \big[\langle\cdot, d\rangle - q\big] = \sup\nolimits_A \big[\langle\cdot, d\rangle - \Phi_A\big] \tag{11}$$

$\big($the equality of the two expressions follows from (4)$\big)$. It is clear from the first expresson in (11), (8) and (2) that

$$\Theta_A \circ \iota = \Phi_A, \tag{12}$$

and so (4) implies that $\Theta_A \in \mathcal{PC}(D)$. We define the function $\Psi_A\colon\ B \to\ ]-\infty, \infty]$ by

$$\Psi_A := \sup\nolimits_{d\in D} \big[\langle\cdot, d\rangle - \Theta_A(d)\big]. \tag{13}$$

In the following lemma, we collect together the basic properties of the functions Θ_A and Ψ_A. The proof of Lemma 3.2(c) below is due to Radu Ioan Boţ.

Lemma 3.2. *Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ be an SSD space, $\big(D, \iota, \langle\cdot,\cdot\rangle\big)$ be a linked external space and A be a nonempty q–positive subset of B. Then:*

(a) $\Psi_A \le q$ *on A.* (Compare Lemma 2.11(a).)

(b) $\Psi_A \in \mathcal{PC}(B)$.

(c) ${\Phi_A}^* \ge \Theta_A = {\Psi_A}^*$ *on D.* (Compare Lemma 2.11(c).)

(d) $\Phi_A = {\Psi_A}^{@}$.

(e) $\Psi_A \ge {\Phi_A}^{@} \ge q$ *on B.*

(f) $A \subset \mathcal{P}_q(\Psi_A) \subset \mathcal{P}_q\big({\Phi_A}^{@}\big)$.

Proof. (a) Let $d \in D$. Then the first expresson in (11) implies that $\langle\cdot, d\rangle - q \le \Theta_A(d)$ on A and so $\langle\cdot, d\rangle - \Theta_A(d) \le q$ on A. (a) follows by taking the supremum over $d \in D$ and using (13), and (b) is immediate from (a).

(c) We note from (9) and the second expression in (11) that, for all $d \in D$,

$${\Phi_A}^*(d) = \sup\nolimits_B \big[\langle\cdot, d\rangle - \Phi_A\big] \ge \sup\nolimits_A \big[\langle\cdot, d\rangle - \Phi_A\big] = \Theta_A(d),$$

thus ${\Phi_A}^* \ge \Theta_A$ on D. From (a) and the second expression in (12), for all $d \in D$,

$${\Psi_A}^*(d) = \sup\nolimits_B \big[\langle\cdot, d\rangle - \Psi_A\big] \ge \sup\nolimits_A \big[\langle\cdot, d\rangle - \Psi_A\big] \ge \sup\nolimits_A \big[\langle\cdot, d\rangle - q\big] = \Theta_A(d).$$

Thus ${\Psi_A}^* \ge \Theta_A$ on B. However, it is obvious that ${\Psi_A}^* \le \Theta_A$ on B, which completes the proof of (c). (d) follows by composing the equality in (c) with ι and using (12) and (10). (d) implies that ${\Psi_A}^{@@} = {\Phi_A}^{@}$, and since $\Psi_A \ge {\Psi_A}^{@@}$ on B, we obtain the first inequality in (e). The second inequality in (e) follows from Lemma 2.11(b).

(f) is immediate from (a) and (e). $\square$

Our next result, which gives a useful maximal property of Ψ_A, is motivated by results originally proved by Burachik and Svaiter in [1] for maximally monotone multifunctions. We will return to this topic in Theorem 5.11. We note that the argument used in Theorem 3.3 is similar to that already used in Lemma 3.2(c).

Theorem 3.3. *Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ be an SSD space, $\big(D, \iota, \langle\cdot,\cdot\rangle\big)$ be a linked external space, A be a nonempty q–positive subset of B, $f \in \mathcal{PC}(B)$ be $w(B,D)$–lower semicontinuous and $f \le q$ on A. Then $f^* \ge \Theta_A$ on D and Ψ_A is the largest $w(B,D)$–lower semicontinuous element of $\mathcal{PC}(B)$ that is dominated by q on A.*

Proof. From (9) and the first expression in (11), for all $d \in D$,

$$f^*(d) = \sup\nolimits_B \big[\langle\cdot,d\rangle - f\big] \ge \sup\nolimits_A \big[\langle\cdot,d\rangle - f\big] \ge \sup\nolimits_A \big[\langle\cdot,d\rangle - q\big] = \Theta_A(d),$$

thus $f^* \ge \Theta_A$ on D, as required. Since f is $w(B,D)$–lower semicontinuous, using the Fenchel–Moreau theorem for the (possibly nonhausdorff) locally convex space $\big(B, w(B,D)\big)$ (see Theorem 10.1), what we have already proved, and (13),

$$f = \sup\nolimits_{d\in D} \big[\langle\cdot,d\rangle - f^*(d)\big] \le \sup\nolimits_{d\in D} \big[\langle\cdot,d\rangle - \Theta_A(d)\big] = \Psi_A \text{ on } B,$$

thus $f \le \Psi_A$ on B. The result follows from Lemma 3.2(a), since Ψ_A is obviously $w(B,D)$–lower semicontinuous. □

Corollary 3.4. *Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ be an SSD space, $\big(D, \iota, \langle\cdot,\cdot\rangle\big)$ be a linked external space, $f \in \mathcal{PC}(B)$ be $w(B,D)$–lower semicontinuous $f \ge q$ on B, and $A := \mathcal{P}_q(f) \ne \emptyset$. Then*

$$\Psi_A \ge f \ge \Phi_A \text{ on } B \quad\text{and}\quad {\Phi_A}^* \ge f^* \ge \Theta_A \text{ on } D. \tag{14}$$

If, further, A is maximally q–positive then

$$\mathcal{P}_q\big(\Psi_A\big) = \mathcal{P}_q\big({\Phi_A}^{@}\big) = \mathcal{P}_q\big(\Phi_A\big) = A. \tag{15}$$

Proof. It is clear from Lemmas 2.9 and 2.12(c) that A is a q–positive subset of B and $f \ge \Phi_A$ on B, from which ${\Phi_A}^* \ge f^*$ on D. Theorem 3.3 implies that $f^* \ge \Theta_A$ on D and $\Psi_A \ge f$ on B, which completes the proof of (14). If we use Theorem 2.14 to strengthen Lemma 3.2(f), we obtain $A \subset \mathcal{P}_q(\Psi_A) \subset \mathcal{P}_q({\Phi_A}^{@}) = \mathcal{P}_q(\Phi_A) = A$, which completes the proof of (15). □

Our next result is a partial converse to Corollary 3.4. We note that f is *not* required to be $w(B,D)$–lower semicontinuous.

Corollary 3.5. *Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ be an SSD space, $\big(D, \iota, \langle\cdot,\cdot\rangle\big)$ be a linked external space, A be a maximally q–positive subset of B, $f \in \mathcal{PC}(B)$ and*

$$\Psi_A \ge f \ge \Phi_A \text{ on } B. \tag{16}$$

Then $f \ge q$ on B, $f^{@} \ge q$ on B and

$$\mathcal{P}_q(f) = \mathcal{P}_q\big(f^{@}\big) = A = \mathcal{P}_q\big(\Psi_A\big) = \mathcal{P}_q\big({\Phi_A}^{@}\big) = \mathcal{P}_q\big(\Phi_A\big).$$

Proof. From (16) and (5), $\Psi_A \ge f \ge \Phi_A \ge q$ on B. Thus, using Lemma 3.2(f),

$$A \subset \mathcal{P}_q(\Psi_A) \subset \mathcal{P}_q(f) \subset \mathcal{P}_q(\Phi_A). \tag{17}$$

Taking conjugates in (16) and using Lemma 3.2(d) and (5), ${\Phi_A}^{@} \ge f^{@} \ge {\Psi_A}^{@} = \Phi_A \ge q$ on B. Thus, using Lemma 3.2(f),

$$A \subset \mathcal{P}_q\big({\Phi_A}^{@}\big) \subset \mathcal{P}_q\big(f^{@}\big) \subset \mathcal{P}_q(\Phi_A). \tag{18}$$

The result now follows from (17), (18) and Theorem 2.14. □

4. SSD–homomorphisms and the Gossez extension

The main result in this section is Theorem 4.5, in which we extend to SSD spaces a concept originally due to Gossez for maximally monotone multifunctions.

Definition 4.1. Let $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ and $\big(D,\lceil\cdot,\cdot\rceil\big)$ be SSD spaces. In this case, we will *always* write $\widetilde q(d) := \frac12\lceil d,d\rceil \quad (d \in D)$. We say that $\iota\colon B \to D$ is a *SSD–homomorphism* if ι is linear and,

$$\text{for all } b,c \in B,\qquad \big\lceil \iota(b),\iota(c)\big\rceil = \lfloor b,c\rfloor, \tag{19}$$

from which

$$\widetilde q\circ\iota = q. \tag{20}$$

Let $\iota\colon B \to D$ be an SSD–homomorhism. We define the bilinear map $\langle\cdot,\cdot\rangle_\iota\colon B\times D \to \mathbb{R}$ by

$$\langle b,d\rangle_\iota := \lceil \iota(b),d\rceil \qquad \big((b,d)\in B\times D\big). \tag{21}$$

We note then from (21) and (19) that if $b,c \in B$ then $\big\langle b,\iota(c)\big\rangle_\iota = \big\lceil\iota(b),\iota(c)\big\rceil = \lfloor b,c\rfloor$, and so $\big(D,\iota,\langle\cdot,\cdot\rangle_\iota\big)$ is a linked external space (see Definition 3.1). If C is a nonempty $\widetilde q$–positive subset of D then, as in (2), (3) and (4),

$$\Phi_C(d) := \sup\nolimits_C \big[\lceil\cdot,d\rceil - \widetilde q\big] = q(d) - \inf \widetilde q(C-d) \quad (d\in D) \quad \text{and} \quad \Phi_C = \widetilde q \text{ on } C. \tag{22}$$

If $f \in \mathcal{PC}(D)$ and $d \in D$ then, as in (7),

$$f^{@}(d) := \sup\nolimits_D \big[\lceil\cdot,d\rceil - f\big]. \tag{23}$$

The next two lemmas contain the preliminary results that we will need. We obtain Lemma 4.2 by transcribing Lemmas 2.9, 2.12(b) and 2.11(b,c) to our present situation.

Lemma 4.2.(a) *Let $f \in \mathcal{PC}(D)$, $f \ge \widetilde q$ on D and $\mathcal{P}_{\widetilde q}(f) \ne \emptyset$. Then $\mathcal{P}_{\widetilde q}(f)$ is a $\widetilde q$–positive subset of D and $f^{@} = \widetilde q$ on $\mathcal{P}_{\widetilde q}(f)$.*

(b) *If C is a nonempty $\widetilde q$–positive subset of D then $\Phi_C{}^{@} \ge \Phi_C$ and $\Phi_C{}^{@@} = \Phi_C$ on D.*

Lemma 4.3. *Let $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ and $\big(D,\lceil\cdot,\cdot\rceil\big)$ be SSD spaces, $\iota\colon B \to D$ be an SSD–homomorphism and A be a nonempty q–positive subset of B. Then:*

(a) *$\iota(A)$ is a nonempty $\widetilde q$–positive subset of D and $\Phi_{\iota(A)} \in \mathcal{PC}(D)$.*

(b) *For all $d\in D$, $\Theta_A(d) = \Phi_{\iota(A)}(d) = \widetilde q(d) - \inf \widetilde q\big(\iota(A)-d\big)$.*

(c) *$\Theta_A{}^{@} \ge \Phi_A{}^{*} \ge \Theta_A$ on D and $\Theta_A{}^{@} \ge \Phi_A{}^{*@} \ge \Theta_A$ on D.*

Proof. If $b,c \in A$ then (20) gives $\widetilde q\big(\iota(b)-\iota(c)\big) = \widetilde q\circ\iota(b-c) = q(b-c) \ge 0$, and (a) follows from (22). If $d \in D$ then the first expression in (11), (21) and (20) give $\Theta_A(d) = \sup_A \big[\langle\cdot,d\rangle_\iota - q\big] = \sup_A \big[\lceil\iota(\cdot),d\rceil - \widetilde q\circ\iota\big] = \sup_{\iota(A)}\big[\lceil\cdot,d\rceil - \widetilde q\big]$, and (b) follows from (22).

Of course, it is immediate from (b) and Lemma 4.2(b) that $\Theta_A{}^{@} = \Phi_{\iota(A)}{}^{@} \ge \Phi_{\iota(A)} = \Theta_A$ on D, but for (c) we need the extra information about $\Phi_A{}^*$ and $\Phi_A{}^{*@}$. So let $d \in D$. Then, from (23), (21), (12), (9) and the second expression in (11),

$$\begin{aligned}\Theta_A{}^{@}(d) &= \sup\nolimits_D \big[\lceil\cdot, d\rceil - \Theta_A\big] \ge \sup\nolimits_B \big[\lceil\iota(\cdot), d\rceil - \Theta_A \circ \iota\big] = \sup\nolimits_B \big[\langle\cdot, d\rangle_\iota - \Phi_A\big]\\ &= \Phi_A{}^*(d) = \sup\nolimits_B \big[\langle\cdot, d\rangle_\iota - \Phi_A\big] \ge \sup\nolimits_A \big[\langle\cdot, d\rangle_\iota - \Phi_A\big] = \Theta_A(d).\end{aligned}$$

This completes the proof of the first assertion in (c). It is clear from this, (b), and Lemma 4.2(b) that $\Theta_A{}^{@} \ge \Phi_A{}^{*@} \ge \Theta_A{}^{@@} = \Phi_{\iota(A)}{}^{@@} = \Phi_{\iota(A)} = \Theta_A$ on D, which gives the second assertion in (c). $\square$

The next concept is a generalization to SSD spaces of an idea originally introduced by Gossez in [3] for maximally monotone multifunctions. The use of the word "extension" in Definition 4.4 will be justified in Theorem 4.5(a).

Definition 4.4. Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ and $\big(D, \lceil\cdot,\cdot\rceil\big)$ be SSD spaces, $\iota\colon B \to D$ be an SSD–homomorphism and A be a nonempty q–positive subset of B. It is clear from Lemma 4.3(b) that if $d \in D$ then $\Theta_A(d) \le \widetilde{q}(d) \iff \inf \widetilde{q}\big(\iota(A) - d\big) \ge 0$. We define the *Gossez extension* of A in D to be the set

$$A^{\mathcal{G}} = \big\{d \in D\colon \inf \widetilde{q}\big(\iota(A) - d\big) \ge 0\big\} = \big\{d \in D\colon \Theta_A(d) \le \widetilde{q}(d)\big\}. \tag{24}$$

Theorem 4.5 will be used in Theorem 6.15, which will be used, in turn, in Theorem 9.5, where we prove that for a maximally monotone set, type (ED), dense type, type (D), type (WD) and type (NI) are all equivalent. Theorem 9.5 will be used in Theorems 9.9 and 9.10.

Theorem 4.5. *Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ and $\big(D, \lceil\cdot,\cdot\rceil\big)$ be SSD spaces, $\iota\colon B \to D$ be an SSD–homomorphism and A be a nonempty q–positive subset of B. Then:*

(a) $\iota(A) \subset A^{\mathcal{G}}$.

(b) *If, for all $d \in A^{\mathcal{G}}$, $\inf \widetilde{q}\big(\iota(A) - d\big) \le 0$ then*

$$\Theta_A \ge \widetilde{q} \text{ on } D. \tag{25}$$

(c) *If (25) is satisfied then $A^{\mathcal{G}} = \mathcal{P}_{\tilde{q}}(\Phi_A{}^{*@}) = \mathcal{P}_{\tilde{q}}(\Phi_A{}^*) = \mathcal{P}_{\tilde{q}}(\Theta_A{}^{@}) = \mathcal{P}_{\tilde{q}}(\Theta_A)$.*

Proof. If $a \in A$ then, from (12), (4) and (20), $\Theta_A\big(\iota(a)\big) = \Phi_A(a) = q(a) = \widetilde{q} \circ \iota(a) = \widetilde{q}\big(\iota(a)\big)$, and so the second expression in (24) implies that $\iota(a) \in A^{\mathcal{G}}$. This gives (a). (b) is immediate from Lemma 4.3(b). Suppose, finally, that (25) is satisfied. It is obvious from (a) and the second expression in (24) that $\emptyset \ne A^{\mathcal{G}} = \mathcal{P}_{\tilde{q}}\big(\Theta_A\big)$ and, from Lemma 4.3(c), that

$$\mathcal{P}_{\tilde{q}}(\Theta_A{}^{@}) \subset \mathcal{P}_{\tilde{q}}(\Phi_A{}^*) \subset \mathcal{P}_{\tilde{q}}(\Theta_A) \quad\text{and}\quad \mathcal{P}_{\tilde{q}}(\Theta_A{}^{@}) \subset \mathcal{P}_{\tilde{q}}(\Phi_A{}^{*@}) \subset \mathcal{P}_{\tilde{q}}(\Theta_A).$$

(c) follows since Lemma 4.2(a) with $f := \Theta_A$ gives $\mathcal{P}_{\tilde{q}}(\Theta_A) \subset \mathcal{P}_{\tilde{q}}(\Theta_A{}^{@})$. $\square$

5. VZ functions on Banach SSD spaces

We note that we do not use anything from Section 4 in this section — we will combine this section with Section 4 in Section 6. The other comment is that the dual of a Banach space is not mentioned explicitly until Remark 5.9, though there is an implicit use of the dual in Theorem 5.6(b) (in the observation that a proper convex lower semicontinuous function dominates a continuous affine function). If X is a nonzero real Banach space, we write $\mathcal{PCLSC}(X)$ for the set

$$\{f \in \mathcal{PC}(X)\colon\ f \text{ is lower semicontinuous on } X\}.$$

Definition 5.1. We say that $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ is a *Banach SSD space* if $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ is an SSD space and $\|\cdot\|$ is a norm on B with respect to which B is a Banach space and

$$\text{for all } b, c \in B, \quad \big|\lfloor b, c\rfloor\big| \le \|b\|\|c\|. \tag{26}$$

If we take $c = b$, we derive that

$$\tfrac{1}{2}\|\cdot\|^2 + q \ge 0 \text{ on } B. \tag{27}$$

Then, for all $d, e \in B$,

$$|q(d) - q(e)| = \tfrac{1}{2}\big|\lfloor d, d\rfloor - \lfloor e, e\rfloor\big| = \tfrac{1}{2}\big|\lfloor d - e, d + e\rfloor\big| \le \tfrac{1}{2}\|d - e\|\|d + e\|. \tag{28}$$

We define the continuous even functions g and p on B by $g := \frac{1}{2}\|\cdot\|^2$ and $p := g + q$, so that $\ p \ge 0$ on B. Since $p(0) = 0$, in fact

$$\inf\nolimits_B p = 0. \tag{29}$$

Also, for all $d, e \in B$, $|g(d) - g(e)| = \frac{1}{2}\big|\|d\| - \|e\|\big|\big(\|d\| + \|e\|\big) \le \frac{1}{2}\|d - e\|\big(\|d\| + \|e\|\big)$. Combining this with (28), for all $d, e \in B$,

$$|p(d) - p(e)| \le \|d - e\|\big(\|d\| + \|e\|\big). \tag{30}$$

Remark 5.2. It is clear from (26) that Example 2.3 is a Banach SSD space if, and only if $\|T\| \le 1$, which is the case with (a), (b) and (c). Looking ahead to Remark 5.9, it then follows from (43) that $\iota = T$.

Example 5.3. We now continue our discussion of Example 2.4. We suppose that E is a nonzero Banach space, $B = E \times E^*$ and, for all $(x, x^*) \in B$, $\|(x, x^*)\|_2 := \sqrt{\|x\|^2 + \|x^*\|^2}$. It is clear from the Cauchy-Schwarz inequality that (26) is satisfied, and so $\big(E \times E^*, \lfloor\cdot,\cdot\rfloor, \|\cdot\|_2\big)$ is a Banach SSD space.

We now introduce the concept of inf–convolution. This certainly goes back as far as [13], but we emphasize that we will be using it here for nonconvex functions.

Definition 5.4. Let X be a vector space and $h, k\colon X \to]-\infty,\infty]$. The *inf–convolution of h and k* is defined by $\quad (h \mathbin{\nabla} k)(x) := \inf_{y\in X}\big[h(y) + k(x-y)\big] \quad (x \in X)$. It is clear that

$$\inf\nolimits_X k = 0 \quad\Longrightarrow\quad \inf\nolimits_X \big[h \mathbin{\nabla} k\big] = \inf\nolimits_X h. \tag{31}$$

Now let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space and $f \in \mathcal{PC}(B)$. We say that f is a *VZ function* if

$$(f - q) \mathbin{\nabla} p = 0 \text{ on } B. \tag{32}$$

It follows from (29) and (31) that

$$\text{if } f \text{ is a VZ function then } \inf\nolimits_B[f-q] = 0. \tag{33}$$

"VZ" stands for "Voisei–Zălinescu", since (32) is an extension to Banach SSD spaces of a condition introduced in [22, Proposition 3]. The following simple inequality will be useful: suppose that $f \in \mathcal{PC}(B)$ and $\quad f \ge q$ on B; $\quad$ then, for all $c \in B$,

$$\left.\begin{aligned}\big((f-q)\mathbin{\nabla} p\big)(c) &= \inf\nolimits_{b\in B}\big[(f-q)(b) + p(c-b)\big]\\ &\le \inf\nolimits_{a\in\mathcal{P}_q(f)}\big[(f-q)(a) + p(c-a)\big]\\ &= \inf p\big(c - \mathcal{P}_q(f)\big) = \inf p\big(\mathcal{P}_q(f) - c\big).\end{aligned}\right\} \tag{34}$$

Definition 5.5. Let A be a subset of a Banach SSD space $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$. We say that A is *p–dense in B* if, for all $c \in B$, $\inf p(A - c) = 0$.

We now come to our main results on VZ functions on Banach SSD spaces. We shall see in Remark 5.12 that the constant $\sqrt{2}$ in (36) is sharp, and also that (36) leads to a strict strengthening of (35). If we take Theorem 5.6(a) into account then a double induction is used to prove Theorem 5.6(c). We do not know if this is actually necessary. Theorem 5.6(d) is an extension to Banach SSD spaces of [22, Theorem 8]. Theorem 5.6 is related to some results proved by Zagrodny in [23]. These are discussed more fully in the comments preceding Problem 9.8.

Theorem 5.6. *Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space and $f \in \mathcal{PCLSC}(B)$ be a VZ function. Then:*

(a) *$\mathcal{P}_q(f)$ is a q–positive subset of B and*

$$d \in \operatorname{dom} f \quad\Longrightarrow\quad \operatorname{dist}(d, \mathcal{P}_q(f)) \le 5\sqrt{(f-q)(d)}. \tag{35}$$

(b) *$\mathcal{P}_q(f)$ is p–dense in B.*

(c) *For all $c \in B$, $\inf q\big(\mathcal{P}_q(f) - c\big) \le 0$ and*

$$c \in B \quad\Longrightarrow\quad \operatorname{dist}\big(c, \mathcal{P}_q(f)\big) \le \sqrt{2}\sqrt{-\inf q\big(\mathcal{P}_q(f) - c\big)}. \tag{36}$$

(d) *$\mathcal{P}_q(f)$ is a maximally q–positive subset of B.*

Proof. (a) (33) implies that $f \ge q$ on B, and so $\mathcal{P}_q(f)$ is defined. Let $d \in \operatorname{dom} f$. We first prove that there exists a Cauchy sequence $\{b_n\}_{n\ge 1}$ such that, for all $n \ge 1$,

$$(f-q)(b_n) \le (f-q)(d)/4^n \quad \text{and} \quad \|d-b_n\| \le 5\sqrt{(f-q)(d)}. \tag{37}$$

Since we can take $b_n = d$ if $(f-q)(d) = 0$, we can and will suppose that

$$\alpha := \sqrt{(f-q)(d)} > 0. \tag{38}$$

Write $b_0 := d$. Then we can choose inductively $b_1, b_2, \ldots \in B$ $\big($using the fact that $\big((f-q)\,\nabla\, p\big)(b_{n-1}) = 0\big)$ such that, for all $n \ge 1$, $(f-q)(b_n) + p(b_{n-1}-b_n) \le (\alpha/2^n)^2$. It follows from (33), (38) and (29) that,

$$\text{for all } n \ge 1, \qquad p(b_{n-1}-b_n) \le (\alpha/2^n)^2, \tag{39}$$

and

$$\text{for all } n \ge 0, \qquad (f-q)(b_n) \le (\alpha/2^n)^2. \tag{40}$$

Substituting this into Lemma 2.6, for all $n \ge 1$,

$$-q(b_{n-1}-b_n) \le \Big[\sqrt{(f-q)(b_{n-1})} + \sqrt{(f-q)(b_n)}\Big]^2 \le [\alpha/2^{n-1} + \alpha/2^n]^2 = 9(\alpha/2^n)^2.$$

Consequently, since $g(b_{n-1}-b_n) = p(b_{n-1}-b_n) - q(b_{n-1}-b_n)$, (39) gives,

$$\text{for all } n \ge 1, \qquad g(b_{n-1}-b_n) \le (\alpha/2^n)^2 + 9(\alpha/2^n)^2 = 10(\alpha/2^n)^2,$$

and so, for all $n \ge 1$, $\|b_{n-1}-b_n\| \le 5\alpha/2^n$. Adding up this inequality for $n = 1, \ldots, m$ and using (38), we derive that, for all $m \ge 1$, $\|d-b_m\| \le 5\alpha = 5\sqrt{(f-q)(d)}$, and (37) now follows from (40). Now set $a = \lim_n b_n$, so that $\|d-a\| \le 5\sqrt{(f-q)(d)}$. (37) and the lower semicontinuity of $f-q$ now imply that $(f-q)(a) \le 0$, that is to say, $a \in \mathcal{P}_q(f)$. Since $\operatorname{dom} f \ne \emptyset$, it follows that $\mathcal{P}_q(f) \ne \emptyset$ and so, from Lemma 2.9, $\mathcal{P}_q(f)$ is a q–positive subset of B, and obviously (35) is satisfied. This completes the proof of (a).

(b) Let $c \in B$. Since $\big((f-q)\,\nabla\, p\big)(c) = 0$, we can choose inductively $d_1, d_2, \ldots \in B$ such that, for all $n \ge 1$,

$$f(d_n) + g(c-d_n) + q(c) - \lfloor c, d_n \rfloor = (f-q)(d_n) + p(c-d_n) < 1/n^2.$$

Consequently, using (29), (33) and (26), for all $n \ge 1$,

$$(f-q)(d_n) < 1/n^2,\ p(c-d_n) < 1/n^2 \tag{41}$$

and

$$f(d_n) + g(c-d_n) + q(c) - \|c\|\|d_n\| < 1/n^2. \tag{42}$$

Since $f \in \mathcal{PCLSC}(B)$, f dominates a continuous affine function, and so (42) and the usual coercivity argument imply that $K := \sup_{n\ge 1}\|d_n\| < \infty$. From (a) and (41), for all $n \ge 1$, there exists $a_n \in \mathcal{P}_q(f)$ such that $\|a_n - d_n\| \le 5/n$. Now, from (30), for all $n \ge 1$,

$$\begin{aligned}|p(c-a_n) - p(c-d_n)| &\le \|a_n - d_n\|(2\|c\| + \|a_n\| + \|d_n\|)\\ &\le \big(2\|c\| + (K+5) + K\big)5/n.\end{aligned}$$

Thus $\lim_{n\to\infty}\big[p(c-a_n) - p(c-d_n)\big] = 0$, and (b) follows by combining this with (41).

(c) Let $c \in B$. Then, from (b),

$$\inf g\big(\mathcal{P}_q(f)-c\big)+\inf q\big(\mathcal{P}_q(f)-c\big)\le \inf(g+q)\big(\mathcal{P}_q(f)-c\big)=\inf p\big(\mathcal{P}_q(f)-c\big)=0.$$

Thus $\frac{1}{2}\operatorname{dist}\big(c,\mathcal{P}_q(f)\big)^2=\inf g\big(\mathcal{P}_q(f)-c\big)\le-\inf q\big(\mathcal{P}_q(f)-c\big)$, from which (36) is an immediate consequence.

(d) We suppose that $c \in B$ and $\inf q\big(\mathcal{P}_q(f)-c\big)\ge 0$, and we must prove that $c \in \mathcal{P}_q(f)$. From (c), in fact $\inf q\big(\mathcal{P}_q(f)-c\big)=0$ and $\operatorname{dist}\big(c,\mathcal{P}_q(f)\big)=0$. The lower semicontinuity of f implies that $\mathcal{P}_q(f)$ is closed, and so $c \in \mathcal{P}_q(f)$. This completes the proof of (d). $\square$

The interest of Theorem 5.7 below is that it tells us that we can determine whether f is a VZ function by simply inspecting $\mathcal{P}_q(f)$.

Theorem 5.7. *Let $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ be a Banach SSD space and $f \in \mathcal{PCLSC}(B)$. Then f is a VZ function $\iff$ $f \ge q$ on B and $\mathcal{P}_q(f)$ is p–dense in B.*

Proof. ($\Longrightarrow$) is immediate from (33) and Theorem 5.6(b). Suppose, conversely, that $f \ge q$ on B and $\mathcal{P}_q(f)$ is p–dense in B. Then from (34), for all $c \in B$, $\big((f-q)\,\nabla\,p\big)(c)\le \inf p\big(\mathcal{P}_q(f)-c\big)=0$, from which $(f-q)\,\nabla\,p\le 0$ on B. On the other hand, since $f-q\ge 0$ on B and, from (29), $p \ge 0$ on B, we have $(f-q)\,\nabla\,p\ge 0$ on B. Thus f is a VZ function, as required. $\square$

We point out that the function h in Theorem 5.8(a) is not required to be lower semicontinuous, so we cannot simply apply Theorem 5.7 with f replaced by h.

Theorem 5.8. *Let $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$. be a Banach SSD space and $f \in \mathcal{PCLSC}(B)$ be a VZ function. Then:*

(a) *Let $h \in \mathcal{PC}(B)$, $h \ge q$ on B, and $\mathcal{P}_q(h)\supset\mathcal{P}_q(f)$. Then $\mathcal{P}_q(h)=\mathcal{P}_q(f)$ and h is a VZ function.*

(b) *$f^@ \in \mathcal{PCLSC}(B)$, $f^@$ is a VZ function and $\mathcal{P}_q\big(f^@\big)=\mathcal{P}_q(f)$.*

Proof. (a) It is clear from Theorem 5.6(d) that $\mathcal{P}_q(h)=\mathcal{P}_q(f)$. From (34) and Theorem 5.6(b), for all $c \in B$, $\big((h-q)\,\nabla\,p\big)(c)\le\inf p\big(\mathcal{P}_q(h)-c\big)=\inf p\big(\mathcal{P}_q(f)-c\big)=0$, from which $(h-q)\,\nabla\,p\le 0$ on B. On the other hand, since $h-q\ge 0$ on B and $p\ge 0$ on B, we have $(h-q)\,\nabla\,p\ge 0$ on B. Thus h is a VZ function.

(b) Let $c \in B$. Then, since $q \le p$ on B, Definition 2.10 gives

$$q(c)-f^@(c)=\inf_{b\in B}\big[f(b)-\lfloor b,c\rfloor+q(c)\big]=\big((f-q)\,\nabla\,q\big)(c)\le\big((f-q)\,\nabla\,p\big)(c)=0,$$

and so $f^@\ge q$ on B. It now follows from Lemma 2.12(b) that $\mathcal{P}_q\big(f^@\big)\supset\mathcal{P}_q(f)$, and so (a) implies that $f^@$ is a VZ function and $\mathcal{P}_q\big(f^@\big)=\mathcal{P}_q(f)$. Since $\mathcal{P}_q(f)\ne\emptyset$, it is evident that $f^@\in\mathcal{PCLSC}(B)$. $\square$

Remark 5.9. Up to this point, we have not mentioned the Banach space dual, B^*, of B. It is easy to see from (26) and standard algebraic arguments that there exists a linear map $\iota\colon\ B \to B^*$ such that $\|\iota\| \le 1$ and

$$\text{for all } b, c \in B, \quad \big\langle b, \iota(c)\big\rangle = \lfloor b, c\rfloor. \tag{43}$$

It follows that $\big(B^*, \iota, \langle\cdot,\cdot\rangle\big)$ is a linked external space (see Definition 3.1) and, if A is a nonempty q–positive subset of B, we define $\Theta_A \in \mathcal{PC}(B^*)$ and $\Psi_A \in \mathcal{PC}(B)$ by (11) and (13), with D replaced by B^*.

The proof of Theorem 5.8 relies heavily on the lower semicontinuity of f. We will show in Corollary 5.10 below that part of Theorem 5.8(b) can be recovered even if f is not assumed to be lower semicontinuous.

Corollary 5.10. *Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space and $f \in \mathcal{PC}(B)$ be a VZ function. Then $f^@ \in \mathcal{PCLSC}(B)$, $f^@$ is a VZ function and $\mathcal{P}_q\big(f^@\big)$ is a maximally q–positive subset of B.*

Proof. Let $\overline{f}$ be the lower semicontinuous envelope of f. Since q is continuous and $f \ge q$ on B, it follows that $f \ge \overline{f} \ge q$ on B. Thus, from (29),

$$0 = (f - q) \mathbin{\nabla} p \ge (\overline{f} - q) \mathbin{\nabla} p \ge 0 \mathbin{\nabla} p = 0 \text{ on } B,$$

and so $\overline{f}$ is a VZ function. Since $\overline{f} \in \mathcal{PCLSC}(B)$, Theorem 5.8(b) implies that $\overline{f}^@$ is a VZ function also. It is well known that $\overline{f}^* = f^*$ on B^* thus, composing with ι and using (10), $\overline{f}^@ = f^@$ on B. The result now follows from Theorem 5.6(d), with f replaced by $f^@$. $\square$

Theorem 5.11. *Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space, $f \in \mathcal{PCLSC}(B)$ be a VZ function and $A := \mathcal{P}_q(f)$. Then*

$$\Psi_A \ge f \ge \Phi_A \ge q \text{ on } B \quad\text{and}\quad {\Phi_A}^* \ge f^* \ge \Theta_A \text{ on } B^*, \tag{44}$$

$$\mathcal{P}_q\big(\Psi_A\big) = \mathcal{P}_q\big({\Phi_A}^@\big) = \mathcal{P}_q\big(\Phi_A\big) = A, \tag{45}$$

and

$$\Phi_A, {\Phi_A}^@ \text{ and } \Psi_A \text{ are all VZ functions.} \tag{46}$$

Now let $h \in \mathcal{PC}(B)$ and $\Psi_A \ge h \ge \Phi_A$ on B. Then h and $h^@$ are VZ functions.

Proof. We first note from (33) and Theorem 5.6(d) that $f \ge q$ on B and A is a maximally q–positive subset of B. From standard normed space theory, f is $w(B, B^*)$–lower semicontinuous and so (44) and (45) follow from Corollary 3.4 and (5). The first assertion in (44), (45) and Theorem 5.7 imply that Ψ_A and Φ_A are VZ functions, and (46) now follows from Theorem 5.8(b), with f replaced by Φ_A. The assertions about h and $h^@$ follow from Theorem 5.8(a) and Corollary 5.10. $\square$

Remark 5.12. Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space and $f \in \mathcal{PCLSC}(B)$ be a VZ function. We know from Theorem 5.11 that $\mathcal{P}_q\big(\Phi_{\mathcal{P}_q(f)}\big) = \mathcal{P}_q(f)$, $\Phi_{\mathcal{P}_q(f)}$ is a VZ function and $\Phi_{\mathcal{P}_q(f)} \le f$ on B. Combining this with (3), for all $c \in B$,

$$-\inf q\big(\mathcal{P}_q(f) - c\big) = \big(\Phi_{\mathcal{P}_q(f)} - q\big)(c) \le (f - q)(c). \tag{47}$$

Thus Theorem 5.6(c) implies that, for all $c \in B$,

$$\operatorname{dist}(c, \mathcal{P}_q(f)) = \operatorname{dist}\big(c, \mathcal{P}_q\big(\Phi_{\mathcal{P}_q(f)}\big)\big) \le \sqrt{2}\sqrt{\big(\Phi_{\mathcal{P}_q(f)} - q\big)(c)} \le \sqrt{2}\sqrt{(f - q)(c)}. \tag{48}$$

This shows that Theorem 5.6(c) is stronger than Theorem 5.6(a). Now consider the Banach SSD space $\big(\mathbb{R}\times\mathbb{R}, \lfloor\cdot,\cdot\rfloor, \|\cdot\|_2\big)$, where the notation is as in Examples 5.3. Define $f \in \mathcal{PCLSC}(B)$ by $f(x_1,x_2) := \frac{1}{2}(x_1^2 + x_2^2)$. Then $(f - q)(x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2) - x_1x_2 = \frac{1}{2}(x_1 - x_2)^2$ and $p(x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2) + x_1x_2 = \frac{1}{2}(x_1 + x_2)^2$. Let $c := (z_1,z_2) \in B$ and $b := \big(\frac{1}{2}(z_1 + z_2), \frac{1}{2}(z_1 + z_2)\big) \in B$. Then $(f - q)(b) = 0$ and $p(c - b) = 0$. Consequently, f is a VZ function. Now $\mathcal{P}_q(f)$ is the diagonal of $\mathbb{R}^2$ and so, by direct computation, for all $c = (x_1,x_2) \in \mathbb{R}^2$, $-\inf q\big(\mathcal{P}_q(f) - c\big) = \frac{1}{4}(x_1 - x_2)^2$. Since $\frac{1}{4}(x_1 - x_2)^2 < \frac{1}{2}(x_1 - x_2)^2$ when $x_1 \ne x_2$, the inequality in (47) is generally strict.

Now let $h := \Phi_{\mathcal{P}_q(f)}$. (3) gives us that, for all $(x_1,x_2) \in B$,

$$\sqrt{(h - q)(x_1,x_2)} = \sqrt{\tfrac{1}{4}(x_1 - x_2)^2} = \tfrac{1}{2}|x_1 - x_2|.$$

On the other hand, by direct computation, $\operatorname{dist}\big((x_1,x_2), \mathcal{P}_q(h)\big) = \frac{1}{\sqrt{2}}|x_1 - x_2|$. Thus the constant $\sqrt{2}$ in the inequalities in (48) is sharp. The genesis of this argument and example can be found in the results of Martínez-Legaz and Théra in [8].

Remark 5.13. The following result follows by applying the inequality between the first and last terms in (48) to Example 5.3: *Let E be a nonzero Banach space, and f be a lower semicontinuous VZ function on $E \times E^*$. Then, for all $(x,x^*) \in E \times E^*$,*

$$\inf\nolimits_{(y,y^*)\in\mathcal{P}_q(f)} \sqrt{\|y - x\|^2 + \|y^* - x^*\|^2} \le \sqrt{2}\sqrt{f(x,x^*) - \langle x,x^*\rangle}.$$

This strengthens the result proved in [22, Theorem 4], namely that

$$\inf\nolimits_{(y,y^*)\in\mathcal{P}_q(f)} \sqrt{\|y - x\|^2 + \|y^* - x^*\|^2} \le 2\sqrt{f(x,x^*) - \langle x,x^*\rangle}.$$

As we observed in Remark 5.12, the constant $\sqrt{2}$ is sharp.

Remark 5.14. We note that the inequalities for B in (44) have four functions, while the inequality for B^* has only three. The reason for this is that we do not have a function on B^* that plays the role that the function q plays on B. The function $\widetilde{q}$, which plays such a role, will be introduced in this context in Definition 6.1.

6. Banach SSD duals

If X is a nonzero real Banach space, we write X^{**} for the bidual of X $\big($with the pairing $\langle\cdot,\cdot\rangle\colon X^*\times X^{**}\to\mathbb{R}\big)$. If $x\in X$, we write $\widehat{x}$ for the canonical image of x in X^{**}, that is to say

$$x\in X \text{ and } x^*\in X^* \quad\Longrightarrow\quad \langle x^*,\widehat{x}\rangle=\langle x,x^*\rangle.$$

Definition 6.1. Let $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ be a Banach SSD space, $(B^*,\|\cdot\|)$ be the Banach space dual of B and the bounded linear map $\iota\colon B\to B^*$ be defined as in (43). Let $\big(B^*,\lceil\cdot,\cdot\rceil,\|\cdot\|\big)$ also be a Banach SSD space. We say that $\big(B^*,\lceil\cdot,\cdot\rceil,\|\cdot\|\big)$ is a *Banach SSD dual* of $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ if $\langle\cdot,\cdot\rangle_\iota=\langle\cdot,\cdot\rangle$ on $B\times B^*$ $\big($see (21)$\big)$, that is to say

$$\text{for all } b\in B \text{ and } c^*\in B^*,\qquad \big\lceil\iota(b),c^*\big\rceil=\langle b,c^*\rangle. \tag{49}$$

We have not required explicitly that ι be an SSD-homomorphism from $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ into $\big(B^*,\lceil\cdot,\cdot\rceil\big)$ $\big($see (19)$\big)$: this is automatically satisfied since (49) and (43) imply that, for all $b,c\in B$, $\big\lceil\iota(b),\iota(c)\big\rceil=\big\langle b,\iota(c)\big\rangle=\lfloor b,c\rfloor$.

Thus if $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ is a Banach SSD space with Banach SSD dual $\big(B^*,\lceil\cdot,\cdot\rceil,\|\cdot\|\big)$, we can use all the results of Sections 3 and 4 (with "D" replaced by "B^*") and Section 5.

By analogy with (43), we define the bounded linear map $\widetilde{\iota}\colon B^*\to B^{**}$ so that

$$\text{for all } c^*,b^*\in B^*,\qquad \big\langle c^*,\widetilde{\iota}(b^*)\big\rangle=\lceil c^*,b^*\rceil. \tag{50}$$

and the function $\widetilde{p}$: $B^*\to\mathbb{R}$ by $\widetilde{p}:=\frac12\|\cdot\|^2+\widetilde{q}$. Thus we have

$$\widetilde{p}\ge 0 \text{ on } B^*. \tag{51}$$

We now show that Definition 6.1 also leads to an automatic factorization of the canonical map from B into B^{**}. Lemma 6.2 will be used in Lemma 8.2.

Lemma 6.2. *Let $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ be a Banach SSD space with Banach SSD dual $\big(B^*,\lceil\cdot,\cdot\rceil,\|\cdot\|\big)$. Then, for all $b\in B$, $\widehat{b}=\widetilde{\iota}\circ\iota(b)$.*

Proof. Let $b\in B$ and $c^*\in B^*$. Then, from the definition of $\widehat{b}$, (49) and (50),

$$\big\langle c^*,\widehat{b}\big\rangle=\langle b,c^*\rangle=\big\lceil\iota(b),c^*\big\rceil=\big\lceil c^*,\iota(b)\big\rceil=\big\langle c^*,\widetilde{\iota}\circ\iota(b)\big\rangle.$$

This gives the required result. $\square$

Remark 6.3. In this remark, we suppose that the notation is as in Example 2.3, and that $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ is a Banach SSD space with Banach SSD dual $\big(B,\lceil\cdot,\cdot\rceil,\|\cdot\|\big)$. We shall show that $\lfloor\cdot,\cdot\rfloor=\lceil\cdot,\cdot\rceil$ on $B\times B$.

We know already from Remarks 5.2 and 5.9 that $\iota=T$ and $\|\iota\|\le 1$. We write I_B for the identity map on B.

It is clear from Lemma 6.2 that $\widetilde{\iota}\circ\iota=I_B$. Now from (50), for all $b,c\in B$, $\big\langle\widetilde{\iota}(b),c\big\rangle=\big\langle c,\widetilde{\iota}(b)\big\rangle=\lceil c,b\rceil=\lceil b,c\rceil=\big\langle b,\widetilde{\iota}(c)\big\rangle$, so $\widetilde{\iota}$ is self–adjoint. For all $b\in B$, we have $\|\widetilde{\iota}(b)-\iota(b)\|^2=\big\langle\widetilde{\iota}(b),\widetilde{\iota}(b)\big\rangle-2\big\langle\widetilde{\iota}(b),\iota(b)\big\rangle+\langle\iota(b),\iota(b)\rangle=\big\|\widetilde{\iota}(b)\big\|^2-2\big\langle b,\widetilde{\iota}\circ\iota(b)\big\rangle+\|\iota(b)\|^2=\big\|\widetilde{\iota}(b)\big\|^2-2\big\langle b,b\big\rangle+\|\iota(b)\|^2=\big\|\widetilde{\iota}(b)\big\|^2-2\|b\|^2+\|\iota(b)\|^2$. Since $\|\iota\|\le 1$ and $\|\widetilde{\iota}\|\le 1$, $\|\widetilde{\iota}(b)-\iota(b)\|^2\le 0$, from which $\widetilde{\iota}(b)=\iota(b)$. Thus $\widetilde{\iota}=\iota$, from which $\lceil\cdot,\cdot\rceil=\lfloor\cdot,\cdot\rfloor$ as required. In other words, $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ is its own Banach SSD dual.

The following concept will be critical in Theorem 6.12.

Definition 6.4. Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space with Banach SSD dual $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$. In line with Definition 5.5, we say that $\iota(B)$ is *$\widetilde{p}$–dense in B^** if

$$\text{for all } b^* \in B^*, \quad \inf \widetilde{p}\big(\iota(B) - b^*\big) = 0. \tag{52}$$

Example 6.5. We now show that the Banach SSD space $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|_2\big)$ of Example 5.3 has a Banach SSD dual, and

$$\iota(B) \text{ is } \widetilde{p}\text{–dense in } B^*. \tag{53}$$

(See Definition 6.4.) We recall that E is a nonzero Banach space and $B = E \times E^*$. We represent B^* by $E^* \times E^{**}$, under the pairing $\big\langle (x, x^*), (y^*, y^{**})\big\rangle := \langle x, y^*\rangle + \langle x^*, y^{**}\rangle$. Then, for all $(x, x^*) \in B$, $q(x, x^*) = \langle x, x^*\rangle$ and, from (43), $\iota(x, x^*) = (x^*, \widehat{x})$. The dual norm on $E^* \times E^{**}$ is given by $\|(y^*, y^{**})\|_2 := \sqrt{\|y^*\|^2 + \|y^{**}\|^2}$.

Replacing E by E^* in Example 5.3, we define the symmetric bilinear form $\lceil\cdot,\cdot\rceil\colon B^* \times B^* \to \mathbb{R}$ by $\big\lceil (x^*, x^{**}), (y^*, y^{**})\big\rceil := \langle y^*, x^{**}\rangle + \langle x^*, y^{**}\rangle$. Then $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|_2\big)$ is a Banach SSD space. We represent $B^{**} = (B^*)^*$ by $E^{**} \times E^{***}$ under the pairing $\big\langle (y^*, y^{**}), (w^{**}, w^{***})\big\rangle := \langle y^*, w^{**}\rangle + \langle y^{**}, w^{***}\rangle$. Then, for all $(y^*, y^{**}) \in B^*$, $\widetilde{q}(y^*, y^{**}) = \langle y^*, y^{**}\rangle$ and $\widetilde{\iota}(y^*, y^{**}) = (y^{**}, \widehat{y^*})$.

We next show that $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|_2\big)$ is a Banach SSD dual of $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|_2\big)$, that is to say (49) is satisfied. To this end, let $(x, x^*) \in B$ and $(y^*, y^{**}) \in B^*$. Then $\big\lceil \iota(x, x^*), (y^*, y^{**})\big\rceil = \big\lceil (x^*, \widehat{x}), (y^*, y^{**})\big\rceil = \langle y^*, \widehat{x}\rangle + \langle x^*, y^{**}\rangle = \langle x, y^*\rangle + \langle x^*, y^{**}\rangle = \big\langle (x, x^*), (y^*, y^{**})\big\rangle$, which gives (49), as required.

We now establish (53). To see this, let $(y^*, y^{**}) \in B^*$ and $\varepsilon > 0$. From the definition of $\|y^{**}\|$, there exists $z^* \in E^*$ such that $\|z^*\| \le \|y^{**}\|$ and $\langle z^*, y^{**}\rangle \ge \|y^{**}\|^2 - \varepsilon$. But then

$$\iota(0, y^* + z^*) - (y^*, y^{**}) = (y^* + z^*, 0) - (y^*, y^{**}) = (z^*, -y^{**}) \in B^*.$$

Since

$$\widetilde{p}(z^*, -y^{**}) = \tfrac{1}{2}\big(\|z^*\|^2 + \|y^{**}\|\big) - \langle z^*, y^{**}\rangle \le \|y^{**}\|^2 - \langle z^*, y^{**}\rangle \le \varepsilon,$$

we have established (53), as required.

The following observation will be useful in our discussion of monotone sets in Section 9: if $(a, a^*) \in B$ and $(y^*, y^{**}) \in B^*$ then $\widetilde{q}\big(\iota(a, a^*) - (y^*, y^{**})\big) = \widetilde{q}\big((a^*, \widehat{a}) - (y^*, y^{**})\big) = \widetilde{q}(a^* - y^*, \widehat{a} - y^{**}) = \langle a^* - y^*, \widehat{a} - y^{**}\rangle$. As a consequence, if $\emptyset \ne A \subset B$ then

$$\inf \widetilde{q}\big(\iota(A) - (y^*, y^{**})\big) = \inf\nolimits_{(a,a^*)\in A} \langle a^* - y^*, \widehat{a} - y^{**}\rangle. \tag{54}$$

Remark 6.6. In the situation of Example 6.5, there are norms $\|\cdot\|$ on B other than $\|\cdot\|_2$ under which $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ has a Banach SSD dual, and $\iota(B)$ is $\widetilde{p}$–dense in B^*. We refer the reader to [21, Example 2.4, p. 6] and [21, Example 4.4, pp. 14–15] for more details. Looking ahead, in all these cases, Theorem 8.3(b) remains true.

Remark 6.7. Let $\big(B_1, \lfloor\cdot,\cdot\rfloor_1, \|\cdot\|_1\big)$ be a Banach SSD space with Banach SSD dual $\big(B_1^*, \lceil\cdot,\cdot\rceil_1, \|\cdot\|_1\big)$ and $\big(B_2, \lfloor\cdot,\cdot\rfloor_2, \|\cdot\|_2\big)$ be a Banach SSD space with Banach SSD dual $\big(B_2^*, \lceil\cdot,\cdot\rceil_2, \|\cdot\|_2\big)$. We define $\|\cdot\|\colon B_1 \times B_2 \to \mathbb{R}$ and $\lfloor\cdot,\cdot\rfloor\colon (B_1 \times B_2) \times (B_1 \times B_2) \to \mathbb{R}$ by $\big\|(b_1, b_2)\big\| := \sqrt{\|b_1\|_1^2 + \|b_2\|_2^2}$ and $\big\lfloor(b_1, b_2), (c_1, c_2)\big\rfloor := \lfloor b_1, c_1\rfloor_1 + \lfloor b_2, c_2\rfloor_2$. Similarly, we define $\|\cdot\|\colon B_1^* \times B_2^* \to \mathbb{R}$ and $\lceil\cdot,\cdot\rceil\colon (B_1^* \times B_2^*) \times (B_1^* \times B_2^*) \to \mathbb{R}$ by $\big\|(b_1^*, b_2^*)\big\| := \sqrt{\|b_1^*\|_1^2 + \|b_2^*\|_2^2}$ and $\big\lceil(b_1^*, b_2^*), (c_1^*, c_2^*)\big\rceil := \lceil b_1^*, c_1^*\rceil_1 + \lceil b_2^*, c_2^*\rceil_2$. Then $\big(B_1 \times B_2, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ is a Banach SSD space with Banach SSD dual $\big(B_1^* \times B_2^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$.

As an example of this construction, we could take $\big(B_1, \lfloor\cdot,\cdot\rfloor_1, \|\cdot\|_1\big)$ to be a Banach SSD space of the kind considered in Remark 6.3, and $\big(B_2, \lfloor\cdot,\cdot\rfloor_2, \|\cdot\|_2\big)$ to be a Banach SSD space of the kind considered in Example 6.5. If B_1 is odd-dimensional and E is finite-dimensional then B is odd–dimensional, and so cannot itself be of the form considered in Example 6.5. Example 2.3(c) is of this form, and a glance at that example shows how pathological the q–positive sets can be.

We now recall Rockafellar's formula for the conjugate of a sum:

Lemma 6.8. *Let X be a nonzero real Banach space and $f \in \mathcal{PC}(X)$, and let $h \in \mathcal{PC}(X)$ be real–valued and continuous. Then, for all $x^* \in X^*$,*

$$(f + h)^*(x^*) = \min\nolimits_{y^* \in X^*} \big[f^*(y^*) + h^*(x^* - y^*)\big].$$

Proof. See Rockafellar, [12, Theorem 3(a), p. 85], Zălinescu, [24, Theorem 2.8.7(iii), p. 127], or [20, Corollary 10.3, p. 52]. $\square$

Remark 6.9. [20, Theorem 7.4, p. 43] contains a version of the Fenchel duality theorem with a sharp lower bound on the functional obtained.

Lemma 6.10. *Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space with Banach SSD dual $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$. Define the function g on B by $g := \frac{1}{2}\|\cdot\|^2$. Let $f \in \mathcal{PC}(B)$. Then*

$$-\big((f - q) \mathbin{\nabla} p\big) = \big((f^* - \widetilde{q}) \mathbin{\nabla} \widetilde{p}\big) \circ \iota \text{ on } B.$$

Proof. Let $c \in B$. Define $h\colon B \to \mathbb{R}$ by $h(b) := g(c - b)$. Then, by direct computation using the fact that g is an even function,

$$\text{for all } c^* \in B^*, \qquad h^*(c^*) = g^*(c^*) + \langle c, c^*\rangle. \tag{55}$$

Then, from (21), the continuity of h, Lemma 6.8, (55), (20) and the fact that, for all $c^* \in B^*$, $g^*(c^*) = \frac{1}{2}\|c^*\|^2$,

$$\begin{aligned}
-\big((f - q) \mathbin{\nabla} p\big)(c) &= \sup\nolimits_{b \in B} \big[-(f - q)(b) - p(c - b)\big] \\
&= \sup\nolimits_{b \in B} \big[\langle b, \iota(c)\rangle - f(b) - h(b)\big] - q(c) = (f + h)^*\big(\iota(c)\big) - q(c) \\
&= \min\nolimits_{b^* \in B^*} \big[f^*(b^*) + h^*\big(\iota(c) - b^*\big)\big] - q(c) \\
&= \min\nolimits_{b^* \in B^*} \big[f^*(b^*) + g^*\big(\iota(c) - b^*\big) + \big\langle c, \iota(c) - b^*\big\rangle\big] - q(c) \\
&= \min\nolimits_{b^* \in B^*} \big[f^*(b^*) + g^*\big(\iota(c) - b^*\big) - \lceil\iota(c), b^*\rceil + \widetilde{q}\big(\iota(c)\big)\big] \\
&= \min\nolimits_{b^* \in B^*} \big[(f^* - \widetilde{q})(b^*) + \widetilde{p}\big(\iota(c) - b^*\big)\big] \\
&= \big((f^* - \widetilde{q}) \mathbin{\nabla} \widetilde{p}\big)\big(\iota(c)\big).
\end{aligned}$$

This completes the proof of Lemma 6.10. $\square$

Definition 6.11. Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space with Banach SSD dual $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$ and $f \in \mathcal{PC}(B)$. We say that f is an *MAS function* if $\quad f \ge q$ on $B\quad$ and $f^* \ge \widetilde{q}$ on B^*. This is an extension to Banach SSD spaces of the concept introduced by Marques Alves and Svaiter in [5, Theorem 4.2, pp. 702–704] for the situation described in Example 6.5.

It is clear from the layout of Section 5 that the main results on VZ functions (that is, up to and including Theorem 5.8) do not depend explicitly on B^*. By contrast, a knowledge of B^* is absolutely essential for even the definition of MAS function. As a consequence, Theorem 6.12 below is rather suprising. Theorem 6.12(a) and its partial converse Theorem 6.12(b) are motivated by various results scattered through [22, Section 2]. Theorem 6.12(c) is motivated by [6]. We recall from (53) that the $\widetilde{p}$–density condition is satisfied in the situation of Example 6.5, and we will discuss the implications of Theorem 6.12 to this example (via Theorem 6.15) in Theorems 9.5, 9.7, 9.9 and 9.10.

Theorem 6.12. *Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space with Banach SSD dual $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$.*

(a) *Let $f \in \mathcal{PC}(B)$ be an MAS function. Then f is a VZ function.*

(b) *Let $\iota(B)$ be $\widetilde{p}$–dense in B^* and $f \in \mathcal{PC}(B)$ be a VZ function. Then f is an MAS function.*

(c) *Let $\iota(B)$ be $\widetilde{p}$–dense in B^* and $A \subset B$. Then there exists an MAS function $f \in \mathcal{PCLSC}(B)$ such that $A = \mathcal{P}_q(f) \iff A$ is maximally q–positive and $\quad \Theta_A \ge \widetilde{q}$ on B^*.*

Proof. (a) Taking together (29), (51) and our hypothesis that f is an MAS function, we have $\inf_B \big[f - q\big] \ge 0$, $\inf_B p \ge 0$, $\inf_{B^*} \big[f^* - \widetilde{q}\big] \ge 0$ and $\inf_{B^*} \widetilde{p} \ge 0$. Consequently, $\inf_B \big[(f - q)\,\nabla\, p\big] \ge 0$ and $\inf_B \big[\big((f^* - \widetilde{q})\,\nabla\,\widetilde{p}\big) \circ \iota\big] \ge 0$, and (a) follows from Lemma 6.10.

(b) We know from (33) that $\quad f \ge q$ on B. Now let $b^* \in B^*$ and $c \in B$. Then, from Lemma 6.10 again,

$$(f^* - \widetilde{q})(b^*) + \widetilde{p}\big(\iota(c) - b^*\big) \ge \big((f^* - \widetilde{q})\,\nabla\,\widetilde{p}\big)\big(\iota(c)\big) = -\big((f - q)\,\nabla\, p\big)(c) = 0.$$

Taking the infimum over $c \in B$ and using (52), $\quad (f^* - \widetilde{q})(b^*) \ge 0$ on B^*. Since this holds for all $b^* \in B^*$, f is an MAS function, giving (b).

(c) ($\Longrightarrow$) Let $f \in \mathcal{PCLSC}(B)$ be an MAS function and $A = \mathcal{P}_q(f)$. From (a), f is a VZ function, and so Theorem 5.6(d) implies that A is maximally q–positive. From (46) and (b), Ψ_A is an MAS function, consequently $\quad {\Psi_A}^* \ge \widetilde{q}$ on B^*, $\quad$ thus Lemma 3.2(c) implies that $\quad \Theta_A \ge \widetilde{q}$ on B^*, $\quad$ as required.

($\Longleftarrow$) Suppose, conversely, that A is maximally q–positive and $\quad \Theta_A \ge \widetilde{q}$ on B^*. From (5), Lemma 3.2(c) and (6), $\quad \Phi_A \ge q$ on B, $\quad {\Phi_A}^* \ge \widetilde{q}$ on $B^*\quad$ and $\mathcal{P}_q(\Phi_A) = A$, and the result follows with $f := \Phi_A$. $\big($We could also use Ψ_A for this part of (c).$\big)$ □

Definition 6.13. Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space with Banach SSD dual $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$. We say that a topology $\mathcal{T}$ on B^* is *compatible* if it satisfies the conditions (a)–(c) below:

(a) $\mathcal{T} \supset w(B^*, B^*)$. ($w(B^*, B^*)$ is the weak topology induced on B^* by the bilinear form $\lceil\cdot,\cdot\rceil$.)

(b) If $f \in \mathcal{PCLSC}(B)$ and $b^* \in B^*$ then there exists a net $\{b_\gamma\}$ of elements of B such that $\iota(b_\gamma) \to b^*$ in $\mathcal{T}$ and $f(b_\gamma) \to f^{*@}(b^*)$.

(c) If $\{b_\gamma\}$ and $\{a_\gamma\}$ are nets of elements of B, $b^* \in B^*$, $\iota(b_\gamma) \to b^*$ in $\mathcal{T}$ and $\|a_\gamma - b_\gamma\| \to 0$ then $\iota(a_\gamma) \to b^*$ in $\mathcal{T}$.

Remark 6.14. Definition 6.13(a) says that $\mathcal{T}$ is not too small, Definition 6.13(b) says that $\mathcal{T}$ is not too large, and Definition 6.13(c) says that $\mathcal{T}$ behaves well under norm perturbations in B. It follows from Lemma 8.2 below that $w(B^*, B^*)$ is compatible. If the norm topology of B^* is compatible then, from (b) above with $f := 0$, $\iota(B)$ is norm–dense in B^*.

Theorem 6.12 leads to the following fundamental result on the Gossez extension of a maximally q–positive set (see Definition 4.4), which will be used in Theorem 9.5, and thus indirectly in Theorems 9.9 and 9.10, which depend on Theorem 9.5. It is actually Theorem 6.15 that provides the incentive for the investigation of the continuity of $\widetilde{q}$ that we will perform in Section 8.

Theorem 6.15 *Let* $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ *be a Banach SSD space with Banach SSD dual* $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$, $\mathcal{T}$ *be a compatible topology on* B^*, $\widetilde{q}$ *be* $\mathcal{T}$*–continuous and* A *be a maximally* q*–positive subset of* B. *Then the conditions* (a)–(c) *below are equivalent.*

(a) $A^{\mathcal{G}} \subset \overline{\iota(A)}^{\mathcal{T}}$, *the closure of* $\iota(A)$ *in the topology* $\mathcal{T}$.

(b) *For all* $b^* \in A^{\mathcal{G}}$, $\inf \widetilde{q}\big(\iota(A) - b^*\big) \le 0$.

(c) $\Theta_A \ge \widetilde{q}$ *on* B^*.

Proof. Suppose that (a) is satisfied and $b^* \in A^{\mathcal{G}}$. Then there exists a net $\{a_\gamma\}$ of elements of A such that $\iota(a_\gamma) \to b^*$ in $\mathcal{T}$. From Definition 6.13(a), $\iota(a_\gamma) \to b^*$ in $w(B^*, B^*)$ and so $\lceil \iota(a_\gamma), b^*\rceil \to \lceil b^*, b^*\rceil = 2\widetilde{q}(b^*)$. From the $\mathcal{T}$–continuity of $\widetilde{q}$, $\widetilde{q}\big(\iota(a_\gamma)\big) \to \widetilde{q}(b^*)$. Thus

$$\widetilde{q}\big(\iota(a_\gamma) - b^*\big) = \widetilde{q}\big(\iota(a_\gamma)\big) - \lceil \iota(a_\gamma), b^*\rceil + \widetilde{q}(b^*) \to \widetilde{q}(b^*) - 2\widetilde{q}(b^*) + \widetilde{q}(b^*) = 0,$$

and so (a) $\Longrightarrow$ (b). It follows from Theorem 4.5(b) that (b) $\Longrightarrow$ (c). So it remains to prove that (c) $\Longrightarrow$ (a).

So suppose that (c) is satisfied and $b^* \in A^{\mathcal{G}}$. (4) implies that $\Phi_A \in \mathcal{PCLSC}(B)$. From Theorem 4.5(c), ${\Phi_A}^{*@}(b^*) = \widetilde{q}(b^*)$. Definition 6.13(b) now gives us a net $\{b_\gamma\}$ of elements of B such that

$$\iota(b_\gamma) \to b^* \text{ in } \mathcal{T} \quad \text{and} \quad \Phi_A(b_\gamma) \to {\Phi_A}^{*@}(b^*) = \widetilde{q}(b^*). \tag{56}$$

It now follows from (20), the first assertion in (56) and the $\mathcal{T}$–continuity of $\widetilde{q}$ that $q(b_\gamma) = \widetilde{q} \circ \iota(b_\gamma) \to \widetilde{q}(b^*)$ and so, using the second assertion in (56),

$$(\Phi_A - q)(b_\gamma) = \Phi_A(b_\gamma) - q(b_\gamma) \to \widetilde{q}(b^*) - \widetilde{q}(b^*) = 0. \tag{57}$$

Now (5) implies that $\Phi_A \ge q$ on B, and, from Lemma 3.2(c), ${\Phi_A}^* \ge \Theta_A \ge \widetilde{q}$ on B^*, from which Φ_A is an MAS fucntion. Thus, from Theorem 6.12(a), Φ_A is a VZ function. Since Φ_A is lower semicontinuous on B, (6) and (48) imply that, for all γ,

$$\operatorname{dist}(b_\gamma, A) = \operatorname{dist}\big(b_\gamma, \mathcal{P}_q(\Phi_A)\big) \le \sqrt{2}\sqrt{(\Phi_A - q)(b_\gamma)}.$$

$\big($We interpret $\sqrt{\infty}$ to be ∞.$\big)$ (57) now gives us $a_\gamma \in A$ such that $\|a_\gamma - b_\gamma\| \to 0$, and so (a) follows from the first assertion in (56) and Definition 6.13(c). $\square$

7. $\mathcal{CLB}(X)$ and $\mathcal{T}_{\mathcal{CLB}}(X^{**})$

Let X be a nonzero real Banach space. Corresponding to the usage outlined in first paragraph of Section 3, if $f \in \mathcal{PC}(X)$ and $f^* \in \mathcal{PC}(X^*)$, we define $f^{**}\colon X^{**} \to]{-\infty}, \infty]$ by $f^{**}(x^{**}) := \sup_{X^*}\big[\langle\cdot, x^{**}\rangle - f^*\big]$. We write $\mathcal{CLB}(X)$ for the set of all convex functions $f\colon X \to \mathbb{R}$ that are Lipschitz on the bounded subsets of X, or equivalently, bounded above on the bounded subsets of X, and we define the topology $\mathcal{T}_{\mathcal{CLB}}(X^{**})$ on X^{**} to be the coarsest topology on X^{**} making all the functions $h^{**}\colon X^{**} \to \mathbb{R}$ $\big(h \in \mathcal{CLB}(X)\big)$ continuous. $\big($See [20, Definition 38.1, p, 155].$\big)$ We write $\mathcal{T}_{\|\ \|}(X)$ for the norm–topology on X. We collect together in the following lemma the basic properties of $\mathcal{T}_{\mathcal{CLB}}(X^{**})$ that we will use. We will discuss subtler properties of the topologies $\mathcal{T}_{\mathcal{CLB}}$ in Lemma 7.3.

Lemma 7.1. *Let X be a nonzero real Banach space.*

(a) *Let $\big\{x_\gamma^{**}\big\}$ be a net of elements of X^{**}, $x^{**} \in X^{**}$ and $x_\gamma^{**} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(X^{**})$. Then $\big\{x_\gamma^{**}\big\}$ is eventually bounded and $x_\gamma^{**} \to x^{**}$ in the weak*–topology $w(X^{**}, X^*)$.*

(b) *Let $f \in \mathcal{PCLSC}(X)$ and $x^{**} \in X^{**}$. Then there exists a net $\{x_\gamma\}$ of elements of X such that $\widehat{x_\gamma} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(X^{**})$ and $f(x_\gamma) \to f^{**}(x^{**})$.* (We note from the Fenchel–Moreau theorem that $f^* \in \mathcal{PC}(X^*)$.)

(c) *Let $\{x_\gamma^{**}\}$ and $\{y_\gamma^{**}\}$ be nets of elements of X^{**}, $x^{**} \in X^{**}$, $x_\gamma^{**} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(X^{**})$ and $\|y_\gamma^{**} - x_\gamma^{**}\| \to 0$. Then $y_\gamma^{**} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(X^{**})$.*

Proof. (a) See [20, Lemma 38.2(b,f), p. 156].

(b) See [20, Lemma 45.9(a), p. 175].

(c) See [20, Lemma 45.15, p. 177]. $\square$

Remark 7.2. Despite the nice properties of $\mathcal{T}_{\mathcal{CLB}}(X^{**})$ exhibited in Lemma 7.1, it is nevertheless quite a pathological topology. For instance, if $\big(B^{**}, \mathcal{T}_{\mathcal{CLB}}(X^{**})\big)$ is a topological vector space then X is reflexive. See [20, Remark 45.13, p. 177].

Lemma 7.3(b) was originally developed in a study of the subdifferentials of saddle functions.

Lemma 7.3. *Let E be a nonzero Banach space.*

(a) *The map $\widetilde{q}$ is continuous from $\big(E^* \times E^{**}, \mathcal{T}_{\|\ \|}(E^*) \times \mathcal{T}_{\mathcal{CLB}}(E^{**})\big)$ into $\mathbb{R}$.*

(b) *Let H also be a nonzero Banach space, $(y^{**}, z) \in E^{**} \times H$ and $\big\{(y_\gamma^{**}, z_\gamma)\big\}$ be a net of elements of $E^{**} \times H$. Then $(y_\gamma^{**}, \widehat{z_\gamma}) \to (y^{**}, \widehat{z})$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**} \times H^{**}) \iff (y_\gamma^{**}, z_\gamma) \to (y^{**}, z)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(H)$.*

Proof. (a) See [20, Lemma 38.2(e), p. 156]. $\big($We note that B^* in that reference was defined to be $E^{**} \times E^*$ rather than $E^* \times E^{**}$ as we have done here, and the topology $\mathcal{T}_{\mathcal{CLBN}}(B^*)$ was defined to be $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$.$\big)$

(b) See [20, Theorem 49.4, pp. 194]. $\square$

8. $\mathcal{T}_{\mathcal{D}}(B^*)$

We suppose throughout this section that $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ is a Banach SSD space with Banach SSD dual $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$. In order to apply Theorem 6.15, we need a compatible topology on B^* with respect to which $\widetilde{q}$ is continuous. To get some insight into this problem, we consider the case of Example 6.5, that is to say, $B^* = E^* \times E^{**}$ and $\widetilde{q}\colon (x^*, x^{**}) \mapsto \langle x^*, x^{**}\rangle$. It has been known since Gossez's work in [3] that $\mathcal{T}_{\|\ \|}(E^* \times E^{**})$ is too large to be of any practical use. (The root of the problem can be found in Remark 6.14.) Gossez considers the topology $\mathcal{T}_{\|\ \|}(E^*) \times w(E^{**}, E^*)$, but this topology does not seem to generalize easily to the case of SSD spaces. In Definition 8.1, we introduce the topology $\mathcal{T}_{\mathcal{D}}(B^*)$ on B^*. We will see in Lemma 8.2 that $\mathcal{T}_{\mathcal{D}}(B^*)$ is sufficiently small that it is compatible, and we will see in Theorem 8.3(b) that $\mathcal{T}_{\mathcal{D}}(B^*)$ is sufficiently large that Theorem 6.15 leads to significant results.

Definition 8.1. Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space with Banach SSD dual $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$. We define the topology $\mathcal{T}_{\mathcal{D}}(B^*)$ on B^* to be the coarsest topology on B^* making the function $\widetilde{\iota}\colon B^* \to \big(B^{**}, \mathcal{T}_{\mathcal{CLB}}(B^{**})\big)$ continuous. This means that if $\{b^*_\gamma\}$ is a net of elements of B^* and $b^* \in B^*$ then

$$b^*_\gamma \to b^* \text{ in } \mathcal{T}_{\mathcal{D}}(B^*) \iff \widetilde{\iota}(b^*_\gamma) \to \widetilde{\iota}(b^*) \text{ in } \mathcal{T}_{\mathcal{CLB}}(B^{**}). \tag{58}$$

Now suppose that $\{b_\gamma\}$ is a net of elements of B and $b^* \in B^*$. Combining (58) with Lemma 6.2, we have

$$\iota(b_\gamma) \to b^* \text{ in } \mathcal{T}_{\mathcal{D}}(B^*) \iff \widehat{b_\gamma} \to \widetilde{\iota}(b^*) \text{ in } \mathcal{T}_{\mathcal{CLB}}(B^{**}). \tag{59}$$

Lemma 8.2. *We suppose that $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ is a Banach SSD space with Banach SSD dual $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$. Then $\mathcal{T}_{\mathcal{D}}(B^*)$ is a compatible topology on B^*.*

Proof. We first verify Definition 6.13(a). Let $\{b^*_\gamma\}$ be a net of elements of B^*, $b^* \in B^*$ and $b^*_\gamma \to b^*$ in $\mathcal{T}_{\mathcal{D}}(B^*)$. (58) implies that $\widetilde{\iota}(b^*_\gamma) \to \widetilde{\iota}(b^*)$ in $\mathcal{T}_{\mathcal{CLB}}(B^{**})$ and so, from Lemma 7.1(a), $\widetilde{\iota}(b^*_\gamma) \to \widetilde{\iota}(b^*)$ in $w(B^{**}, B^*)$. (50) now gives us that

$$\text{for all } c^* \in B^*, \quad \lceil c^*, b^*_\gamma\rceil = \big\langle c^*, \widetilde{\iota}(b^*_\gamma)\big\rangle \to \big\langle c^*, \widetilde{\iota}(b^*)\big\rangle = \lceil c^*, b^*\rceil,$$

and so $b^*_\gamma \to b^*$ in $w(B^*, B^*)$. This completes the proof of Definition 6.13(a).

We next verify Definition 6.13(b). To this end, let $f \in \mathcal{PCLSC}(B)$ and $b^* \in B^*$. Lemma 7.1(b) provides us with a net $\{b_\gamma\}$ of elements of B such that $\widehat{b_\gamma} \to \widetilde{\iota}(b^*)$ in $\mathcal{T}_{\mathcal{CLB}}(B^{**})$ and $f(b_\gamma) \to f^{**} \circ \widetilde{\iota}(b^*)$. (59) gives $\iota(b_\gamma) \to b^*$ in $\mathcal{T}_{\mathcal{D}}(B^*)$, and the analog of (10) gives $f^{**} \circ \widetilde{\iota}(b^*) = f^{*@}(b^*)$. This completes the proof of Definition 6.13(b).

Finally, we verify Definition 6.13(c). To this end, let $\{b_\gamma\}$ and $\{a_\gamma\}$ be nets of elements of B, $b^* \in B^*$, $\iota(b_\gamma) \to b^*$ in $\mathcal{T}_{\mathcal{D}}(B^*)$ and $\|a_\gamma - b_\gamma\| \to 0$. From (59), $\widehat{b_\gamma} \to \widetilde{\iota}(b^*)$ in $\mathcal{T}_{\mathcal{CLB}}(B^{**})$. Since $\widehat{\ }$ is a norm–isometry, $\|\widehat{a_\gamma} - \widehat{b_\gamma}\| \to 0$, and so Lemma 7.1(c) implies that $\widehat{a_\gamma} \to \widetilde{\iota}(b^*)$ in $\mathcal{T}_{\mathcal{CLB}}(B^{**})$. It now follows from another application of (59) that $\iota(a_\gamma) \to b^*$ in $\mathcal{T}_{\mathcal{D}}(B^*)$. This completes the proof of Definition 6.13(c). $\big($In fact, one can prove in a similar way, using (58) instead of (59), the stronger result that if $\{b_\gamma^*\}$ and $\{a_\gamma^*\}$ are nets of elements of B^*, $b^* \in B^*$, $b_\gamma^* \to b^*$ in $\mathcal{T}_{\mathcal{D}}(B^*)$ and $\|a_\gamma^* - b_\gamma^*\| \to 0$ then $a_\gamma^* \to b^*$ in $\mathcal{T}_{\mathcal{D}}(B^*)$.$\big)$ □

Theorem 8.3 below will be used in Theorems 9.5, 9.9, and 9.10.

Theorem 8.3 *Let E be a nonzero Banach space and $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|_2\big)$ and $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|_2\big)$ be as in Example 6.5.*

(a) *The topologies $\mathcal{T}_{\mathcal{D}}(B^*)$ and $\mathcal{T}_{\|\ \|}(E^*) \times \mathcal{T}_{\mathcal{CLB}}(E^{**})$ on $B^* = E^* \times E^{**}$ are identical.*

(b) *$\widetilde{q}$ is $\mathcal{T}_{\mathcal{D}}(B^*)$–continuous.*

Proof. We recall from Example 6.5 that, for all $(y^*, y^{**}) \in B^*$, $\widetilde{\iota}(y^*, y^{**}) = \big(y^{**}, \widehat{y^*}\big)$ and $\widetilde{q}(y^*, y^{**}) = \langle y^*, y^{**}\rangle$. Let $\big\{(y_\gamma^*, y_\gamma^{**})\big\}$ be a net of elements of B^* and $(y^*, y^{**}) \in B^*$. Then, from (58),

$$(y_\gamma^*, y_\gamma^{**}) \to (y^*, y^{**}) \text{ in } \mathcal{T}_{\mathcal{D}}(B^*) \iff \big(y_\gamma^{**}, \widehat{y_\gamma^*}\big) \to \big(y^{**}, \widehat{y^*}\big) \text{ in } \mathcal{T}_{\mathcal{CLB}}(B^{**}).$$

(a) is now immediate from Lemma 7.3(b) with $H := E^*$, and (b) is immediate from (a) and Lemma 7.3(a). □

Remark 8.4. A hidden bonus of Theorem 8.3 is that, despite the fact that $B^{**} = E^{**} \times E^{***}$, we do not actually have to deal with E^{***}.

9. Classes of monotone sets

We suppose in this section that E is a nonzero Banach space. For most of the time it will be convenient to work in terms of subsets of $E \times E^*$ and $E^* \times E^{**}$ rather than multifunctions $E \rightrightarrows E^*$ and $E^{**} \rightrightarrows E^*$, and we leave it to the reader to verify the consistence between our versions and the multifunction versions.

The motivation for the consideration of the various classes of sets described below was to see how many of the properties of maximally monotone sets on reflexive spaces can be recovered in the nonreflexive case. Historically, the first such classes of sets were the class of sets of "dense type" and "type (D)". These were essentially introduced by Gossez in [3, Lemme 2.1, p. 375] — see Phelps, [11, Section 3] for an exposition. If A is a monotone subset of $E \times E^*$, Gossez defines $\overline{A} \subset E^* \times E^{**}$ by:

$$\overline{A} := \big\{(y^*, y^{**}) \in E^* \times E^{**}\colon\ \inf_{(a,a^*)\in A}\langle a^* - y^*, \widehat{a} - y^{**}\rangle \geq 0\big\}. \tag{60}$$

Definition 9.1. Let $A \subset E \times E^*$. We say that A is *maximally monotone of type (D)* if A is maximally monotone and, for all $(y^*, y^{**}) \in \overline{A}$, there exists a bounded net $\{(a_\gamma, a_\gamma^*)\}$ of elements of A such that $(a_\gamma^*, \widehat{a_\gamma}) \to (y^*, y^{**})$ in $\mathcal{T}_{\|\ \|}(E^*) \times w(E^{**}, E^*)$. We say that A is *maximally monotone of dense type* if the topology $w(E^{**}, E^*)$ in the definition above is replaced by the topology $\mathcal{T}_1$ defined to be the upper bound of $w(E^{**}, E^*)$ and the coarsest topology making the function $\|\cdot\|\colon\ E^{**} \to \mathbb{R}$ continuous.

The next classes of monotone sets in our discussion are the classes of sets of type (NI) and (WD), which were introduced in [14, Definition 10, p. 183] and [14, Definition 14, p. 187].

Definition 9.2. Let $A \subset E \times E^*$. We say that A is *maximally monotone of type (NI)* if A is maximally monotone and,

$$\text{for all } (y^*, y^{**}) \in E^* \times E^{**}, \quad \inf\nolimits_{(a,a^*)\in A}\langle a^* - y^*, \widehat{a} - y^{**}\rangle \le 0.$$

We say that A is *maximally monotone of type (WD)* if A is maximally monotone and, for all $(y^*, y^{**}) \in \overline{A}$, there exists a bounded net $\{(a_\gamma, a_\gamma^*)\}$ of elements of A such that $a_\gamma^* \to y^*$ in $\mathcal{T}_{\|\ \|}(E^*)$. Clearly,

$$\textit{if } A \textit{ is maximally monotone of type (D) then } A \textit{ is of type (WD)}, \tag{61}$$

and it was proved in [14, Lemma 15, pp. 187–188] that

$$\textit{if } A \textit{ is maximally monotone of type (WD) then } A \textit{ is of type (NI)}. \tag{62}$$

The next class of monotone sets in our discussion is the class of sets of type (ED), which was introduced in [15, Definition 35.1, p. 138] under the name "type (DS)".

Definition 9.3. Let $A \subset E \times E^*$. We say that A is *maximally monotone of type (ED)* if A is maximally monotone and, for all $(y^*, y^{**}) \in \overline{A}$, there exists a net $\{(a_\gamma, a_\gamma^*)\}$ of elements of A such that $(a_\gamma^*, \widehat{a_\gamma}) \to (y^*, y^{**})$ in $\mathcal{T}_{\|\ \|}(E^*) \times \mathcal{T}_{\mathcal{CLB}}(E^{**})$. It is clear from Lemma 7.1(a) and Definition 9.1 that

$$\textit{if } A \textit{ is maximally monotone of type (ED) then } A \textit{ is of dense type}. \tag{63}$$

We now recast the above definitions in the more compact notation of SSD spaces, using the notation of Example 6.5.

Lemma 9.4 *Let A be a maximally monotone subset of $E \times E^*$. Then:*
(a) $\overline{A} = A^{\mathcal{G}}$.
(b) *A is of type (NI) $\iff$ $\Theta_A \ge \widetilde{q}$ on $E^* \times E^{**}$.*
(c) *A is of type (ED) $\iff$ for all $(y^*, y^{**}) \in A^{\mathcal{G}}$, there exists a net $\{(a_\gamma, a_\gamma^*)\}$ of elements of A such that $\iota(a_\gamma, a_\gamma^*) \to (y^*, y^{**})$ in $\mathcal{T}_{\mathcal{D}}(E^* \times E^{**})$.*

Proof. (a) is immediate from (60), (54) and the first expression in (24), and (b) is immediate from Definition 9.2, (54) and Lemma 4.3(b). As for (c), from Definition 9.3 and (a), A is of type (ED) exactly when, for all $(y^*, y^{**}) \in A^{\mathcal{G}}$, there exists a net $\{(a_\gamma, a_\gamma^*)\}$ of elements of A such that $(a_\gamma^*, \widehat{a_\gamma}) \to (y^*, y^{**})$ in $\mathcal{T}_{\|\ \|}(E^*) \times \mathcal{T}_{\mathcal{CLB}}(E^{**})$ and (c) follows from Theorem 8.3(a). $\square$

Now it is clear from (63), (61) and (62) that, for maximally monotone sets,

$$\textit{type (ED)} \implies \textit{dense type} \implies \textit{type (D)} \implies \textit{type (WD)} \implies \textit{type (NI)},$$

and the question arises naturally whether there are any result in the reverse direction. Considerable progress was made recently by Marques Alves and Svaiter in [7, Theorem 4.4, pp. 10–11], where it was established that

$$\textit{if } A \textit{ is maximally monotone of type (NI) then } A \textit{ is of type (D)}, \tag{64}$$

thus for maximally monotone sets, type (D), type (WD) and type (NI) are equivalent. Consequently, the conjecture on [14, p. 187] and the first conjecture on [14, p. 188] are false while, as we will see in Theorem 9.9(f), the second conjecture on [14, p. 188] (on the convexity of the closure of the range) is true. This latter result was actually established by Zagrodny in [23] (see [20, Problem 43.3, p. 168]). (64) also provides a positive answer to [20, Problem 36.4, p. 149]. The following result extends (64), and provides an (unexpected positive) answer to [16, Problem 4.3, p. 268]:

Theorem 9.5 *Let E be a nonzero Banach space. Then for maximally monotone subsets of $E \times E^*$, type (ED), dense type, type (D), type (WD) and type (NI) are equivalent.*

Proof. By virtue of the remarks above, we only have to prove that type (NI) $\Longrightarrow$ type (ED). So let A be a maximally monotone subset of $E \times E^*$ of type (NI). Lemma 9.4(b) implies that $\Theta_A \ge \widetilde{q}$ on $E^* \times E^{**}$, and then, from Lemma 9.4(a) and Theorem 6.15((c)$\Longrightarrow$(a)), for all $(y^*, y^{**}) \in A^{\mathcal{G}}$, there exists a net $\{(a_\gamma, a_\gamma^*)\}$ of elements of A such that $\iota(a_\gamma, a_\gamma^*) \to (y^*, y^{**})$ in $\mathcal{T}_{\mathcal{D}}(E^* \times E^{**})$. Thus, from Lemma 9.4(c), A is of type (ED). $\square$

The next class of monotone sets in our discussion is the class of strongly representable sets, which was introduced and studied in [5], [6] and [22].

Definition 9.6. Let E be a nonzero Banach space and $A \subset E \times E^*$. We say that A is *strongly representable* if there exists an MAS function $f \in \mathcal{PCLSC}(E \times E^*)$ such that $A = \mathcal{P}_q(f)$.

We now give a proof using SSD spaces of the following result, which was established by Marques Alves and Svaiter in [5, Theorem 4.2, pp. 702–704] and [6, Theorem 1.2].

Theorem 9.7. *Let E be a nonzero Banach space and $A \subset E \times E^*$. Then A is strongly representable $\iff$ A is maximally monotone of type (NI).*

Proof. This is immediate from (53), Theorem 6.12(c) and Lemma 9.4(b). $\square$

The "maximally monotone" assertion of Theorem 9.9(a) was obtained in [22, Theorem 8] under the VZ hypothesis and, in [5, Theorem 4.2(2)] under the MAS hypothesis.

Theorem 9.9(c) extends the result proved in [22, Corollary 25] that $\mathcal{P}_q(f)$ is of type (ANA).

Theorem 9.9(d) extends the result proved in [5, Theorem 4.2(2)].

Theorem 9.9(f) was obtained in [22, Corollary 7]. This is a very significant result, because maximally monotone sets A of $E \times E^*$ are known such that $\overline{\pi_{E^*}(A)}$ is not convex. (The first such example was given by Gossez in [4, Proposition, p. 360]). Thus (as was first observed in [22]) Theorem 9.9(f) implies that there exist maximally monotone sets A that are not of the form $\mathcal{P}_q(f)$ for any lower semicontinuous VZ function on $E \times E^*$ or, equivalently, not of the form $\mathcal{P}_q(f)$ for any lower semicontinuous MAS function on $E \times E^*$.

In [23, Section 3, pp. 775–783], Zagrodny considers subsets S of $E \times E^*$ such that, writing φ_S for the Fitzpatrick function of S, $\varphi_S \in \mathcal{PC}(E \times E^*)$, $\quad \varphi_S \geq q$ on $E \times E^*$ $\quad$ and

$$(a^*, a^{**}) \in \partial\varphi_S(x, x^*) \quad \Longrightarrow \quad \varphi_S(x, x^*) \leq \langle x, a^* \rangle + \langle x^*, a^{**} \rangle - \langle a^*, a^{**} \rangle. \tag{65}$$

Since the analysis in [23] leans heavily on ε–enlargements, it is hard to correlate it on a step by step basis with what we have presented here. Nevertheless, we note the following consequences if A is a maximally monotone subset of $E \times E^*$ of type (NI):

• In [23, (20), p. 776], Zagrodny deduces the second assertion in (66).

• In [23, Corollary 3.4, p. 780], Zagrodny deduces that A is p–dense in $E \times E^*$ and an additional boundedness conclusion on the approximants. (Compare Theorem 5.6(b).)

• In [23, Corollary 3.5, pp. 781–782] and [23, Corollary 3.6, pp. 782–783], Zagrodny deduces that the sets $\overline{\pi_E(A)}$ and $\overline{\pi_{E^*}(A)}$ are convex. As we have already noted, this is a very significant result.

Problem 9.8. We note from the definition of $\partial\varphi_S$ that (65) can be put in the form $(a^*, a^{**}) \in \partial\varphi_S(x, x^*) \Longrightarrow {\varphi_S}^*(a^*, a^{**}) \geq \langle a^*, a^{**} \rangle$ that is to say $(a^*, a^{**}) \in R(\partial\varphi_S)$ $\Longrightarrow {\varphi_S}^*(a^*, a^{**}) \geq \langle a^*, a^{**} \rangle$. This leads to the following question: *is Theorem 6.12(*a*) true if, instead of assuming that f is an MAS function, we assume that $f \geq q$ on B and $f^* \geq \widetilde{q}$ on $R(\partial f)$?* Given the applications of Theorem 6.12(a) that we make, it is probably no restriction to assume that $f \in \mathcal{PCLSC}(B)$.

Theorem 9.9. *Let E be a nonzero Banach space and $f \in \mathcal{PCLSC}(E \times E^*)$. Assume either that f is a VZ function or, equivalently* (bearing in mind (53) and Theorem 6.12(a,b)), *an MAS function, and let $A := \mathcal{P}_q(f)$. Then:*

(a) *A is a maximally monotone subset of $E \times E^*$ of type (ED).*

(b) *Let $(x, x^*) \in E \times E^*$ and $\alpha, \beta > 0$. Then there exists a unique value of $\tau \geq 0$ for which there exists a bounded sequence $\{(y_n, y_n^*)\}_{n \geq 1}$ of elements of A such that,*

$$\lim_{n\to\infty} \|y_n - x\| = \alpha\tau, \quad \lim_{n\to\infty} \|y_n^* - x^*\| = \beta\tau \quad \text{and} \quad \lim_{n\to\infty} \langle y_n - x, y_n^* - x^* \rangle = -\alpha\beta\tau^2.$$

(c) *Let $(x, x^*) \in E \times E^* \setminus A$ and $\alpha, \beta > 0$. Then there exists a bounded sequence $\{(y_n, y_n^*)\}_{n \geq 1}$ of elements of $\quad A \cap \big[(E \setminus \{x\}) \times (E^* \setminus \{x^*\})\big]$ such that,*

$$\lim_{n\to\infty} \frac{\|y_n - x\|}{\|y_n^* - x^*\|} = \frac{\alpha}{\beta} \quad \text{and} \quad \lim_{n\to\infty} \frac{\langle y_n - x, y_n^* - x^* \rangle}{\|y_n - x\|\|y_n^* - x^*\|} = -1. \tag{66}$$

In particular, A is of type (ANA) (see [20, Definition 36.11, p. 152]).

(d) *Let $(x, x^*) \in E \times E^* \setminus A$, $\alpha, \beta > 0$ and $\inf_{(y,y^*)\in A} \langle y - x, y^* - x^* \rangle > -\alpha\beta$. Then there exists a bounded sequence $\{(y_n, y_n^*)\}_{n \geq 1}$ in $\quad A \cap \big[(E \setminus \{x\}) \times (E^* \setminus \{x^*\})\big]$ such that (66) is satisfied, $\lim_{n\to\infty} \|y_n - x\| < \alpha$ and $\lim_{n\to\infty} \|y_n^* - x^*\| < \beta$. In particular, A is of type (BR)* (see [20, Definition 36.13, p. 153]).

(e) *Let* $(x,x^*) \in E \times E^* \setminus A$, $\alpha,\beta > 0$ *and* $f(x,x^*) < \langle x,x^*\rangle + \alpha\beta$. *Then there exists a bounded sequence* $\big\{(y_n,y_n^*)\big\}_{n\ge 1}$ *of elements of* $A \cap \big[(E\setminus\{x\}) \times (E^*\setminus\{x^*\})\big]$ *such that* (66) *is satisfied,* $\lim_{n\to\infty}\|y_n - x\| < \alpha$ *and* $\lim_{n\to\infty}\|y_n^* - x^*\| < \beta$.

(f) *We define the* projection maps $\pi_E\colon E\times E^* \to E$ *and* $\pi_{E^*}\colon E\times E^* \to E^*$ *by* $\pi_E(x,x^*) := x$ *and* $\pi_{E^*}(x,x^*) := x^*$. *Then* $\overline{\pi_E(A)} = \overline{\pi_E(\operatorname{dom} f)}$ *and* $\overline{\pi_{E^*}(A)} = \overline{\pi_{E^*}(\operatorname{dom} f)}$. *Consequently, the sets* $\overline{\pi_E(A)}$ *and* $\overline{\pi_{E^*}(A)}$ *are convex.*

Proof. (a) is immediate from Theorems 5.6(d), 9.7 and 9.5.

(b), (c) and (d) are immediate from (a) and either [16, Theorem 8.6, pp. 277–278] or [20, Theorem 42.6, pp. 163–164].

(e) is immediate from (d) and the observation in (47) that, for all $(x,x^*) \in E \times E^*$, $-\inf_{(y,y^*)\in A}\langle y-x, y^*-x^*\rangle \le f(x,x^*) - \langle x,x^*\rangle$.

(f) If $x \in \pi_E(\operatorname{dom} f)$ then there exists $x^* \in E^*$ such that $f(x,x^*) < \infty$, and so it follows from (e) that there exists $(y,y^*) \in A$ such that $\|y-x\| < 1/n$. Consequently, $x \in \overline{\pi_E(A)}$. Thus we have proved that $\pi_E(\operatorname{dom} f) \subset \overline{\pi_E(A)}$. On the other hand, $A \subset \operatorname{dom} f$, and so $\overline{\pi_E(A)} = \overline{\pi_E(\operatorname{dom} f)}$. We can prove in an exactly similar way that $\overline{\pi_{E^*}(A)} = \overline{\pi_{E^*}(\operatorname{dom} f)}$. The convexity of the sets $\overline{\pi_E(A)}$ and $\overline{\pi_{E^*}(A)}$ now follows immediately. $\square$

In the final results of this section, which are more conveniently stated in terms of multifunctions, we give other consequences of Theorem 9.5. "Type (FP)" (= "locally maximally monotone") was defined in [17, Definition 6, p. 394] and [20, Definition 36.5, p. 149], "type (FPV)" (= "maximally monotone locally") was defined in [17, Definition 7, p. 395] and [20, Definition 36.7, p. 150], "strongly maximally monotone" was defined in [17, Definition 8, pp. 395–396] and [20, Definition 36.9, p. 151], and the statement "$S + \lambda J_\eta$ is surjective" was defined in [20, (42.2), p. 164]. The facts that strongly representable maximally monotone multifunctions are of type (FP) (type (FPV) and strongly maximally monotone, respectively) were observed in [22, Theorem 22], ([22, Remark 6] and [22, Theorem 23], respectively). In the above acronyms, "F" stands for "Fiztpatrick", "P" stands for "Phelps" and "V" stands for "Veronas".

Theorem 9.10. *Let* E *be a nonzero Banach space, and* $S\colon E \rightrightarrows E^*$ *be maximally monotone of type (NI). Then:*

(a) S *is of type (FP).*

(b) S *is of type (FPV).*

(c) S *is strongly maximally monotone. If, further,* $S^{-1}\colon E^* \rightrightarrows E$ *is coercive, that is to say* $\inf\langle S^{-1}x^*, x^*\rangle/\|x^*\| \to \infty$ *as* $\|x^*\| \to \infty$, *then* $D(S) = E$.

(d) *For all* $\lambda,\eta > 0$, $S + \lambda J_\eta$ *is surjective.*

Proof. (a) follows from Theorem 9.5 and [20, Theorem 37.1, pp. 153–154]. (b) follows from Theorem 9.5 and [20, Theorem 39.1, pp. 157–158]. The first assertion in (c) follows from Theorem 9.5 and [20, Theorem 40.1, pp. 158–159], and the second assertion in (c) follows from [20, Corollary 41.2, p. 160]. (d) follows from Theorem 9.5 and [20, Theorem 42.8, pp. 164]. $\square$

10. Appendix: a nonhausdorff Fenchel–Moreau theorem

In Theorem 3.3, we referred to the Fenchel–Moreau theorem for (possibly nonhausdorff) locally convex spaces. We shall give a proof of this result in Theorem 10.1. When we say that X is a *locally convex space*, we mean that X is a nonzero real vector space endowed with a topology compatible with its vector structure and a base of neighborhoods of 0 of the form $\big\{x \in X\colon\ S(x) \le 1\big\}_{S\in\mathcal{S}(X)}$, where $\mathcal{S}(X)$ is a family of seminorms on X such that if $S_1 \in \mathcal{S}(X)$ and $S_2 \in \mathcal{S}(X)$ then $S_1 \vee S_2 \in \mathcal{S}(X)$; and if $S \in \mathcal{S}(X)$ and $\lambda \ge 0$ then $\lambda S \in \mathcal{S}(X)$. If L is a linear functional on X then L is continuous if, and only if, there exists $S \in \mathcal{S}(X)$ such that $L \le S$ on X.

As an example of the construction above, we can suppose that X and Y are vector spaces paired by a bilinear form $\langle\cdot,\cdot\rangle$. Then $\big(X, w(X,Y)\big)$ is a locally convex space with determining family of seminorms $\big\{|\langle\cdot,y_1\rangle| \vee \cdots \vee |\langle\cdot,y_n\rangle|\big\}_{n\ge 1,\ y_1,\ldots,y_n\in Y}$.

The author is grateful to Constantin Zălinescu for showing him a proof of Theorem 10.1 based on the standard (Hausdorff) result and a quotient construction. The proof we give here is a simplification of the result on *Fenchel–Moreau points* of [19, Theorem 5.3, pp. 157–158] or [20, Theorem 12.2, pp. 59–60] (which is also valid in the nonhausdorff setting).

Theorem 10.1. *Let X be a locally convex space with defining family of seminorms $\mathcal{S}(X)$, and $f \in \mathcal{PC}(X)$ be lower semicontinuous. Write X^* for the set of continuous linear functionals on X. If $L \in X^*$, define $f^*(L) := \sup_X \big[L - f\big]$. Let $y \in X$. Then*

$$f(y) = \sup\nolimits_{L\in X^*} \big[L(y) - f^*(L)\big]. \tag{67}$$

Proof. Since, for all $L \in X^*$, $L(y) - f^*(L) = \inf_{x\in X} \big[L(y) - L(x) + f(x)\big] = (f \mathbin{\nabla} L)(y)$ and the inequality "$\ge$" in (67) is obvious from the definition of $f^*(L)$, we only have to prove that

$$f(y) \le \sup\nolimits_{L\in X^*} (f \mathbin{\nabla} L)(y)\big]. \tag{68}$$

Let $\lambda \in \mathbb{R}$ and $\lambda < f(y)$. Since f is proper, there exists $z \in \operatorname{dom} f$. Choose $Q \in \mathcal{S}(X)$ such that

$$Q(z - x) \le 1 \quad\Longrightarrow\quad f(x) > f(z) - 1 \tag{69}$$

and

$$Q(y - x) \le 1 \quad\Longrightarrow\quad f(x) > \lambda. \tag{70}$$

We first prove that

$$(f \mathbin{\nabla} Q)(z) \ge f(z) - 1. \tag{71}$$

To this end, let x be an arbitrary element of X. If $Q(z - x) \le 1$ then (69) implies that $f(x) + Q(z - x) \ge f(x) > f(z) - 1$. If, on the other hand, $Q(z - x) > 1$, let $\gamma := 1/Q(z - x) \in\,]0, 1[$ and put $u := \gamma x + (1 - \gamma)z$. Then $Q(z - u) = \gamma Q(z - x) = 1$ and so, from the convexity of f, and (69) with x replaced by u,

$$\gamma f(x) + (1 - \gamma) f(z) \ge f\big(\gamma x + (1 - \gamma)z\big) = f(u) > f(z) - 1.$$

Substituting in the formula for γ and clearing of fractions yields $f(x) + Q(z - x) \ge f(z)$. This completes the proof of (71).

Now let $M \geq 1$ and $M \geq \lambda + 2 + Q(z-y) - f(z)$. We will prove that

$$(f \mathbin{\nabla} MQ)(y) \geq \lambda. \tag{72}$$

To this end, let x be an arbitrary element of X. If $Q(y-x) \leq 1$ then (70) implies that $f(x) + MQ(y-x) \geq f(x) > \lambda$. If, on the other hand, $Q(y-x) > 1$ then, from (71),

$$\begin{aligned} f(x) + MQ(y-x) &= f(x) + Q(y-x) + (M-1)Q(y-x) \\ &\geq f(x) + Q(z-x) - Q(z-y) + (M-1) \\ &\geq f(z) - 1 - Q(z-y) + M - 1 \geq \lambda, \end{aligned}$$

which completes the proof of (72). The *Hahn–Banach–Lagrange theorem* of [19, Theorem 2.9, p. 153] or [20, Theorem 1.11, p. 21] now provides us with a linear functional L on X such that $L \leq MQ$ on X and $(f \mathbin{\nabla} L)(y) \geq \lambda$, and (68) follows by letting $\lambda \to f(y)$. $\square$

References

[1] R. S. Burachik and B. F. Svaiter, *Maximal monotonicity, conjugation and the duality product*, Proc. Amer. Math. Soc. **131** (2003), 2379–2383.

[2] S. Fitzpatrick, *Representing monotone operators by convex functions*, Workshop/ Miniconference on Functional Analysis and Optimization (Canberra, 1988), 59–65, Proc. Centre Math. Anal. Austral. Nat. Univ., **20**, Austral. Nat. Univ., Canberra, 1988.

[3] J.- P. Gossez, *Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs*, J. Math. Anal. Appl. **34** (1971), 371–395.

[4] J.- P. Gossez, *On a convexity property of the range of a maximal monotone operator*, Proc. Amer. Math. Soc. **55** (1976), 359–360.

[5] M. Marques Alves and B. F. Svaiter, *Brøndsted–Rockafellar property and maximality of monotone operators representable by convex functions in non–reflexive Banach spaces.*, J. of Convex Anal., **15** (2008), 693–706.

[6] —–, *A new old class of maximal monotone operators.*, J. of Convex Anal., **16** (2009), 881–890.

[7] —–, *On Gossez type (D) maximal monotone operators.*, http://arxiv.org/abs/0903.5332v2, posted April 6, 2009, to appear in J. of Convex Anal., **17** (2010).

[8] J.–E. Martínez-Legaz and M. Théra, *ε–Subdifferentials in terms of subdifferentials*, Set–Valued Anal. **4** (1996), 327–332.

[9] —–, *A convex representation of maximal monotone operators*, J. Nonlinear Convex Anal. **2** (2001), 243–247.

[10] J.–P. Penot, *The relevance of convex analysis for the study of monotonicity*, Nonlinear Anal. **58** (2004), 855–871.

[11] R. R. Phelps, *Lectures on Maximal Monotone Operators*, Extracta Mathematicae **12** (1997), 193–230.

[12] R. T. Rockafellar, *Extension of Fenchel's duality theorem for convex functions*, Duke Math. J. **33** (1966), 81–89.

[13] —–, *Convex analysis*, Princeton University Press, Princeton, N.J., 1970.

[14] S. Simons, *The range of a monotone operator*, J. Math. Anal. Appl. **199** (1996), 176–201.
[15] —–, *Minimax and monotonicity*, Lecture Notes in Mathematics **1693** (1998), Springer–Verlag.
[16] —–, *Maximal monotone multifunctions of Brøndsted–Rockafellar type*, Set–Valued Anal. **7** (1999), 255–294.
[17] —–, *Five kinds of maximal monotonicity*, Set–Valued Anal. **9** (2001), 391–409.
[18] —–, *Positive sets and Monotone sets*, J. of Convex Anal., **14** (2007), 297–317.
[19] —–, *The Hahn–Banach–Lagrange theorem*, Optimization, **56** (2007), 149–169.
[20] —–, *From Hahn–Banach to monotonicity*, Lecture Notes in Mathematics, **1693**, second edition, (2008), Springer–Verlag.
[21] —–, *Nonreflexive Banach SSD spaces*, http://arxiv.org/abs/0810.4579v2, posted November 3, 2008.
[22] M. D. Voisei and C. Zălinescu, *Strongly–representable operators*, J. of Convex Anal., **16** (2009), 1011–1033.
[23] D. Zagrodny, *The convexity of the closure of the domain and the range of a maximal monotone multifunction of Type NI*, Set–Valued Anal, **16** (2008), 759–783.
[24] C. Zălinescu, *Convex analysis in general vector spaces*, (2002), World Scientific.