LOCAL ENERGY DECAY FOR SOLUTIONS OF MULTI-DIMENSIONAL ISOTROPIC SYMMETRIC HYPERBOLIC SYSTEMS

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1. Introduction

Decay estimates for solutions to hyperbolic initial value problems play a central role in the perturbative existence theory for nonlinear equations. Traditionally, such estimates are proved either by considering the representation formula for the solution or by using the Fourier transform. The generalized energy method offers an alternative approach in problems which possess enough symmetry. The first examples required Lorentz invariance [4], and more recently Galilean invariant examples from continuum mechanics have been successfully treated using this method [5, 8, 9, 10]. The nonlinear analysis in these works relied on key linear estimates which establish, in an *ad hoc* manner, local energy decay.

This paper attempts to provide a unified view of these local energy decay estimates. The general framework is based on symmetric hyperbolic systems. Many problems can be reduced to this form in combination with a system of constraint equations. The constraints are essential because they rule out time-independent solutions for which decay cannot hold. The other key ingredient is an isotropy assumption on the symbol associated to the problem, guaranteeing the existence of commuting vector fields. An additional artificial dissipation term can be included at no extra cost.

The main result, appearing in section 4, shows that solutions decompose into individual wave families, corresponding to the eigenstates of the symbol. Thanks to the isotropy assumption, the characteristic cones are standard, and the components related to the positive eigenvalues concentrate along these cones. This is reminiscent of the one-dimensional picture where wave families propagate along characteristics. The remaining components, associated to the nonpositive eigenvalues, actually decay uniformly in L^2. In the anisotropic case, solutions still decay uniformly in L^2 in a region which is strictly interior to all characteristic cones, but detailed information along the cones is lost.

Several examples appear in section 5. The prototypical example is the wave equation. Maxwell's equation offers a bit more complexity, and we also briefly discuss the anisotropic case, for which only the weaker decay result holds. The main examples come from continuum mechanics. We consider in detail the constrained first order system that was derived in [10] in the study of the incompressible limit in elastic solids. We obtain a sharp result which we then use to systematically re-derive our previous estimates. The results obtained here will also be applied in a forthcoming paper on nonlinear incompressible elastodynamics [11].

2. Preliminaries

Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional inner product spaces over $\mathbb{R}$. We will be concerned with $\mathcal{V}$-valued strong solutions $u : [0,T) \times \mathbb{R}^n \to \mathcal{V}$ of the linear system

$$L(\partial)u - \nu\Delta u = f \quad \text{with} \quad L(\partial) = \partial_t + A(\nabla), \quad A(\nabla) = A_k\partial_k \tag{1a}$$

together a system of constraints

$$B(\nabla)u = g \quad \text{with} \quad B(\nabla) = B_k\partial_k. \tag{1b}$$

Here, we suppose that the coefficients are constant linear maps

$$A_k \in \mathcal{L}(\mathcal{V},\mathcal{V}), \quad B_k \in \mathcal{L}(\mathcal{V},\mathcal{W}), \quad k = 1,\dots,n$$

and

$$f : [0,T) \times \mathbb{R}^n \to \mathcal{V}, \quad g : [0,T) \times \mathbb{R}^n \to \mathcal{W}$$

are functions whose required regularity will become clear below. The viscosity parameter ν is nonnegative and constant.

The first assumption is the symmetry of the coefficients of (1a) as elements of $\mathcal{L}(\mathcal{V},\mathcal{V})$

$$A_k = A_k^*, \quad k = 1,\dots,n. \tag{2}$$

Associated to the differential operators $A(\nabla)$ and $B(\nabla)$, define the symbols

$$A(\xi) = A_k\xi^k \quad \text{and} \quad B(\xi) = B_k\xi^k, \quad \xi \in \mathbb{R}^n.$$

The second assumption is that

$$\ker B(\xi) \cap \ker A(\xi) = \{0\}, \quad \text{for every} \quad 0 \neq \xi \in \mathbb{R}^n. \tag{3}$$

The third assumption is that there exist smooth maps taking the identity to the identity[1] such that

$$V : SO(\mathbb{R}^n) \to SO(\mathcal{V}) \quad \text{and} \quad W : SO(\mathbb{R}^n) \to \mathcal{L}(\mathcal{W},\mathcal{W})$$

such that for every $\xi \in \mathbb{R}^n$ and $R \in SO(\mathbb{R}^n)$

$$A(R\xi) = V(R)A(\xi)V(R)^* \tag{4a}$$

and

$$B(R\xi) = W(R)B(\xi)V(R)^*. \tag{4b}$$

We will see momentarily that these assumptions imply that, in a certain sense, the system is isotropic and that there exists a useful collection of commuting vector fields. Examples will be given in Section 5.

3. Consequences of the assumptions

Isospectral property.

Lemma 1. *The spectrum of $A(\omega)$ is real and independent of $\omega \in S^{n-1}$.*

[1] In the applications they will be homomorphisms.

Proof. The symmetry of the map $A(\omega)$, from (2), implies that its eigenvalues are real.

Suppose that ω and ω' are distinct points in S^{n-1}. Select $v \in S^{n-1}$ orthogonal to ω (in the standard inner product on $\mathbb{R}^n$) so that ω, ω', and v are linearly dependent. Define the anti-symmetric map $S = \omega \otimes v - v \otimes \omega$. Then $R(t) = \exp tS$ determines a 2π-periodic one-parameter family in $SO(\mathbb{R}^n)$ such that $R(\theta)\omega = \omega'$, for some $\theta \in (0, 2\pi)$. It follows from (4a) that

$$A(R(t)\omega) = V(R(t))A(\omega)V(R(t))^*.$$

Setting $t = \theta$, we find that $A(\omega)$ and $A(\omega')$ are similar, and so they have the same spectrum. □

Let us denote the spectrum of an operator A by $\sigma(A)$.

Corollary 1. *The nonzero eigenvalues of $A(\omega)$ occur in plus/minus pairs.*

Proof. Lemma 1 shows that $\sigma(A(\omega)) = \sigma(A(-\omega)) = \sigma(-A(\omega))$. On the other hand, we have in general that $\sigma(-A(\omega)) = -\sigma(A(\omega))$. □

Invariance property.

Lemma 2. *For any smooth function $u : \mathbb{R}^n \to \mathcal{V}$ and any $R \in SO(\mathbb{R}^n)$, the following hold*

$$A(\nabla)[V(R)u(R^*x)] = V(R)[A(\nabla)u](R^*x), \tag{5a}$$

$$B(\nabla)[V(R)u(R^*x)] = W(R)[B(\nabla)u](R^*x), \tag{5b}$$

$$\Delta[V(R)u(R^*x)] = V(R)[\Delta u](R^*x). \tag{5c}$$

Proof. These equations follow immediately from the chain rule and, in the first two cases, the assumptions (4a),(4b). □

Vector fields. Let $\{e_i\}_{i=1}^n$ be the standard basis on $\mathbb{R}^n$, and define the anti-symmetric maps

$$S_{ij} = e_i \otimes e_j - e_j \otimes e_i \quad 1 \leq i < j \leq n. \tag{6}$$

Then $R_{ij}(\tau) = \exp(\tau S_{ij})$ is a smooth one-parameter family in $SO(\mathbb{R}^n)$. It is natural to consider the vector fields arising as the infinitesimal generators of the invariants

$$\frac{d}{d\tau} V(R_{ij}(\tau))u(R_{ij}(\tau)^*x)|_{\tau=0} = \Omega_{ij}u(x) + Z_{ij}u(x) \equiv \widetilde{\Omega}_{ij}u(x),$$

where $\Omega_{ij} = x^i\partial_j - x^j\partial_i$ are the standard angular momentum operators (note that we have used the fact that $V(I) = I$) and

$$Z_{ij} = \frac{d}{d\tau} V(R_{ij}(\tau))|_{\tau=0} \in \mathcal{L}(\mathcal{V}, \mathcal{V}).$$

Further, we define

$$Y_{ij} = \frac{d}{d\tau} W(R_{ij}(\tau))|_{\tau=0} \in \mathcal{L}(\mathcal{W}, \mathcal{W}).$$

We shall also make use of the scaling vector field

$$S = t\partial_t + r\partial_r.$$

Commutation properties.

Lemma 3. *If u is a sufficiently regular solution to* (1a),(1b)*, then*

$$[L(\partial)-\nu\Delta]\widetilde{\Omega}_{ij}u=\widetilde{\Omega}_{ij}f \tag{7a}$$

and

$$B(\nabla)\widetilde{\Omega}_{ij}u=(\Omega_{ij}+Y_{ij})g. \tag{7b}$$

In addition, for any positive integer p,

$$[L(\partial)-\nu\Delta]S^pu=(S+1)^pf-\sum_{j=0}^{p-1}(-1)^{p-j}\binom{p}{j}\nu\Delta S^ju, \tag{8a}$$

and

$$B(\nabla)S^pu=(S+1)^pg. \tag{8b}$$

Proof. It follows from (5a), (5c) that

$$[L(\partial)-\nu\Delta][V(R_{ij}(\tau))u(t,R_{ij}(\tau)^*x)]=V(R_{ij}(\tau))f(t,R_{ij}(\tau)^*x)$$

and from (5b) that

$$B(\nabla)[V(R_{ij}(\tau))u(t,R_{ij}(\tau)^*x)]=W(R_{ij}(\tau))g(t,R_{ij}(\tau)^*x).$$

The relations (7a), (7b) follow from these by taking the derivative in τ and evaluating at $\tau=0$ (and using $V(I)=I$, $W(I)=I$).

It is easily seen that

$$L(\partial)S=(S+1)L(\partial)\quad\text{and}\quad\Delta S=(S+2)\Delta,$$

and so by induction

$$L(\partial)S^p=(S+1)^pL(\partial)\quad\text{and}\quad\Delta S^p=(S+2)^p\Delta.$$

Therefore, using the binomial theorem, we have

$$\begin{aligned}[L(\partial)-\nu\Delta]S^pu&=[(S+1)^pL(\partial)-(S+2)^p\nu\Delta]u\\&=(S+1)^pf-[(S+2)^p-(S+1)^p]\nu\Delta u\\&=(S+1)^pf-\sum_{j=0}^{p-1}(-1)^{p-j}\binom{p}{j}(S+2)^j\nu\Delta u\\&=(S+1)^pf-\sum_{j=0}^{p-1}(-1)^{p-j}\binom{p}{j}\nu\Delta S^ju.\end{aligned}$$

This proves (8a).

The statement (8b) follows easily from the fact that $B(\nabla)S=(S+1)B(\nabla)$. □

Spectral projections. For each $\omega\in S^{n-1}$, let $\mathcal{P}_\beta(\omega)$ be the orthogonal projection of $\mathcal{V}$ onto the eigenspace of $A(\omega)$ corresponding to the eigenvalue λ_β.

Lemma 4. *The orthogonal projections $\mathcal{P}_\beta(\omega)$ are smooth functions of $\omega=x/|x|$ on S^{n-1} which satisfy the commutation property $[\widetilde{\Omega}_{ij},\mathcal{P}_\beta(\omega)]=0$.*

Proof. The smoothness of $\mathcal{P}_\beta(\omega)$ follows from the formula

$$\mathcal{P}_\beta(\omega) = \frac{1}{2\pi i} \int\limits_{|\zeta-\lambda_\beta|=\rho} (\zeta I - A(\omega))^{-1} d\zeta, \tag{9}$$

where ρ is chosen so that $|\lambda_\beta - \lambda_{\beta'}| > \rho$ for $\beta \neq \beta'$.

We have from (4a) and (9) that

$$V(R)\mathcal{P}_\beta(R^*\omega) = \mathcal{P}_\beta(\omega)V(R),$$

for every $R \in SO(\mathbb{R}^n)$. Therefore, we obtain

$$\begin{aligned}\widetilde{\Omega}_{ij}[\mathcal{P}_\beta(\omega)u(x)] &= \frac{d}{d\tau}[V(R_{ij}(\tau))\mathcal{P}_\beta(R_{ij}(\tau)^*\omega)u(R_{ij}(\tau)^*x)]|_{\tau=0} \\ &= \frac{d}{d\tau}[\mathcal{P}_\beta(\omega)V(R_{ij}(\tau))u(R_{ij}(\tau)^*x)]|_{\tau=0} \\ &= \mathcal{P}_\beta(\omega)\widetilde{\Omega}_{ij}u(x).\end{aligned}$$

□

Plane waves. Consider a plane wave solution of the operator $L(\partial)$:

$$u(t,x) = \phi(\lambda_\beta t - \langle\omega, x\rangle)\psi_\beta(\omega),$$

in which $\psi_\beta(\omega)$ is an eigenvector of $A(\omega)$ for λ_β. Our assumptions imply that the propagation speed λ_β is independent of the direction of propagation ω and that a rotation R of the propagation direction produces a corresponding rotation $V(R)$ of the eigenspace of the polarization vector $\psi_\beta(\omega)$. In this sense, the system is isotropic.

4. Main result

Throughout the remainder of the paper, we regard the projections $\{\mathcal{P}_\beta(\omega)\}$ onto the eigenspaces of $A(\omega)$ as homogeneous functions of degree zero on $\mathbb{R}^n$, by setting $\omega = x/|x|$.

Theorem 1. *Let $n \geq 2$ and $j = 1, \dots, n$. Assume that conditions* (2) *and* (3) *hold. There are positive constants α and C, depending on the coefficients A_k and B_k, such that all sufficiently regular solutions of* (1a), (1b) *satisfy the estimate*

$$\begin{aligned}&\alpha t\|\partial_j u\|_{L^2(\{r\leq\alpha t\},\mathcal{V})} + (\nu t)^{1/2}\|\nabla u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \nu t\|\Delta u\|_{L^2(\mathbb{R}^n,\mathcal{V})} \\ &\qquad \leq C\|u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|Su\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|f\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|g\|_{L^2(\mathbb{R}^n,\mathcal{W})}.\end{aligned}\tag{10}$$

If, in addition, conditions (4a), (4b) *hold, then*

$$\begin{aligned}&\|(\lambda_\beta t - r)\mathcal{P}_\beta\partial_j u\|_{L^2(\{r\geq\alpha t\},\mathcal{V})} \\ &\qquad \leq C\left[\|\widetilde{\Omega}u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|u\|_{L^2(\mathbb{R}^n,\mathcal{V})}\right] + \|Su\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|f\|_{L^2(\mathbb{R}^n,\mathcal{V})},\end{aligned}\tag{11}$$

and

$$\|rB(\omega)\partial_j u\|_{L^2(\{r\geq\alpha t\},\mathcal{W})} \leq C\left[\|\widetilde{\Omega}u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|u\|_{L^2(\mathbb{R}^n,\mathcal{V})}\right] + \|rg\|_{L^2(\mathbb{R}^n,\mathcal{W})}.\tag{12}$$

The proof will be given below, but first we isolate the key step.

Lemma 5. *Let $n \geq 2$. Suppose that conditions* (2) *and* (3) *hold. All sufficiently regular solutions of* (1a) *satisfy the estimate*

$$\|(tA(\nabla) - r\partial_r)u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} + (n-2)\nu t\|\nabla u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} + (\nu t)^2\|\Delta u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} \leq \|Su - tf\|^2_{L^2(\mathbb{R}^n,\mathcal{V})}.$$

Proof of Lemma 5. Here and later on we may assume that solutions are smooth and that they decay rapidly at infinity. A density argument can then be used to pass to solutions for which the norms appearing are finite.

Using the definition of S, we may rewrite (1a) as

$$tA(\nabla)u - r\partial_r u - \nu t\Delta u = -Su + tf.$$

Taking the L^2-norm of both sides, this immediately gives

$$\|tA(\nabla)u - r\partial_r u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} + 2\langle r\partial_r u - tA(\nabla)u, \nu t\Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} + \|\nu t\Delta u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} \leq \|Su - tf\|^2_{L^2(\mathbb{R}^n,\mathcal{V})}.$$

Thanks to the symmetry of the coefficient matrices, we find using integration by parts that

$$\langle A(\nabla)u, \Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} = \langle A_k\partial_k u, \Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} = 0.$$

Again using integration by parts, we can rewrite the remaining cross term as follows:

$$\begin{aligned}
2\langle r\partial_r u, \nu t\Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} =& 2\nu t\langle x^j\partial_j u, \Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})}\\
=& -2\nu t\langle x^j\partial_j\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} - 2\nu t\langle\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})}\\
=& n\nu t\langle\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} - 2\nu t\langle\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})}\\
=& (n-2)\nu t\langle\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})}\\
=& (n-2)\nu t\|\nabla u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})}.
\end{aligned}$$

□

We now continue with the proof of Theorem 1.

Proof of (10). Let $n \geq 2$, and suppose that (2) and (3) hold.

By (3), the expression $|A(\omega)u|^2_{\mathcal{V}} + |B(\omega)u|^2_{\mathcal{W}}$ vanishes if and only if $u = 0$. In other words, the map $A(\omega)^2 + B(\omega)^*B(\omega)$ in $\mathcal{L}(\mathcal{V},\mathcal{V})$ is positive definite and symmetric, by (2). If we let

$$(3\alpha)^2 = \min\{\lambda : \lambda \in \sigma(A(\omega)^2 + B(\omega)^*B(\omega)) \text{ for some } \omega \in S^{n-1}\},$$

then

$$(3\alpha)^2|u|^2_{\mathcal{V}} \leq |A(\omega)u|^2_{\mathcal{V}} + |B(\omega)u|^2_{\mathcal{W}},$$

for all $u \in \mathcal{V}$ and $\omega \in S^{n-1}$. Therefore, using the Fourier transform, we obtain

$$3\alpha\|\nabla u\|_{L^2(\mathbb{R}^n,\mathcal{V})} \leq \|A(\nabla)u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|B(\nabla)u\|_{L^2(\mathbb{R}^n,\mathcal{W})}, \tag{13}$$

for all sufficiently regular functions $u : \mathbb{R}^n \to \mathcal{V}$.

Introduce a cut-off function $\zeta \in C^\infty(\mathbb{R})$ with $0 \leq \zeta \leq 1$ and

$$\zeta(s) = \begin{cases} 1, & \text{if } s \leq 1 \\ 0, & \text{if } s \geq 2. \end{cases}$$

Fixing α as in (13), define $\eta_\alpha(t,r) = \zeta(r/(\alpha t))$.

Let u solve (1a), (1b), and set $v = \eta_\alpha u$, so that v is supported in $\{r \le 2\alpha t\}$. By (13), we obtain

$$\begin{aligned}3\alpha t\|\nabla v\|_{L^2(\mathbb{R}^n,\mathcal{V})} &\le t\|A(\nabla)v\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|B(\nabla)v\|_{L^2(\mathbb{R}^n,\mathcal{V})}\\ &\le \|(tA(\nabla) - r\partial_r)v\|_{L^2(\mathbb{R}^n,\mathcal{V})} + 2\alpha t\|\nabla v\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|B(\nabla)v\|_{L^2(\mathbb{R}^n,\mathcal{V})}.\end{aligned}$$

This yields the bound

$$\alpha t\|\nabla v\|_{L^2(\mathbb{R}^n,\mathcal{V})} \le \|(tA(\nabla) - r\partial_r)v\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|B(\nabla)v\|_{L^2(\mathbb{R}^n,\mathcal{V})}. \tag{14}$$

Since $(\alpha t + r)|\partial_j\eta_\alpha| \le C$, we have from (14)

$$\begin{aligned}&\alpha t\|\partial_j u\|_{L^2(\{r\le\alpha t\},\mathcal{V})}\\ &\quad\le \alpha t\|\eta_\alpha\partial_j u\|_{L^2(\mathbb{R}^n,\mathcal{V})}\\ &\quad\le \alpha t\|\partial_j v\|_{L^2(\mathbb{R}^n,\mathcal{V})} + C\|u\|_{L^2(\mathbb{R}^n,\mathcal{V})}\\ &\quad\le \|(tA(\nabla) - r\partial_r)v\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|B(\nabla)v\|_{L^2(\mathbb{R}^n,\mathcal{V})} + C\|u\|_{L^2(\mathbb{R}^n,\mathcal{V})}\\ &\quad\le \|(tA(\nabla) - r\partial_r)u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|B(\nabla)u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + C\|u\|_{L^2(\mathbb{R}^n,\mathcal{V})}.\end{aligned}$$

The dependence of the constant C on the coefficients occurs upon differentiation of the cut-off function in the last inequality.

The estimate (10) now follows by an application of Lemma 5 and (1b). □

Proof of (11). Let $n \ge 2$ and suppose that (2), (3), and (4a), (4b) hold.

Using the orthogonal projections and the vector fields Ω_{ij} defined in Section 3, we have the pointwise estimate

$$\begin{aligned}&|(\lambda_\beta t - r)\mathcal{P}_\beta(\omega)\partial_j u(t,x)|_{\mathcal{V}}\\ &\qquad= |\mathcal{P}_\beta(\omega)(tA(\omega) - rI)\partial_j u(t,x)|_{\mathcal{V}}\\ &\qquad\le |(tA(\omega) - rI)\partial_j u(t,x)|_{\mathcal{V}}\\ &\qquad= |(tA_k - r\omega^k I)\omega^k\partial_j u(t,x)|_{\mathcal{V}}\\ &\qquad= |(tA_k - r\omega^k I)(\omega^j\partial_k + \frac{1}{r}\Omega_{kj})u(t,x)|_{\mathcal{V}}\\ &\qquad= \left|\left[\omega^j(tA(\nabla) - r\partial_r) + (tA_k - r\omega^k I)\frac{1}{r}\Omega_{kj}\right]u(t,x)\right|_{\mathcal{V}}\\ &\qquad\le |(tA(\nabla) - r\partial_r)u(t,x)|_{\mathcal{V}} + C\left|\frac{t}{r} + 1\right||\Omega u(t,x)|_{\mathcal{V}}.\end{aligned}$$

The inequality (11) follows from this after an integration over the region $\{r \ge \alpha t\}$ and an application of Lemma 5.

□

Note that away from the origin, the symbol is used with the physical variables in proving this last result.

Proof of (12). Let u solve (1b). As above, we have

$$\begin{aligned}rB(\omega)\partial_j u(t,x) &= B_k x^k\partial_j u(t,x)\\ &= B_k[\Omega_{kj} + x^j\partial_k]u(t,x)\\ &= B_k\Omega_{kj}u(t,x) + x^j B(\nabla)u(t,x),\end{aligned}$$

from which (12) follows immediately. □

Time derivatives. Time derivatives can also be estimated using the pde, however the quantity $t+r$ appears as a weight with the inhomogeneity in the exterior region.

Corollary 2. *Let $n \geq 2$ and $j = 1, \dots, n$. Assume that conditions* (2) *and* (3) *hold. There are positive constants α and C, depending on the coefficients A_k and B_k, such that all sufficiently regular solutions of* (1a), (1b) *satisfy the estimate*

$$(15) \quad \alpha t \|\partial_t u\|_{L^2(\{r\leq\alpha t\},\mathcal{V})} \leq C\|u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|Su\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|f\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|g\|_{L^2(\mathbb{R}^n,\mathcal{W})}.$$

If, in addition, conditions (4a), (4b) *hold and also $\nu = 0$, then*

$$(16) \quad \|(\lambda_\beta t - r)\mathcal{P}_\beta \partial_t u\|_{L^2(\{r\geq\alpha t\},\mathcal{V})} \leq C\left[\|\widetilde{\Omega} u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|u\|_{L^2(\mathbb{R}^n,\mathcal{V})}\right] + \|Su\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|(t+r)f\|_{L^2(\mathbb{R}^n,\mathcal{V})},$$

Proof. The inequality (15) is easily shown by rewriting (1a) as

$$\partial_t u = -A(\nabla)u + \nu\Delta u + f,$$

and then applying (10).

Now let $nu = 0$. Inequality (16) follows from (11) applied to Noticing that

$$\partial_t u = -A(\nabla)u + f = A(\omega)\partial_r u + \mathcal{O}(\frac{1}{r}|\Omega u|) + f,$$

we now get (16) from (11). ∎

Higher-order estimates. Let $\Gamma = (\Gamma_1, \dots, \Gamma_q)$ denote our list of $q = n+1+n(n-1)/2$ vector fields ∂_j, S, $\widetilde{\Omega}_{ij}$. Let $a = (a_1, \dots, a_m)$ be an m-tuple in $\{1, \dots, q\}$. We denote by Γ^a the m^{th}-order operator $\Gamma_{a_1} \cdots \Gamma_{a_m}$. From Lemma 3, it follows that if u is a sufficiently regular solution of (1a), (1b), then $\Gamma^a u$ satisfies a system of the same type. Although we will not write the estimates down explicitly, it is clear that Theorem 1 can be applied to obtain weighted estimates for the higher-order derivatives $\partial_j \Gamma^a u$.

The utility of the estimates resulting from Theorem 1 (and its higher-order version) depends on being able to control norms of form $\sum_{|a|\leq\ell} \|\Gamma^a u(t,\cdot)\|_{L^2(\mathbb{R}^n,\mathcal{V})}$. This can be accomplished using the standard energy method thanks to the symmetry of the coefficients (2) and the commutation properties from Lemma 3.

5. Examples

In the following examples, the artificial viscosity term has been dropped, since the physically interesting cases include more complicated dissipative mechanisms that do not fall within the framework of our result.

Wave equation. Let's look a simple test case, namely the wave equation for $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$:

$$\partial_t^2 \phi - c^2 \Delta\phi = h.$$

The first step is to rewrite the problem in the form (1a), (1b). The setup uses $\mathcal{V} = \mathbb{R}^{n+1}$ with the inner product $\text{diag}(c^{-2}, 1, \dots, 1)$. Also, let $\{e_a\}_{a=0}^n$ be the standard basis on $\mathcal{V}$.

The wave equation can be written in first order form (1a) (with $\nu = 0$) in the standard way:

$$u = u^a e_a = \partial_t \phi e_0 + \partial_k \phi e_k, \quad f = h e_0,$$

and

$$-A_k = c^2 e_0 \otimes e_k + e_k \otimes e_0, \quad k = 1, \dots, n.$$

Notice that the symmetry condition (2) holds in the chosen inner product.

Let $\omega = (\omega^k) \in S^{n-1}$, and set $\hat{\omega} = \omega^k e_k \in \mathbb{R}^{n+1}$. Then

$$A(\omega) = -(c^2 e_0 \otimes \hat{\omega} + \hat{\omega} \otimes e_0).$$

Given $R \in SO(\mathbb{R}^n)$, we define $V(R) \in SO(\mathcal{V})$ by $V(R)e_0 = e_0$ and $V(R)e_i = Re_i$, $i \neq 0$. Then

$$A(R\omega) = -(c^2 e_0 \otimes V(R)\hat{\omega} + V(R)\hat{\omega} \otimes e_0) = V(R)A(\omega)V(R)^*,$$

and we have verified (4a).

In coordinates, the constraints are simply

$$\partial_\ell u^m - \partial_m u^\ell = 0, \quad 1 \leq \ell < m \leq n.$$

In order to express this in the form (1b), with $g = 0$, we use (6) in defining

$$\mathcal{W} = \operatorname{span}\{S_{ij} : 1 \leq i, j \leq n\} \subset \mathcal{L}(\mathcal{V}, \mathcal{V}).$$

and then define $B_k \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, $k = 1, \dots, n$, by

$$u \mapsto B_k u = \tfrac{1}{2} \sum_{\ell,m} \langle u, S_{\ell m} e_k \rangle_{\mathcal{V}} S_{\ell m}.$$

In particular, we have

$$B(\omega)u = \tfrac{1}{2} \sum_{\ell,m} \langle u, S_{\ell m} \hat{\omega} \rangle_{\mathcal{V}} S_{\ell m}, \tag{17}$$

and therefore,

$$\begin{aligned} B(R\omega)V(R)u =& \tfrac{1}{2} \sum_{\ell,m} \langle V(R)u, S_{\ell m} V(R)\hat{\omega} \rangle_{\mathcal{V}} S_{\ell m} \\ =& \tfrac{1}{2} \sum_{\ell,m} \langle u, V(R)^* S_{\ell m} V(R)\hat{\omega} \rangle_{\mathcal{V}} S_{\ell m}. \end{aligned}$$

Since $V(R)^* S_{\ell m} V(R)$ is antisymmetric, it lies in the span of the S_{ij}, and so we see that the last expression has the form $W(R)B(\omega)u$, for some $W(R) \in \mathcal{L}(\mathcal{W}, \mathcal{W})$. This verifies (4b).

Clearly $\psi_\pm(\omega) = (ce_0 \mp \hat{\omega})/\sqrt{c^2+1}$ serve as unit eigenvectors of $A(\omega)$ with eigenvalues $\lambda_\pm = \pm c$. The remaining eigenvalues are all zero, and

$$\ker A(\omega) = \{e_0, \hat{\omega}\}^\perp.$$

On the other hand, we have from (17) that $\ker B(\omega) = \operatorname{span}\{e_0, \hat{\omega}\} = \ker A(\omega)^\perp$.

Having thus verified the hypotheses of Theorem 1, using (11), we get bounds for

$$\|(r \mp ct)\langle \psi_\pm(\omega), \partial_j u \rangle_{\mathcal{V}}\|_{L^2(\{r > \alpha t\})}, \tag{18a}$$

and from (10) we have

$$\alpha t \|\nabla u\|_{L^2(\{r < \alpha t\})}. \tag{18b}$$

In terms of the original variable, we get from (18a) an estimate for

$$\|(r \pm ct)(c^{-1}\partial_j \partial_t \phi \pm \partial_j \partial_r \phi)\|_{L^2(\{r > \alpha t\})},$$

but since

$$\partial_i = \omega^i \partial_r - \frac{1}{r} \sum_{j=1}^n \omega^j \Omega_{ij},$$

this implies a bound for

$$\|(r \pm ct)(\omega_i c^{-1}\partial_t \pm \partial_i)\partial_j \phi)\|_{L^2(\{r>\alpha t\})}.$$

Using (18b), this, in turn, implies bounds for

$$\|(r \pm ct)(\omega_i c^{-1}\partial_t \pm \partial_i)\partial_j \phi)\|_{L^2(\mathbb{R}^n)}.$$

This estimate for first derivatives has been derived by Alinhac using what he calls the ghost weighted energy method, [1]. Bounds of the type were also given in [12]. From here, we easily recover the weaker estimates for

$$\|(r-ct)\partial_t\partial_j\phi\|_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \|(r-ct)\partial_i\partial_j\phi\|_{L^2(\mathbb{R}^n)}$$

given in [5].

Maxwell's equation. Maxwell's system for an electromagnetic field $(E, H) \in \mathbb{R}^3 \times \mathbb{R}^3$ takes the form

$$\begin{aligned}\varepsilon\partial_t E - c\nabla \wedge H &= J\\ \mu\partial_t H + c\nabla \wedge E &= 0\end{aligned}$$

with the constraints

$$\begin{aligned}\nabla \cdot \varepsilon E &= \rho\\ \nabla \cdot \mu H &= 0.\end{aligned}$$

The current density J and the charge density ρ are regarded as known inhomogeneities. For simplicity, we shall set the light speed c, the permittivity ε, and the magnetic permeability μ equal to unity.

Taking $\mathcal{V} = \mathbb{R}^3 \times \mathbb{R}^3$ with the standard inner product, the system verifies the assumptions of Theorem 1, however we shall not go through the details of the formalism in this case.

In this case, the symbol is given by

$$A(\omega)u = (-\omega \wedge H, \omega \wedge E).$$

The nonzero eigenvalues coincide with the propagation speeds $\lambda_\pm = \pm 1$. Here and later on, it will be convenient to use the notation

$$P(\omega) = \omega \otimes \omega, \quad Q(\omega) = I - \omega \otimes \omega. \tag{19}$$

The orthogonal projections onto the eigenspaces for $\lambda_\pm$ are then

$$\mathcal{P}_\pm(\omega)(E,H) = (1/2)(Q(\omega)E \mp \omega \wedge H, Q(\omega)H \pm \omega \wedge E),$$

and the projection onto the zero eigenspace is

$$\mathcal{P}_0(\omega)(E,H) = (P(\omega)E, P(\omega)H).$$

Application of Theorem 1 gives a bound for

$$\|(t-r)[Q(\omega)\partial_j E - \omega \wedge \partial_j H]\|_{L^2} + \|(t-r)[Q(\omega)\partial_j H + \omega \wedge \partial_j E]\|_{L^2}.$$

The remaining components decay uniformly in L^2, see also [1, 2, 3].

Anisotropic Maxwell Equation. In the anisotropic case, where the permittivity has the form $\varepsilon = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, the conditions (2) and (3) hold with the inner product

$$\langle (E,H),(E,' H')\rangle_{\mathcal{V}} = \langle \varepsilon E, E'\rangle_{\mathbb{R}^3} + \mu \langle H, H'\rangle_{\mathbb{R}^3}$$

However, the conditions (4a) and (4b) fail. Still, the interior estimate (10) of Theorem 1 holds giving a uniform bound for

$$t\left(\|\partial_j E\|_{L^2(r<ct/2)} + \|\partial_j H\|_{L^2(r<ct/2)}\right),$$

as well as for higher derivatives. Thus, via the Sobolev lemma, we obtain decay in $L^\infty(r < t/2)$ at a rate of t^{-1}. This improves slightly upon the result [6] where it was shown that solutions decay uniformly in $L^\infty(\mathbb{R}^3)$ at a rate of $t^{-1/2}$.

Linearized elasticity. The motion of an elastic body is typically described by a one-parameter family of orientation preserving deformations $x(t,X)$ taking a point X in the reference configuration to its position x at time t. The reference map is the inverse $X(t,x)$. For the case of homogeneous isotropic materials, a first order system was derived in [10] for the couple $(H(t,x), v(t,x), \rho(t,x))$ where

$$H^i_\ell(t,x) = \partial_\ell X^i(t,x), \quad v(t,x) = D_t x(t,X)|_{X=X(t,x)}, \quad \rho(t,x) = \det H(t,x).$$

Here H is the inverse of the deformation gradient, v is the velocity, and ρ is (proportional to) the density. The natural vector space to describe the motion in these variables is

$$\mathcal{V} = (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}.$$

The linearized equations of motion take the form

$$\partial_t H + \nabla v = f_H \tag{20a}$$

$$\partial_t v + \nabla \cdot TH + \mu^2 \nabla \rho = f_v \tag{20b}$$

$$\partial_t \rho + \nabla \cdot v = f_\rho, \tag{20c}$$

in which $T \in \mathcal{L}(\mathbb{R}^3 \otimes \mathbb{R}^3, \mathbb{R}^3 \otimes \mathbb{R}^3)$ is defined by

$$TH = c_2^2 H + (c_1^2 - c_2^2) \operatorname{tr} H\, I, \tag{20d}$$

and

$$(\nabla \cdot TH)^i = \partial_\ell (TH)^i_\ell.$$

The material parameters are assumed to satisfy $c_1 > c_2 > 0$ and $\mu > 0$. As a consequence, the mapping T is positive definite and symmetric on $\mathbb{R}^3 \otimes \mathbb{R}^3$, with the standard inner product, and T induces a new inner product on $\mathbb{R}^3 \otimes \mathbb{R}^3$ through

$$\langle H, \bar{H}\rangle_T = \langle TH, \bar{H}\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3} = \operatorname{tr}[(TH)\bar{H}^*].$$

The inner product on $\mathcal{V}$ is defined by

$$\begin{aligned}\langle u, \bar{u}\rangle_{\mathcal{V}} &= \langle (H,v,\rho),(\bar{H},\bar{v},\bar{\rho})\rangle_{\mathcal{V}} \\ &= \langle H, \bar{H}\rangle_T + \langle v, \bar{v}\rangle_{\mathbb{R}^3} + \mu^2 \rho\bar{\rho}.\end{aligned}$$

The equations (20a)-(20c) are equivalent to (1a) with the symbol $A(\omega) \in \mathcal{L}(\mathcal{V},\mathcal{V})$ defined by

$$A(\omega)u = A(\omega)(H,v,\rho) = (v \otimes \omega, TH\omega + \mu^2 \rho\omega, \langle v, \omega\rangle_{\mathbb{R}^3}). \tag{21}$$

The inhomogeneity is $f = (f_H, f_v, f_\rho)$. Thus defined, the symbol $A(\omega)$ (and hence also each A_k) satisfies the symmetry condition (2).

The constraint equations are

$$\partial_m H^i_\ell - \partial_\ell H^i_m = 0, \quad \text{and} \quad \nabla(\rho - \text{ tr } H) = g_\rho. \tag{22}$$

The second constraint is simply the linearization of the relation $\rho = \det H$.

The constraint system can be formulated as in (1b) using

$$\mathcal{W} = (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Using the maps (6), the symbol $B(\omega) \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ then given by

$$B(\omega)u = B(\omega)(H, v, \rho) = \sum_{\ell,m,n} \langle H, e_n \otimes S_{\ell m}\omega\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3}\ (S_{\ell m}, e_n, 0) + (\rho - \text{ tr } H)(0, 0, \omega).$$

Define the map $V : SO(\mathbb{R}^3) \to \mathcal{L}(\mathcal{V}, \mathcal{V})$ by

$$V(R)u = V(R)(H, v, \rho) = (RHR^*, Rv, \rho), \quad R \in SO(\mathbb{R}^3), \quad u \in \mathcal{V}.$$

It is straightforward to verify that, in fact, $V : SO(\mathbb{R}^3) \to SO(\mathcal{V})$.

The map V also satisfies the conditions (4a), (4b). For example, let us verify (4b). By the definitions

$$\begin{aligned} B(R\omega)V(R)u &= \sum_{\ell,m,n} \langle RHR^*, e_n \otimes S_{\ell m}R\omega\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3}\ (S_{\ell m}, e_n, 0) \\ &\qquad + (\rho - \text{ tr } RHR^*)(0, 0, R\omega) \\ &= \sum_{\ell,m,n} \langle H, R^*e_n \otimes R^*S_{\ell m}R\omega\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3}\ (S_{\ell m}, e_n, 0) \\ &\qquad + (\rho - \text{ tr } H)(0, 0, R\omega) \end{aligned}$$

Since $R^*S_{\ell m}R$ is antisymmetric, we see that the last expression above depends linearly on the coordinates of $B(\omega)u$. This implies the existence of a map $W(R) \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ for which (4b) is valid.

We now verify (3) by showing that that $\ker A(\omega) = \ker B(\omega)^\perp$. We have that

$$\ker A(\omega) = \{u = (H, v, \rho) : TH\omega + \mu^2\rho\omega = 0,\ v = 0\}$$

and, using the notation (19) again,

$$\begin{aligned} \ker B(\omega) &= \{u = (H, v, \rho) : HS_{\ell m}\omega = 0,\ \ell, m = 1, 2, 3,\ \rho = \text{ tr } H\} \\ &= \{u = (H, v, \rho) : H = HP(\omega),\ \rho = \text{ tr } H\}. \end{aligned}$$

Take $u = (H, v, \rho) \in \ker A(\omega)$ and $\bar{u} = (\bar{H}, \bar{v}, \bar{\rho}) \in \ker B(\omega)$. Then

$$\begin{aligned} \langle u, \bar{u}\rangle_{\mathcal{V}} &= \langle H, \bar{H}\rangle_T + \mu^2\rho\bar{\rho} \\ &= \langle H, \bar{H}P(\omega)\rangle_T + \mu^2\rho\bar{\rho} \\ &= \langle TH, \bar{H}P(\omega)\rangle_{\mathbb{R}^3\otimes\mathbb{R}^3} + \mu^2\rho\bar{\rho} \\ &= \langle TH\omega, \bar{H}\omega\rangle_{\mathbb{R}^3} + \mu^2\rho\bar{\rho} \\ &= \mu^2\rho[-\langle\omega, \bar{H}\omega\rangle_{\mathbb{R}^3} + \bar{\rho}] \\ &= \mu^2\rho[-\text{ tr } \bar{H}P(\omega) + \bar{\rho}] \\ &= \mu^2\rho[-\text{ tr } \bar{H} + \bar{\rho}] \\ &= 0. \end{aligned}$$

This shows that $\ker A(\omega) \subset \ker B(\omega)^\perp$.

In the other direction, suppose that $u = (H, v, \rho) \in \ker B(\omega)^\perp$. Note that if

$$\bar{u} = (\bar{H}, \bar{v}, \bar{\rho}) = (THP(\omega) + \mu^2 \rho P(\omega), v, \text{ tr } THP(\omega) + \mu^2 \rho),$$

then $\bar{u} \in \ker B(\omega)$. So we have

$$\begin{aligned}
0 &= \langle u, \bar{u} \rangle_{\mathcal{V}} \\
&= \langle TH, THP(\omega) + \mu^2 \rho P(\omega) \rangle_{\mathbb{R}^3 \otimes \mathbb{R}^3} + \langle v, v \rangle_{\mathbb{R}^3} + \mu^2 \rho(\text{ tr } THP(\omega) + \mu^2 \rho) \\
&= |TH\omega|^2_{\mathbb{R}^3} + 2\mu^2 \rho \langle TH\omega, \omega \rangle_{\mathbb{R}^3} + \mu^4 \rho^2 + |v|^2_{\mathbb{R}^3} \\
&= |TH\omega + \mu^2 \rho \omega|^2_{\mathbb{R}^3} + |v|^2_{\mathbb{R}^3},
\end{aligned}$$

and thus, $u \in \ker A(\omega)$.

We have shown that the system (20a),(20b),(20c) can be rewritten so as to satisfy the assumptions of Theorem 1.

The nonzero eigenvalues of $A(\omega)$, representing slow and fast propagation speeds, are

$$\lambda_s^\pm = \pm \lambda_s = \pm c_2, \quad \text{and} \quad \lambda_f^\pm = \pm \lambda_f = \pm (c_1^2 + \mu^2)^{1/2}.$$

The nonzero eigenspaces form a 6 dimensional subspace of $\mathcal{V}$. Using (19) again, the corresponding orthogonal projections are

$$\mathcal{P}_s^\pm(\omega) u = \frac{1}{2} \left(Q(\omega) H P(\omega) \pm \lambda_s^{-1} Q(\omega) v \otimes \omega, \pm \lambda_s Q(\omega) H \omega + Q(\omega) v, 0 \right),$$

and

$$\mathcal{P}_f^\pm(\omega) u = \langle u, z^\pm(\omega) \rangle_{\mathcal{V}} \, z^\pm(\omega), \quad z^\pm(\omega) = \frac{1}{\sqrt{2}\lambda_f} \left(P(\omega), \pm \lambda_f \omega, 1 \right)$$

We let $\mathcal{P}_0(\omega)$ be the remaining orthogonal projection onto $\ker A(\omega)$. Theorem 1 ensures that the quanties

$$\|(\lambda_s t \mp r) \mathcal{P}_s^\pm \partial_j u\|_{L^2(\mathbb{R}^3, \mathcal{V})}, \tag{23a}$$

$$\|(\lambda_f t \mp r) \mathcal{P}_f^\pm \partial_j u\|_{L^2(\mathbb{R}^3, \mathcal{V})}, \tag{23b}$$

$$\|(t + r) \mathcal{P}_0 \partial_j u\|_{L^2(\mathbb{R}^3, \mathcal{V})}, \tag{23c}$$

are bounded by

$$C[\|\Gamma u\|_{L^2(\mathbb{R}^3, \mathcal{V})} + \|u\|_{L^2(\mathbb{R}^3, \mathcal{V})} + t\|f\|_{L^2(\mathbb{R}^3, \mathcal{V})} + \|(t + r) g\|_{L^2(\mathbb{R}^3, \mathcal{V})}]. \tag{24}$$

We do not attempt to make explicit the dependence of the constant C on the material parameters. All components of the solution decay uniformly in L^2 except for those in the 3 dimensional subspace corresponding to the positive eigenvalues.

These estimates can be unravelled to a less precise, but still useful form. Set

$$\mathcal{P}_s(\omega) = \mathcal{P}_s^+(\omega) + \mathcal{P}_s^-(\omega), \quad \text{and} \quad \mathcal{P}_f(\omega) = \mathcal{P}_f^+(\omega) + \mathcal{P}_f^-(\omega),$$

and notice that

$$\mathcal{P}_s(\omega) u = (Q(\omega) H P(\omega), Q(\omega) v, 0),$$

and

$$\mathcal{P}_f(\omega) u = \langle u, y(\omega) \rangle_{\mathcal{V}} \, y(\omega) + (0, P(\omega) v, 0), \quad y(\omega) = \frac{1}{\lambda_f} (P(\omega), 0, 1).$$

Then by (23b) and (23a), we obtain that the quantities

$$\begin{aligned}(25a)\quad &\|(\lambda_s t - r)\mathcal{P}_s\partial_j u\|^2_{L^2(\mathbb{R}^3,\mathcal{V})}\\ &= \|(\lambda_s t - r)Q\partial_j HP\|^2_{L^2(\mathbb{R}^3,\mathcal{V})} + \|(\lambda_s t - r)Q\partial_j v\|^2_{L^2(\mathbb{R}^3,\mathcal{V})},\end{aligned}$$

and

$$\begin{aligned}(25b)\quad &\|(\lambda_f t - r)\mathcal{P}_f\partial_j u\|^2_{L^2(\mathbb{R}^3,\mathcal{V})}\\ &= \|(\lambda_f t - r)\langle\partial_j u, y\rangle_{\mathcal{V}}\|^2_{L^2(\mathbb{R}^3)} + \|(\lambda_f t - r)P\partial_j v\|^2_{L^2(\mathbb{R}^3,\mathcal{V})},\end{aligned}$$

are also bounded by the square of (24). In particular, we see that the longitudinal and transverse components of the velocity v concentrate along the fast and slow cones, respectively.

Further simplification comes from taking into account the component of u in the kernel. We have

$$\begin{aligned}(26)\quad |\mathcal{P}_0(\omega)u|^2_{\mathcal{V}} &= |(I - \mathcal{P}_f(\omega) - \mathcal{P}_s(\omega))u|^2_{\mathcal{V}}\\ &= |u|^2_{\mathcal{V}} - |\mathcal{P}_f(\omega)u|^2_{\mathcal{V}} - |\mathcal{P}_s(\omega)u|^2_{\mathcal{V}}\\ &= |H|^2_T + |v|^2_{\mathbb{R}^3} + \mu^2\rho^2 - (\langle u, y(\omega)\rangle^2_{\mathcal{V}} + |P(\omega)v|^2_{\mathbb{R}^3})\\ &\quad - (|Q(\omega)HP(\omega)|^2_T + |Q(\omega)v|^2_{\mathbb{R}^3})\\ &= c_2^2|HQ(\omega)|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} + [c_2^2(c_1^2 - c_2^2)/\lambda_f^2][\operatorname{tr} HQ(\omega)]^2\\ &\quad + [\mu^2(c_1^2 - c_2^2)/\lambda_f^2][\operatorname{tr} H - \rho]^2 + [c_2^2\mu^2/\lambda_f^2][\operatorname{tr} HP(\omega) - \rho]^2.\end{aligned}$$

Now, since we have that

$$\lambda_f\rho = \langle y(\omega), u\rangle_{\mathcal{V}} + (c_1^2/\lambda_f)(\rho - \operatorname{tr} HP(\omega)) - [(c_1^2 - c_2^2)/\lambda_f] \operatorname{tr} HQ(\omega),$$

it follows from (26), (23c), and(25b) that

$$\|(\lambda_f t - r)\lambda_f\partial_j\rho\|_{L^2(\mathbb{R}^3)} \tag{27}$$

is bounded by (24).

Next, writing

$$\operatorname{tr} HP(\omega) = \rho + (\operatorname{tr} HP(\omega) - \rho),$$

we deduce from (27), (26), and (23c) that

$$\|(\lambda_f t - r)\lambda_f^{-1} \operatorname{tr} \partial_j HP\|_{L^2(\mathbb{R}^3)} \tag{28}$$

is bounded by (24). Going back to (25b), we find that the same bound holds for $\operatorname{tr} \partial_j H$.

Since

$$P(\omega)H = P(\omega)HP(\omega) + P(\omega)HQ(\omega) = (\operatorname{tr} HP(\omega))P(\omega) + P(\omega)HQ(\omega),$$

we have

$$|P(\omega)H|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} = (\operatorname{tr} HP(\omega))^2 + |P(\omega)HQ(\omega)|^2_{\mathbb{R}^3\otimes\mathbb{R}^3},$$

and therefore from (27), (26), and (23c),

$$\|(\lambda_f t - r)\lambda_f^{-1}P\partial_j H\|_{L^2(\mathbb{R}^3)}$$

is bounded by (24).

Finally, since

$$|Q(\omega)H|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} = |Q(\omega)HP(\omega)|^2_{\mathbb{R}^3\otimes\mathbb{R}^3} + |Q(\omega)HQ(\omega)|^2_{\mathbb{R}^3\otimes\mathbb{R}^3},$$

we find from (26), (25a), and (23c) that

$$\|(\lambda_s t - r) Q\partial_j H\|_{L^2(\mathbb{R}^3)}$$

is bounded by (24).

We summarize these results in

Corollary 3. *Let u be a C^1 solution of* (20a)-(20c), (22). *Then the quantity*

$$\|(\lambda_s t - r)(Q\partial_j H, Q\partial_j v, 0)\|_{L^2(\mathbb{R}^3,\mathcal{V})} + \|(\lambda_f t - r)(\lambda_f^{-1} P\partial_j H, P\partial_j v, \rho)\|_{L^2(\mathbb{R}^3,\mathcal{V})}$$

is bounded by (24).

With a bit more care, one can show that the constant in (24) grows linearly with λ_f. This estimate was used in [10] in the study of the incompressible limit, $\mu \to 0$.

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