NONLINEAR HYPERBOLIC SYSTEMS AND ELASTODYNAMICS

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1. Introduction

These notes offer an informal exposition of some recent results on the existence of global solutions to the equations of motion for isotropic elastic solids. The discussion includes a brief account of the physical background of the the equations with an emphasis on their field theoretic structure as well as an outline of developments in the theory of nonlinear systems of hyperbolic partial differential equations leading up to the case of elastodynamics.

We begin with a bare description of the dynamical problem for elastic deformations as a field equation arising via the principal of stationary action from a Lagrangian. This approach has several advantages. It allows for a simple introduction to the notions of Galilean invariance and material symmetry and their consequences for the equations of motion. It easily accommodates internal constraints, such as incompressibility, through the inclusion of Lagrange multipliers. It also yields a transparent description of the formalism behind the *generalized energy method* with the use of the energy-momentum tensor and the derivation of commuting vector fields as infinitesimal generators of the symmetry transformations.

The theory of elasticity is an old subject, about which much has been written. The books of Gurtin [4], Marsden-Hughes [16], and Ogden [18] all serve as fine introductions to the subject.

After the background material, there follow the statements of the main results concerning the existence of solutions to the initial value problem for the nonlinear system of isotropic elastodynamics in the case of small displacements from equilibrium without boundaries. The main feature in elasticity, distinguishing it from the case of scalar nonlinear wave equations in 3D, is the presence of two propagation speeds for the linearized system corresponding to fast pressure waves and slow

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shear waves. For the construction of global solutions to the nonlinear problem, an important structural assumption, the *null condition*, must be imposed on the nonlinearity. Well-known in the context of nonlinear wave equations, the null condition turns out to be compatible with the structure of the equations of isotropic elasticity. It can be explained as a nonresonance condition which eliminates the leading order production of waves of a given family through the nonlinear interaction of waves of the same type. For shear waves, it holds automatically, while for pressure waves it can be imposed with the loss of only one additional degree of freedom.

The first main result (Theorem 1), which originally appeared in [21], asserts the existence of global small solutions under the null condition. The other main result (Theorem 3) gives the corresponding global existence result for incompressible materials. Such materials can be thought of as a limiting case where the speed of the pressure waves tends to infinity and at the same time their amplitude tends to zero, and this is precisely how we construct the solutions. However, to implement this strategy, we first need a stability result for an appropriate class of slightly compressible materials which says that the lifespan of solutions tends to infinity with the speed of the pressure waves (Theorem 2). This is not routine because the degree of smallness of the displacement required in Theorem 1 is not uniform in the propagation speeds. In this case, the null condition for the pressure waves is not imposed since, in any case, we can only construct long time local solutions to the approximate equations and in the limit there are only shear waves. Theorems 2 and 3 appeared in [22].

The rest of the article develops the main ideas used in the proofs of Theorems 1 and 2. Energy estimates form the core of the arguments, but in order to obtain global or even long time local existence, they must be supplemented by dispersive estimates. For the scalar case, this is most easily done using Lorentz invariance and the generalized Sobolev lemma. For Galilean invariant systems with multiple speeds an additional series of weighted estimates based on scaling is required to obtain the dispersive estimate because of the weakened version of the Sobolev inequality. The elasticity system presents further difficulties through the form of the nonlinear coupling, since the wave families are nonlocal functions of the displacement.

Although these methods, originally developed in the context of nonlinear wave equations, exhibit a certain degree of flexibility, it is not clear how much further they can be pushed. For example, the introduction of boundaries or other sorts of anisotropy causes difficulties. The case of fluids likewise remains untouchable because of the degeneracy of the shear wave speed.

2. A brief formulation of elastodynamics

2.1. **Deformations.** A *body* or *reference configuration* is a (possibly unbounded) regular domain $\mathcal{B} \subset \mathbb{R}^3$. Points $X \in \mathcal{B}$ are called *material points.*

A *deformation* of $\mathcal{B}$ is a smooth, one-to-one, orientation preserving map $x : \mathcal{B} \to x(\mathcal{B}) \subset \mathbb{R}^3$. The domain $x(\mathcal{B})$ is called the *deformed configuration.* The *displacement* from the reference configuration is measured by $u(X) = x(X) - X$.

The *deformation gradient* is defined by $F = Dx$, or in coordinates, $F^i_\ell = D_\ell x^i$. Since x is one-to-one and orientation preserving, we must have $J(X) \equiv \det F(X) > 0$, for every $X \in \mathcal{B}$. We denote by $\mathbf{GL}^3_+$ the admissible class for the deformation gradient, namely the set of 3×3 matrices over $\mathbb{R}$ with positive determinant.

A deformation is said to be *incompressible* if it preserves volumes, that is, $\operatorname{vol}(\Omega) = \operatorname{vol}(x(\Omega))$, for all subdomains $\Omega \subset \mathcal{B}$. This is equivalent to having $J(X) = 1$, for every $X \in \mathcal{B}$. An example of an incompressible deformation is *simple shear*, given by the linear map $x(X) = [I + \alpha\, e_1 \otimes e_2]X$, with $e_k \in \mathbb{R}^3$ being the standard unit vectors.

2.2. **Motions.** A *motion* of a body $\mathcal{B}$ is a C^2 map

$$x : [0, T) \times \mathcal{B} \to \mathbb{R}^3$$

such that $x(t, \cdot)$ is a deformation for each $t \in [0, T)$.

For $X \in \mathcal{B}$, the curve $t \mapsto x(t, X)$ is called the *path* or the *trajectory* of the material point X. The set of trajectories will be denoted by $\mathcal{T}$. Thus, $\mathcal{T}$ is the image of the cylinder $\mathcal{C} = [0, T) \times \mathcal{B}$ under the mapping $(t, X) \mapsto (t, x(t, X))$, which, by the inverse function theorem, is a diffeomorphism. The inverse map will be written as $(t, x) \mapsto (t, X(t, x))$. It is called the *reference map.* Notice that the lateral boundary of $\mathcal{T}$ is free; it depends on the particular motion.

The coordinates (t, X) on $\mathcal{C}$ are called *material* or *Lagrangian* coordinates, while (t, x) on $\mathcal{T}$ are known as the *spatial* or *Eulerian* coordinates. Material fields can be converted to spatial fields, and vice versa, using the diffeomorphism between $\mathcal{C}$ and $\mathcal{T}$.

We shall use the notation (D_t, D) for material derivatives on $\mathcal{C}$, and $\partial = (\partial_t, \nabla)$ for derivatives in the spatial coordinates on $\mathcal{T}$.

2.3. **The velocity field.** Let $x(t,X)$ be a motion. The *material velocity* is $D_t x(t,X)$, the tangent vector along the particle trajectory. The *spatial velocity* is defined as

$$v(t,x) = D_t x(t, X(t,x)).$$

Recalling that $F = Dx$, we have by the chain rule

$$\begin{aligned} D_t F(t,X) &= D\ D_t x(t,X) \\ &= D[v(t,x(t,X))] \\ &= \nabla v(t,x(t,X)) F(t,X). \end{aligned} \tag{2.1}$$

Regarding this as a linear system for F, it follows that $J = \det F$ satisfies

$$\begin{aligned} D_t J(t,X) &= \mathrm{tr}\,[\nabla v(t,x(t,X))] J(t,X) \\ &= \nabla \cdot v(t,x(t,X)) J(t,X). \end{aligned} \tag{2.2}$$

Therefore, a motion is incompressible if and only if $J(0,X) = 1$ on $\mathcal{B}$, and $\nabla \cdot v(t,x) = 0$ on $\mathcal{T}$.

2.4. **Conservation of mass.** A *mass distribution* or *density* on $\mathcal{B}$ is a nonnegative C^1 function $\rho_0(X)$ on $\mathcal{B}$. Given a subdomain $\Omega \subset \mathcal{B}$, define

$$\mathrm{mass}(\Omega) = \int_\Omega \rho_0(X) dX.$$

Let $\Omega_t = x(t,\Omega)$ be the image of Ω under some motion. Since Ω_t consists of the same collection of material points as Ω, conservation of mass demands that

$$\mathrm{mass}(\Omega_t) = \mathrm{mass}(\Omega).$$

We define a function $\rho(t,x)$ on $\mathcal{T}$ by first giving its material description

$$\rho(t, x(t,X)) = \rho_0(X)/J(t,X).$$

Using a change of variables, it is easy to see that

$$\mathrm{mass}(\Omega_t) = \int_{\Omega_t} \rho(t,x) dx,$$

and thus, $\rho(t,\cdot)$ serves as a density on each time slice of $\mathcal{T}$. It also follows from (2.2) and the chain rule, that ρ satisfies the *continuity equation*

$$\partial_t \rho + v \cdot \nabla \rho + \rho\ \nabla \cdot v = 0.$$

2.5. **Equations of motion for elastodynamics.** The equations are most conveniently derived via the principle of stationary action. We will reconcile this approach with a more traditional balance of forces argument based on Newton's law in the next section.

An elastic material is characterized by a smooth *strain energy* function

$$W : \mathcal{B} \times \mathbf{GL}^3_+ \to \mathbb{R}_+.$$

For motions $x(t, X)$, define the Lagrangian

$$\mathcal{L}[x] = \int_0^T \int_{\mathcal{B}} [\tfrac{1}{2}\rho_0(X)|D_t x(t, X)|^2 - W(X, Dx(t, X))] dX dt.$$

The corresponding Euler-Lagrange equations are

$$\rho_0(X) D_t^2 x^i(t, X) - D_\ell [S^i_\ell(X, Dx(t, X))] = 0,$$

in which

$$S^i_\ell(X, F) = \frac{\partial W}{\partial F^i_\ell}(X, F)$$

is the *Piola-Kirchhoff* stress.

From now on we shall assume that the material is *homogeneous*, so that

$$\rho_0(X) = \rho_0, \quad W(X, F) = W(F), \quad S(X, F) = S(F) = \frac{\partial W}{\partial F}(F).$$

In addition, we impose the standard conditions that the reference configuration $F = I$ has minimum energy and is stress free

$$W(I) = 0, \quad S(I) = 0.$$

With the notation

$$[D \cdot S(Dx)]^i = D_\ell [S^i_\ell(X, Dx(t, X))]$$

the equations of motion take the form

$$\rho_0 D_t^2 x - D \cdot S(Dx) = 0.$$

2.6. **Equations of motion via balance of forces.** Here we shall give a brief derivation of the equations of motion using a classical balance of forces argument. This will provide us with a physical interpretation for the Piola-Kirchhoff stress.

Suppose that $\Omega \subset \mathcal{B}$ is any subregion within the body, and let $\Omega_t = x(t, \Omega)$ be its image under the motion. According to Newton's Law, the rate of change of the momentum of the subregion Ω_t must balance with the total force acting on it. The implicit assumption is that there is a set of spatial coordinates (an inertial frame) in which these quantities can be measured.

The forces on Ω_t consist of internal contact forces acting on $\partial\Omega_t$ and external body forces. We shall ignore the latter, as we have already done in the previous section. The *Cauchy Stress Principle* states that surface contact forces can be expressed as

$$\int_{\partial\Omega_t} \tau(t,x,N_{\Omega_t}(x))d\Sigma(\partial\Omega_t),$$

in which $N_{\Omega_t}(x)$ is the unit outward normal at the point $x \in \partial\Omega_t$ and $d\Sigma(\partial\Omega_t)$ is surface measure. The stress vector τ is measured in units of force/area. *Cauchy's Theorem* then states that there is a symmetric matrix $T(t,x)$, called the *Cauchy* stress tensor, such that

$$\tau(t,x,N) = T(t,x)N.$$

Balance of linear momentum requires that

$$\frac{d}{dt}\int_{\Omega_t} \rho v(t,x)dx = \int_{\partial\Omega_t} T(t,x)N_{\Omega_t}(x)d\Sigma(\partial\Omega_t), \tag{2.3}$$

and since this must hold for every subdomain, this implies the balance law

$$\rho(\partial_t v + v\cdot\nabla v) - \nabla\cdot T = 0.$$

In material coordinates, (2.3) can be expressed as

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} &\rho_0 D_t x(t,X)dX \\ &= \int_{\partial\Omega} J(t,X)T(t,x(t,X))F(t,X)^{-T}N_{\Omega}(X)d\Sigma(\Omega),\end{aligned}$$

with $F = Dx$ and $J = \det F$. This is equivalent to the equation

$$\rho_0 D_t^2 x - D\cdot(JTF^{-T}) = 0.$$

Comparing this with the previous section, we find that

$$T(t,x(t,X)) = \hat{T}(F(t,X)) \equiv J^{-1}S(F)F^T(t,X). \tag{2.4}$$

Thus, the Lagrangian formulation includes the constitutive assumption that the Cauchy stress depends on the deformation only through its gradient. The Piola-Kirchhoff stress S allows the internal surface contact forces to be expressed in the reference configuration. To fully reconcile these two formulations however, further assumptions must be placed on the strain energy function W so that in (2.4) S is consistent with the symmetry of T.

The traditional balance of forces argument leads to a more general system of equations, but only the variational (*hyperelastic*) case conserves energy and can be adapted to include thermodynamics.

2.7. **Natural boundary conditions.** Having introduced the idea of surface stress, we can now pause to write down a typical initial boundary value problem. With homogeneity and no external forces, the equations of motion in material coordinates are

$$\rho_0 D_t^2 x - D\cdot S(Dx) = 0, \quad \text{on} \quad [0,T)\times\mathcal{B}.$$

In the absence of external forces, there are no contact forces on $\partial\mathcal{B}$, so we should have

$$S^i_\ell(Dx)N_\ell = 0, \quad \text{on} \quad [0,T)\times\partial\mathcal{B},$$

with N the normal on $\partial\mathcal{B}$. And finally the initial deformation $x(0,X)$ and velocity $D_t x(0,X)$ should be given on $\mathcal{B}$ so as to be consistent with the boundary conditions.

Of course, for this to be a well-posed initial boundary value problem, further assumptions on W are needed to ensure the hyperbolicity of the pde's.

The basic local existence theory for problems of this type has been thoroughly investigated, see [15], [6].

2.8. **Galiliean invariance.** Classical mechanics is characterized by the Galiliean invariance of equations of motion. This means that the transformation

$$x(t,X)\mapsto x^*(t,X) = x_0 + Ux(t-t_0,X), \quad U\in\mathbf{O}^3_+$$

leaves solutions invariant. Here, $\mathbf{O}^3_+$ is the rotation group:

$$\mathbf{O}^3_+ = \{U\in\mathbf{GL}^3_+ : U^T = U^{-1},\ \det U = 1\}.$$

This property holds provided that the strain energy function satisfies

$$W(F) = W(UF), \quad F\in\mathbf{GL}^3_+, \quad U\in\mathbf{O}^3_+, \tag{2.5}$$

and so we now add (2.5) to our list of assumptions.

By the polar decomposition theorem, any $F\in\mathbf{GL}^3_+$ can be written as

$$F = U\sqrt{R}, \quad R = F^TF, \quad U\in\mathbf{O}^3_+.$$

Thus by (2.5), we have

$$W(F) = W(U\sqrt{R}) = W(\sqrt{R}) \equiv \hat{W}(R). \tag{2.6}$$

Differentiating this last equation with respect to F gives

$$S^i_\ell = \frac{\partial W}{\partial F^i_\ell} = \frac{\partial\hat{W}}{\partial R_{ab}}\frac{\partial R_{ab}}{\partial F^i_\ell} = 2F^i_a\frac{\partial\hat{W}}{\partial R_{a\ell}},$$

This shows that, under the assumption of Galilean invariance, the Cauchy stress $\hat{T}$ in (2.4) is indeed symmetric, as required.

We have already defined

$$S^i_\ell(F) = \frac{\partial W}{\partial F^i_\ell}(F).$$

For future reference, we introduce the following notation for the higher derivatives

$$A^{ij}_{\ell m}(F) = \frac{\partial^2 W(F)}{\partial F^i_\ell \partial F^j_m}$$
$$B^{ijk}_{\ell mn}(F) = \frac{\partial^3 W(F)}{\partial F^i_\ell \partial F^j_m \partial F^k_n}.$$

Since the order of differentiation can be interchanged, the tensors $A^{ij}_{\ell m}(F)$ and $B^{ijk}_{\ell mn}(F)$ are symmetric with respect to the interchange of pairs of indices.

Taking successive derivatives in F of the relation (2.5) yields the following important symmetry properties of these derivatives

$$\begin{aligned} S^i_\ell(F) &= S^I_\ell(UF)U^{Ii} \\ A^{ij}_{\ell m}(F) &= A^{IJ}_{\ell m}(UF)U^{Ii}U^{Jj} \qquad\qquad F \in \mathbf{GL}^3_+, \quad U \in \mathbf{O}^3_+ \\ B^{ijk}_{\ell mn}(F) &= B^{IJK}_{\ell mn}(UF)U^{Ii}U^{Jj}U^{Kk}. \end{aligned}$$

2.9. **Material symmetry.** Let $\varphi : \mathcal{B} \to \mathcal{B}$ be a volume preserving deformation with $\varphi(X_0) = X_0$ for some $X_0 \in \mathcal{B}$. Set $G = D\varphi(X_0)$. Then we have

$$G \in \mathbf{SL}^3 \equiv \{G \in \mathbf{GL}^3_+ : \det G = +1\}.$$

Suppose that $x(t,X)$ is any motion, and define the new motion $x^*(t,X) = x(t,\varphi(X))$. We say that G is a *symmetry* of $\mathcal{B}$ (at X_0) if

$$W(Dx(t,X_0)) = W(Dx^*(t,X_0)),$$

for every motion x. This is equivalent to

$$W(F) = W(FG), \quad F \in \mathbf{GL}^3_+.$$

Since we have assumed that W is homogeneous, the set of symmetries does not depend on the point X_0. The set of symmetries of $\mathcal{B}$ forms a subgroup $\mathcal{G} \subset \mathbf{SL}^3$. A material for which $\mathcal{G} \subset \mathbf{O}^3_+$ is called an *elastic solid*. A *perfect fluid* is characterized by the condition $\mathcal{G} = \mathbf{SL}^3$.

2.9.1. *Isotropic solids.* A solid is *isotropic* if $\mathcal{G} = \mathbf{O}^3_+$, which means that the material has no preferred directions. Rubber and steel are examples of isotropic solids. In this case we have,

$$W(F) = W(FU), \quad F \in \mathbf{GL}^3_+, \quad U \in \mathbf{O}^3_+. \tag{2.7}$$

Remembering (2.6), we can also write

$$\hat{W}(R) = \hat{W}(U^T R U), \quad R = F^T F, \quad F \in \mathbf{GL}^3_+, \quad U \in \mathbf{O}^3_+.$$

Now R is a positive definite symmetric matrix. This last equation implies that $\hat{W}$ depends on R only through its principal invariants i_α, that is, the elementary symmetric functions in the eigenvalues λ_α of R.

$$\begin{aligned}
\mathsf{i}_1 &= \lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}\ R \\
\mathsf{i}_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \tfrac{1}{2}[(\mathrm{tr}\ R)^2 - \mathrm{tr}\ (R^2)] \\
\mathsf{i}_3 &= \lambda_1\lambda_2\lambda_3 = \det R = (\det F)^2 = J^2.
\end{aligned}$$

Since we are going to eventually consider small displacements from the reference configuration, it is convenient to introduce the invariants j_α of $R - I$ which are linearly related to i_α through

$$\begin{aligned}
\mathsf{i}_1 &= 3 + \mathsf{j}_1 \\
\mathsf{i}_2 &= 3 + 2\mathsf{j}_1 + \mathsf{j}_2 \\
\mathsf{i}_3 &= 1 + \mathsf{j}_1 + \mathsf{j}_2 + \mathsf{j}_3.
\end{aligned}$$

This discussion shows that for isotropic (and Galilean invariant) solids, the strain energy function depends on F through the invariants j_α of $R - I$:

$$W(F) = \hat{W}(R) = \tilde{W}(\mathsf{j}).$$

Using the relation (2.4) we can now compute the form of the Cauchy stress for isotropic materials

$$\hat{T}(F) = \varphi_0(\mathsf{j})I + \varphi_1(\mathsf{j})P + \varphi_3(\mathsf{j})P^2, \quad P = FF^T.$$

The presence of the last two terms on the right means that in general the stress vector TN will have a nonzero tangential component, called the *shear stress*.

For future reference, we record the following new set of symmetry relations which follow from (2.7) by differentiation in F:

$$\begin{aligned}
S^i_\ell(F) &= S^i_L(FU)U^{\ell L} \\
A^{ij}_{\ell m}(F) &= A^{ij}_{LM}(FU)U^{\ell L}U^{mM}, \qquad\qquad F \in \mathbf{GL}^3_+, \quad U \in \mathbf{O}^3_+ \\
B^{ijk}_{\ell mn}(F) &= B^{ijk}_{LMN}(FU)U^{\ell L}U^{mM}U^{nN}.
\end{aligned}$$

2.9.2. *Perfect fluids.* As stated above, a perfect fluid is a material with the largest possible symmetry group $\mathcal{G} = \mathbf{SL}^3$. Thus, we have

$$W(F) = W(FG), \quad F \in \mathbf{GL}^3_+, \quad G \in \mathbf{SL}^3.$$

However, given $F \in \mathbf{GL}^3_+$ we can always write $F = J^3 I \cdot J^{-3} F$. Since $J^{-3}F \in \mathbf{SL}^3$, we have that

$$W(F) = W(J^3 I) = \tilde{W}(J).$$

Again calculating from (2.4), we obtain

$$\hat{T}(F) = \tilde{W}'(J) I.$$

Thus, perfect fluids do not support shear stresses.

2.10. **Linearized isotropic elasticity.** We are going to study small displacements, $u(t,X) = x(t,X) - X$, from the reference configuration, and so an important first step is to understand the structure of the linearized equations about the trivial solution $x(t,X) = X$.

Using the notation introduced above, the equations of motion can be written as

$$\rho_0 D_t^2 x^i - A^{ij}_{\ell m}(F) D_\ell D_m x^j = 0.$$

We have already noted the symmetry of $A^{ij}_{\ell m}(F)$ with respect to pairs of indices, and because of the form of the pde, we may assume without loss of generality that $A^{ij}_{\ell m}(F)$ is also symmetric in the lower indices. But then symmetry in the upper indices holds as well.

Then the linearized equations are

$$\rho_0 D_t^2 u^i - A^{ij}(D) u^j = 0, \quad A^{ij}(D) = A^{ij}_{\ell m}(I) D_\ell D_m.$$

We will show that the coefficients in the isotropic case have only two degrees of freedom.

The coefficients $A^{ij}_{\ell m}(I)$ can be determined by considering rank one displacements

$$F^\varepsilon = I + \varepsilon \xi \otimes \eta, \quad 0 \neq \xi, \eta \in \mathbb{R}^3.$$

Using the formulas, the invariants of $R^\varepsilon - I$ are

$$\begin{aligned}
\mathsf{j}_1^\varepsilon &= 2\varepsilon\langle \xi, \eta\rangle + \varepsilon^2 |\xi|^2|\eta|^2 \\
\mathsf{j}_2^\varepsilon &= -\varepsilon^2 |\xi \wedge \eta|^2 = -\varepsilon^2[|\xi|^2|\eta|^2 - \langle \xi, \eta\rangle^2] \\
\mathsf{j}_3^\varepsilon &= 0,
\end{aligned}$$

and we have that

$$W(F^\varepsilon) = \tilde{W}(\mathsf{j}^\varepsilon). \tag{2.8}$$

Recall that the reference configuration is stress free: $S(I) = 0$. Therefore, we see that

$$D_\varepsilon W(F^\varepsilon)|_{\varepsilon=0} = S^i_\ell(I)\xi_i\eta_\ell = 0, \quad \xi,\eta\in\mathbb{R}^3.$$

Using the other side of (2.8), we get

$$0 = D_\varepsilon \tilde{W}(\mathsf{j}^\varepsilon)|_{\varepsilon=0} = 2\langle\xi,\eta\rangle\tilde{W}_1(0).$$

Thus, $\tilde{W}_1(0) = 0$, the subscript denoting the derivative in j_1.

Next, in order to fix the type of the linearized equation, we impose the Legendre-Hadamard ellipticity condition on the operator $A(D)$:

$$D^2_\varepsilon W(F^\varepsilon)|_{\varepsilon=0} = A^{ij}_{\ell m}(I)\eta_i\eta_j\xi_\ell\xi_m > 0, \quad 0\neq\xi,\eta\in\mathbb{R}^3,$$

which has the physical interpretation that performing the deformation $F^\varepsilon X$ costs a positive amount of energy. By (2.8), this implies the condition

$$D^2_\varepsilon\tilde{W}(\mathsf{j}^\varepsilon)|_{\varepsilon=0} = 4\langle\xi,\eta\rangle^2\tilde{W}_{11}(0) - 2[|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2]\tilde{W}_2(0) > 0, \qquad 0\neq\xi,\eta\in\mathbb{R}^3.$$

Thus, we assume that

$$\tilde{W}_{11}(0) > 0 \quad\text{and}\quad -\tilde{W}_2(0) > 0.$$

This assumption makes sense only for solids, because in the case of perfect fluids, $W_2(0) = 0$. Setting

$$\rho_0 c_1^2 = 4W_{11}(0) \quad\text{and}\quad \rho_0 c_2^2 = -2\tilde{W}_2(0),$$

we see that

$$A^{ij}(D) = A^{ij}_{\ell m}(I)D_\ell D_m = \rho_0(c_1^2 - c_2^2)D_iD_j + \rho_0 c_2^2\delta_{ij}\Delta.$$

Finally, consider the family of deformations λX, $\lambda > 0$. Again supposing that this deformation costs energy when $\lambda\neq 1$, we obtain the condition

$$D^2_\lambda W(\lambda I)|_{\lambda=1} = \tfrac{3}{4}c_1^2 - c_2^2 > 0.$$

In conclusion, the linearized equation in the isotropic case is given by

$$(Lu)^i = \rho_0[D_t^2u^i - c_2^2\Delta u^i - (c_1^2 - c_2^2)D_iD_ju^j] = 0, \quad c_1^2 > \tfrac{4}{3}c_2^2 > 0,$$

and this equation is hyperbolic.

We remark that the linearized equation about any dilation $x(X) = \lambda X$ has the same form as above, but other backgrounds lead to complicated characteristic geometry, see [17], [26].

2.11. **Plane wave solutions.** A plane wave with orientation $\bar{u}$ propagating in the direction $\xi \in S^2$ with speed c is a function of the form

$$u(t, X) = f(ct + \langle \xi, X \rangle)\bar{u}, \quad \xi \in S^2.$$

By direct substitution, such a plane wave will solve the linearized equation $Lu = 0$ if and only if $\bar{u}$ is an eigenvector of the matrix

$$A^{ij}(\xi) = A^{ij}_{\ell m}(I)\xi_\ell \xi_m = c_2^2 \delta_{ij} + (c_1^2 - c_2^2)\xi_i \xi_j$$

with eigenvalue c. For a given propagation direction $\xi \in S^2$, there are two such wave families $\mathcal{W}_a(\xi)$, $a = 1, 2$.

The first of these occurs when $c = c_1$ and $\bar{u} = \xi$. Thus, we have

$$\mathcal{W}_1(\xi) = \{f(c_1 t + \langle \xi, X \rangle)\xi\}.$$

Since the wave orientation is aligned with the direction of propagation, we shall refer to these as *pressure* waves.

In the second case, we have $c = c_2$ and $\langle \bar{u}, \xi \rangle = 0$, so that

$$\mathcal{W}_2(\xi) = \{f(c_2 t + \langle \xi, X \rangle)\bar{u} : \langle \bar{u}, \xi \rangle = 0\}.$$

Because the wave orientation is orthogonal to the propagation direction, these are called *shear* waves.

2.12. **The null condition.** The null condition is a compatibility or nonresonance condition between the linearized operator and the nonlinearity. To explain this we need to consider the quadratic portion of the nonlinearity in the equations. Since

$$A^{ij}_{\ell m}(F) = A^{ij}_{\ell m}(I) + B^{ijk}_{\ell mn}(I)(F - I)^k_n + \cdots,$$

we can expand the equations to second order in the displacement u

$$D_t^2 u^i - A^{ij}_{\ell m}(I) D_\ell D_m u^j = B^{ijk}_{\ell mn}(I) D_n u^k D_\ell D_m u^j + \cdots.$$

The missing terms involve nonlinearities in u of at least third order.

For smooth maps define

$$N^i(u, v) = B^{ijk}_{\ell mn}(I) D_n u^k D_\ell D_m v^j.$$

The *null condition* can be formulated as follows:

$$\langle N(u, v), w \rangle_{\mathbb{R}^3} \equiv 0, \quad (u, v, w) \in \mathcal{W}_a(\xi)^3, \quad a = 1, 2.$$

This says that the nonlinear interaction of two elementary waves u, v produces no additional waves of the same type.

The connection between plane waves and the null condition for the nonlinear wave equation was first noted by Shatah, see [10].

Written out explicitly, the conditions become

$$B^{ijk}_{\ell mn}(I)\xi_i\xi_j\xi_k\xi_\ell\xi_m\xi_n = 0, \quad \xi \in S^2$$
$$B^{ijk}_{\ell mn}(I)\eta^{(1)}_i\eta^{(2)}_j\eta^{(3)}_k\xi_\ell\xi_m\xi_n = 0, \quad \xi, \eta^{(\alpha)} \in S^2, \quad \langle \xi, \eta^{(\alpha)}\rangle = 0,$$

for $a = 1, 2$ respectively.

For $a = 1$, this can be expressed as

$$D^3_\varepsilon W(F^\varepsilon)|_{\varepsilon=0} = 0, \quad F^\varepsilon = I + \varepsilon\xi \otimes \xi.$$

On the other hand, the invariants j^ε of $(F^\varepsilon)^T F^\varepsilon - I$ are $\mathsf{j}^\varepsilon_1 = 2\varepsilon + \varepsilon^2$, $\mathsf{j}^\varepsilon_2 = \mathsf{j}^\varepsilon_3 = 0$. Thus, we have

$$0 = D^3_\varepsilon \tilde{W}(\mathsf{j}^\varepsilon)|_{\varepsilon=0} = 8(\tilde{W}_{11}(0) + \tilde{W}_{111}(0)).$$

When $a = 2$, the condition is

$$D_{\varepsilon_1}D_{\varepsilon_2}D_{\varepsilon_3}W(F^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)})|_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)=0} = 0, \quad F^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} = I + (\varepsilon_\alpha\eta^{(\alpha)}) \otimes \xi.$$

In this case, the invariants $\mathsf{j}^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}$ are quadratic in $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, and so the third derivatives of $\tilde{W}(\mathsf{j}^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)})$ vanish for $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0$.

So for isotropic materials the null condition for pressure waves ($a = 1$) is fulfilled provided that $\tilde{W}_{11}(0) + \tilde{W}_{111}(0) = 0$, and the null condition for shear waves ($a = 2$) is automatically satisfied without further restrictions.

2.13. **Reformulation as a first order system.** Given a motion $x(t, X)$, let $X(t, x)$ be the reference map, and define

$$H(t, x) = \nabla X(t, x).$$

As explained before, we can use the deformation x and the reference map X to switch between material and spatial fields. Here is a brief summary.

Material to Spatial Dictionary

ρ_0/J	ρ
$D_t x$	v
F	H^{-1}
$J^{-1}S(F)F^T$	$T = \hat{T}(H^{-1}) = \tilde{T}(H)$
D_t	$\partial_t + v\cdot\nabla$
D_ℓ	$H^k_\ell \partial_k$
$F^k_\ell D_k$	∂_ℓ
$\rho_0^{-1} D_\ell S^i_\ell(F)$	$\rho^{-1}\partial_\ell \tilde{T}^i_\ell(H)$

Recall that in Lagrangian coordinates, the equations of motion are

$$\rho_0 D_t^2 x - D\cdot S(Dx) = 0.$$

In spatial coordinates, we have the first order system

$$\begin{aligned}
&\rho(\partial_t v + v\cdot\nabla v) - \nabla\cdot\tilde{T}(H) = 0\\
&\partial_t\rho + v\cdot\nabla\rho + \rho\,\nabla\cdot v = 0\\
&\partial_t H + v\cdot\nabla H + H\nabla v = 0.
\end{aligned}$$

These equations have already been discussed. The first expresses conservation of linear momentum, the second conservation of mass, and the third is the evolution of F^{-1} in spatial coordinates arising from (2.1). This system is equivalent to the original one under the natural constraints

$$\partial_\ell H^i_k = \partial_k H^i_\ell, \quad \rho = \det H. \tag{2.9}$$

In the special case of a perfect fluid, we have $\tilde{T}(H) = h(\rho)I$, so we recover the compressible Euler equations. The last equation is then unnecessary.

2.14. **Incompressible motion.** As we have seen, incompressible motion is characterized by the internal constraint $J = \det Dx = 1$. This can be imposed by adding a Lagrange multiplier term to the variational problem. Thus we consider the modified Lagrangian

$$\mathcal{L}[x] = \int_0^t \int_{\mathcal{B}} [\tfrac{1}{2}\rho_0|D_t x|^2 - W(Dx) + \Lambda(J-1)]dXdt,$$

with $\Lambda(t,X)$ a Lagrange multiplier. The corresponding Euler-Lagrange equations are

$$\rho_0 D_t^2 x^i - D_\ell[S^i_\ell(Dx)] + (F^{-T})^i_\ell D_\ell \Lambda = 0,$$

with the constraint $J = 1$.

Using the dictionary, this can also be converted to a first order system in Eulerian coordinates:

$$\rho_0(\partial_t v + v\cdot\nabla v) - \nabla\cdot\tilde{T}(H) + \nabla\Lambda = 0, \tag{2.10}$$

$$\nabla\cdot v = 0, \tag{2.11}$$

$$\partial_t H + v\cdot\nabla H + H\nabla v = 0. \tag{2.12}$$

In the case of fluids, $\nabla\cdot\tilde{T}(H)$ is a gradient, and so we recover the incompressible Euler equations.

3. Global existence results

3.1. **Summary of assumptions on the strain energy.** We pause to collect our assumptions on W.

- Homogeneity

$$W(X,F) = W(F) \geq 0, \quad \rho_0(X) = \rho_0 > 0$$

- Stress free reference configuration of minimum energy

$$W(I) = 0, \quad S(I) = 0$$

- Galilean invariance

$$W(F) = W(UF), \quad F \in \mathbf{GL}^3_+, \quad U \in \mathbf{O}^3_+$$

- Isotropy

$$W(F) = W(FU), \quad F \in \mathbf{GL}^3_+, \quad U \in \mathbf{O}^3_+$$

- Legendre-Hadamard condition

$$D^2_\varepsilon W(I + \varepsilon\eta\otimes\xi)|_{\varepsilon=0} > 0, \quad 0 \neq \eta,\xi \in \mathbb{R}^3$$

- Null condition

$$D^3_\varepsilon W(I + \varepsilon\xi\otimes\xi)|_{\varepsilon=0} = 0, \quad \xi \in \mathbb{R}^3$$

3.2. **Global existence – compressible case.** The following is a special case of the result in [21].

Theorem 1. *Suppose that W satisfies the assumptions of the previous section, and let $u_0(X)$ and $u_1(X)$ be given functions on $\mathbb{R}^3$ which are sufficiently small in an appropriate energy norm. Then the initial value problem*

$$\begin{aligned} &\rho_0 D_t^2 x - D\cdot S(Dx) = 0\\ &x(0,X) = X + u_0(X)\\ &D_t x(0,X) = u_1(X)\end{aligned}$$

has a unique global classical solution defined on $[0,\infty)\times\mathbb{R}^3$ with small displacement $x(t,X)-X$ in an appropriate norm.

The norm referred to in the theorem will be specified later in Section 7.3. In particular, the solution space ensures that the solution is classical, and the smallness condition for the displacement ensures that the deformation is one-to-one at all times.

The smallness of the initial data and the null condition are both necessary assumptions. There are examples of singularity formation in finite time for solutions with large initial data [27] and for solutions with arbitrarily small (spherically symmetric) initial data when the strain energy violates the null condition [8]. In the absence of the null condition, solutions exist *almost globally*, see [9],[14].

3.3. **Long time local existence and stability – nearly incompressible case.** The following result appears in [22].

Theorem 2. *Consider a penalized strain energy function of the form*

$$W^\lambda(F) = W^\infty(F) + \lambda^2 h(\rho).$$

Assume that W^∞ satisfies all of the conditions in Section 3.1 except the null condition, and define the Cauchy stress $\tilde{T}(H) = \hat{T}(H^{-1})$ using W^∞ in (2.4)*. Define*

$$h(\rho) = \frac{(\rho+2)}{6\rho}(\rho-1)^2.$$

Suppose that initial data $(H(0,x), v(0,x), \rho(0,x))$ are given on $\mathbb{R}^3$ which satisfy the constraints (2.9) *and such that*

$$H(0,x) - I, \quad v(0,x), \quad \lambda(\rho(0,x)-\rho_0)$$

are sufficiently small (independently of λ) in an appropriate energy norm.

Then the initial value problem for the corresponding first order system

$$\begin{aligned} &\rho(\partial_t v + v\cdot\nabla v) - \nabla\cdot\tilde{T}(H) + \lambda^2\nabla[\rho^2 h'(\rho)] = 0 \\ &\partial_t\rho + v\cdot\nabla\rho + \rho\,\nabla\cdot v = 0 \\ &\partial_t H + v\cdot\nabla H + H\nabla v = 0 \end{aligned} \tag{3.1}$$

has a unique classical local solution $(H^\lambda(0,x), v^\lambda(0,x), \rho^\lambda(0,x))$ *defined on* $[0,\lambda)\times\mathbb{R}^3$*, and*

$$H^\lambda(t,x) - I, \quad v^\lambda(t,x), \quad \lambda(\rho^\lambda(t,x) - \rho_0)$$

remain uniformly small.

The hydrodynamical form of the penalization term is physically reasonable, and it has been chosen so that the singular term has a simple form

$$\lambda^2\rho^{-1}\nabla[\rho^2 h'(\rho)] = \lambda^2\rho\nabla\rho.$$

We shall see in the next section that the limiting equations $(\lambda\to\infty)$ are general, although the approximating equations are not.

Local existence and stability on a fixed uniform time interval $[0,T]\times\mathbb{R}^3$ was shown in [19] using energy estimates alone. The present result combines energy estimates with some new dispersive estimates.

3.4. **Incompressible limit.** The next theorem is a special case of the main result in [22]. A weaker convergence result can also be given in the case where the initial data is not incompressible.

Theorem 3. *In addition to the assumptions of Theorem 2, suppose that the initial data is incompressible:*

$$\rho_0(0,x) = \rho_0, \quad \nabla\cdot v(0,x) = 0.$$

Then the full solution family $(H^\lambda(t,x), v^\lambda(t,x), \rho^\lambda(t,x))$ *converges in*

$$C^0_{loc}([0,\infty)\times\mathbb{R}^3)\cap C^1_{loc}((0,\infty)\times\mathbb{R}^3)$$

to a unique classical finite energy solution $(H(t,x), v(t,x), \rho_0)$ *of the limiting incompressible system*

$$\begin{aligned} \rho_0(\partial_t v + v\cdot\nabla v) - \nabla\cdot\tilde{T}(H) + \nabla\Lambda = 0, \\ \nabla\cdot v = 0, \\ \partial_t H + v\cdot\nabla H + H\nabla v = 0. \end{aligned}$$

4. The energy identity

As a preparation for our development of the generalized energy method from the scalar case, to nonrelativistic systems, and finally to elastodynamics, we momentarily digress, in this section, to discuss the energy method from a general standpoint.

We first introduce the following notation:

$$y = (y^0, \dots, y^{n-1}) \in \mathbb{R}^n, \quad \partial_\alpha = \partial/\partial y_\alpha,$$
$$u : \mathbb{R}^n \to \mathbb{R}^m,$$
$$p = \partial u \in \mathcal{M}^{m\times n} = \{m \times n \text{ matrices}\},$$
$$e : \mathcal{M}^{m\times n} \to \mathbb{R}, \quad e(0) = 0, \quad \frac{\partial e}{\partial p}(0) = 0.$$

The formal variational problem for

$$\mathcal{L}[u] = \int_{\mathbb{R}^n} e(\partial u(y))dy,$$

yields the system of pde's

$$\partial_\alpha \left[\frac{\partial e}{\partial p^i_\alpha}(\partial u(y))\right] = 0. \tag{4.1}$$

For each $\beta = 0, \dots, n-1$, we have the conservation law

$$\partial_\alpha[T_{\alpha\beta}(\partial u(y))] = \partial_\beta u^i(y)\partial_\alpha \left[\frac{\partial e}{\partial p^i_\alpha}(\partial u(y))\right] = 0, \tag{4.2}$$

in which

$$T_{\alpha\beta}(p) = p^i_\beta \frac{\partial e}{\partial p^i_\alpha}(p) - e(p)\delta_{\alpha\beta} \tag{4.3}$$

is called the *energy momentum tensor*.

More generally, suppose now that that the Lagrangian density is inhomogeneous

$$\bar{e} : \mathbb{R}^n \times \mathcal{M}^{m\times n} \to \mathbb{R}, \quad \bar{e}(y, 0) = 0, \quad \frac{\partial \bar{e}}{\partial p}(y, 0) = 0,$$

and that $v(y)$ solves

$$\partial_\alpha \left[\frac{\partial \bar{e}}{\partial p^i_\alpha}(y, \partial v(y))\right] = f^i(y). \tag{4.4}$$

This serves as a model for the type of equations satisfied by the derivatives of the solution of (4.1). Although the equation (4.4) is no longer conservative, we still have that

$$\partial_\alpha[\bar{T}_{\alpha\beta}(y, \partial v(y))] + \frac{\partial \bar{e}}{\partial y^\beta}(y, \partial v(y)) = \partial_\beta v^i(y) f^i(y), \tag{4.5}$$

with

$$\bar{T}_{\alpha\beta}(y,p) = p^i_\beta \frac{\partial \bar{e}}{\partial p^i_\alpha}(y,p) - \bar{e}(y,p)\delta_{\alpha\beta}. \tag{4.6}$$

For a systematic introduction to the field theoretic point of view, see [24].

5. Scalar nonlinear wave equations

5.1. **Local existence.** In order to make the analogy with elasticity, we consider the variational problem associated with the Lagrangian

$$\mathcal{L}[u] = \int\int_{\mathbb{R}^3} \{\tfrac{1}{2}[(\partial_t u)^2 - c^2|\nabla u|^2] - \tfrac{1}{6}B_{\lambda\mu\nu}\partial_\lambda u \partial_\mu u \partial_\nu u\} dxdt.$$

Here $(t,x) \in \mathbb{R}\times\mathbb{R}^3$, $u : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, $\partial = (\partial_0, \dots, \partial_3) = (\partial_t, \nabla)$, and the coefficients $B_{\lambda\mu\nu}$ are symmetric in the indices. The corresponding initial value problem is

$$\begin{aligned} \Box u = (\partial_t^2 - c^2\Delta)u &= \tfrac{1}{2}B_{\lambda\mu\nu}\partial_\lambda(\partial_\mu u \partial_\nu u) \\ u(0,x) = u_0(x), &\quad \partial_t u(0,x) = u_1(x). \end{aligned} \tag{5.1}$$

For small solutions, this problem is hyperbolic, and we have the following classical local existence result.

Theorem 4. *Given* $u_0 \in \dot{H}^k(\mathbb{R}^3)$ *and* $u_1 \in H^{k-1}(\mathbb{R}^3)$, *with* $k \geq 5$

$$\|u_0\|_{\dot{H}^k(\mathbb{R}^3)} + \|u_1\|_{H^{k-1}(\mathbb{R}^3)} \leq \varepsilon$$

sufficiently small, there exists

$$T_0 = T_0(\varepsilon)$$

such that the initial value problem has a unique local solution u *with*

$$\partial u \in \cap_{j=0}^{k-1} C^j([0,T_0); H^{k-j}(\mathbb{R}^3)).$$

Moreover, if $T_0 < \infty$, *then* $\limsup_{t\uparrow T_0} \|\partial u(t)\|_{H^{k-1}} > 2\varepsilon$.

The last point of the theorem allows the estimation of the lifespan T from below by means of *a priori* estimates. In particular, if the norm of the solution remains small, then the solution is global in time.

The regularity assumptions in this theorem are made for convenience only. As we shall see, the main requirement is the control of second derivatives in L^∞, which leads to the choice H^s with $s > 7/2$ in $\mathbb{R}^3$. In certain cases, this can be further relaxed to $s > 3$, again in $\mathbb{R}^3$. See [2], [13], [25] for the treatment of local existence for quasilinear problems with rough data.

5.2. **Energy estimates in the scalar case.** Given a solution of (5.1), the goal is to obtain an a priori estimate for $\|\partial u(t)\|^2_{\mathcal{H}^{k-1}}$, where $\|u\|^2_{\mathcal{H}^m} = \sum_{|a|\le m} \|\partial^a u(t)\|^2_{L^2}$ is not the standard Sobolev space because ∂ includes time derivatives .

First, note that the Lagrangian is $\mathcal{L}[u] = \int e(\partial u(y))dy$, with density

$$e(p) = \tfrac{1}{2}[p_0^2 - c^2(p_1^2 + p_2^2 + p_3^2)] - \tfrac{1}{6} B_{\lambda\mu\nu} p_\lambda p_\mu p_\nu.$$

From (4.2) with $\beta = 0$, we obtain

$$\partial_0 \int_{\mathbb{R}^3} T_{00}(\partial u(y))dx = 0,$$

in which

$$T_{00}(p) = \tfrac{1}{2}[p_0^2 + c^2(p_1^2 + p_2^2 + p_3^2)] + \mathcal{O}(|p|^3).$$

To obtain an estimate for the derivatives of u, we apply ∂^a to the pde and use the Leibnitz rule on the right hand side. Setting $v = \partial^a u$, we get

$$\Box v - B_{\lambda\mu\nu}\partial_\lambda(\partial_\mu v \partial_\nu u) = \tfrac{1}{2} \sum_{\substack{b+c=a \\ b,c\neq a}} \binom{a}{b} B_{\lambda\mu\nu}\partial_\lambda(\partial_\mu \partial^b u \partial_\nu \partial^c u).$$

This has the form

$$\partial_\alpha \Big[\frac{\partial \bar{e}}{\partial p_\alpha}(y, \partial v(y)) \Big] = f(y), \quad y = (t,x) \in \mathbb{R}^4,$$

with

$$\begin{aligned} \bar{e}(y,p) = \frac{1}{2} \frac{\partial^2 e}{\partial p_\lambda \partial p_\mu}(\partial u(y)) p_\lambda p_\mu \\ &= \tfrac{1}{2}[p_0^2 - c^2(p_1^2 + p_2^2 + p_3^2)] - \tfrac{1}{2} B_{\lambda\mu\nu} p_\lambda p_\mu \partial_\nu u(y). \end{aligned}$$

Thus, we may use (4.5) to write

$$\partial_0 \int_{\mathbb{R}^3} \bar{T}_{00}(y, \partial v(y))dx + \int_{\mathbb{R}^3} \frac{\partial \bar{e}}{\partial y^0}(y, \partial v(y))dx = \int_{\mathbb{R}^3} \partial_0 v(y) f(y) dx,$$

and

$$\bar{T}_{00}(y,p) = \tfrac{1}{2}[p_0^2 + c^2(p_1^2 + p_2^2 + p_3^2)] + \mathcal{O}(|\partial u(y)||p|^2).$$

Define the energy (for $k > 1$)

$$E_k[u(t)] = \int_{\mathbb{R}^3} T_{00}(\partial u(y))dx + \sum_{1\le|a|\le k-1} \int_{\mathbb{R}^3} \bar{T}_{00}(y, \partial\partial^a u(y))dx.$$

Since $T_{00}(\partial u) \sim |\partial u|^2$ and $\bar{T}_{00}(y, \partial v) \sim |\partial v|^2$, for $|\partial u| \ll 1$, we have that

$$E_k[u(t)] \sim \|\partial u(t)\|^2_{\mathcal{H}^{k-1}}, \quad |\partial u| \ll 1.$$

We have derived

$$\partial_t E_k[u(t)] = \sum_{1\le|a|\le k-1} \Big\{ \tfrac{1}{2} B_{\lambda\mu\nu} \int_{\mathbb{R}^3} \partial_\lambda \partial^a u \partial_\mu \partial^a u \partial_0 \partial_\nu u dx + \tfrac{1}{2} \sum_{\substack{b+c=a\\ b,c\neq a}} \binom{a}{b} B_{\lambda\mu\nu} \int_{\mathbb{R}^3} \partial_\lambda(\partial_\mu \partial^b u \partial_\nu \partial^c u)\partial_0 \partial^a u dx \Big\}. \tag{5.2}$$

The terms on the right of (5.2) have the form

$$\int_{\mathbb{R}^3} \partial\partial^a u\, \partial^b u\, \partial^2 \partial^c u\, dx,$$

with $|a| \le k-1$, $|b+c| \le k-1$, $|b|, |c| < k-1$. Thus, there is one factor with at most $\frac{k+2}{2}$ derivatives. This term is estimated in L^∞ and the others in L^2. The result is that, for $|\partial u| \ll 1$, the right hand side is bounded by

$$C \sum_{|a|\le[\frac{k}{2}]} \|\partial\partial^a u(t)\|_{L^\infty} \|\partial u(t)\|^2_{\mathcal{H}^{k-1}} \sim C \|\partial u(t)\|_{W^{[\frac{k}{2}],\infty}} E_k[u(t)],$$

where $\|u(t)\|_{W^{m,\infty}} = \sum_{|a|\le m} \|\partial^a u(t)\|_{L^\infty}$. So as long as $|\partial u| \ll 1$, we have the energy inequality

$$\partial_t E_k[u(t)] \le C \|\partial u(t)\|_{W^{[\frac{k}{2}],\infty}} E_k[u(t)],$$

and then

$$E_k[u(t)] \le E_k[u(0)] \exp C \int_0^t \|\partial u(s)\|_{W^{[\frac{k}{2}],\infty}} ds.$$

The estimate is closed by means of the classical Sobolev lemma which in $\mathbb{R}^3$ gives

$$\begin{aligned} \|\partial u(t)\|_{W^{[\frac{k}{2}],\infty}} &\le C \|\partial u(t)\|_{\mathcal{H}^{[\frac{k}{2}]+2}} \\ &\le C \|\partial u(t)\|_{\mathcal{H}^{k-1}} \quad (k \ge 5) \\ &\sim C E_k^{1/2}[u(t)], \end{aligned}$$

provided $|\partial u| \ll 1$. At this point however, we see that smallness of the energy implies the pointwise smallness of the gradient.

The conclusion is that

$$E_k[u(t)] \le E_k[u(0)] \exp Ct \sup_{0\le s\le t} E_k^{1/2}[u(s)],$$

which shows that $E_k[u(t)] \le 2E_k[u(0)] \le 2\varepsilon^2$, for $0 \le t \le C/\varepsilon$.

To improve this result, we need dispersive estimates to better control

$$\int_0^t \|\partial u(s)\|_{W^{[\frac{k}{2}],\infty}} ds.$$

5.3. **The generalized energy method.** The linear wave equation is invariant under Lorentz transformations, the infinitesimal generators of which are spanned by ∂ together with the following set of vector fields (in $\mathbb{R}^3$)

$$\begin{aligned} \Omega &= (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla, \\ L &= (L_1, L_2, L_3) = c^2 \nabla + x\partial_t, \\ S &= t\partial_t + r\partial_r = t\partial_t + x_i\partial_i. \end{aligned} \tag{5.3}$$

The following commutation relations hold

$$[\Omega, \Box] = 0, \quad [L, \Box] = 0, \quad [S, \Box] = -2\Box.$$

Define $\Gamma = (\partial, \Omega, L, S)$. The commutation properties for Γ allows us to repeat the calculation in the previous section using now the derivatives $\Gamma^a u$ instead of only $\partial^a u$.

Define the generalized energy

$$\tilde{E}_k[u(t)] = \int_{\mathbb{R}^3} T_{00}(\partial u(y))dx + \sum_{1 \le |a| \le k-1} \int_{\mathbb{R}^3} \bar{T}_{00}(y, \partial\Gamma^a u(y))dx. \tag{5.4}$$

Then for $|\partial u| \ll 1$, we have the energy inequality

$$\tilde{E}_k[u(t)] \le C\tilde{E}_k[u(0)] \exp C \int_0^t \sum_{|a| \le [\frac{k}{2}]} \|\partial\Gamma^a u(s)\|_{L^\infty} ds.$$

The inclusion of these vector fields leads to a dispersive estimate through the following *generalized Sobolev inequality.*

Lemma 1 ([11])**.** *For any function $u(t,x)$ on $[0,T) \times \mathbb{R}^3$,*

$$\langle t + |x| \rangle \langle ct - |x| \rangle^{1/2} |\partial u(t,x)| \le C \sum_{|a| \le 2} \|\partial\Gamma^a u(t)\|_{L^2(\mathbb{R}^3)},$$

provided the norm on the right is finite.

Here, $\langle z \rangle = (1 + |z|^2)^{1/2}$.

For small solutions, we apply this result to get

$$\begin{aligned} \sum_{|a| \le [\frac{k}{2}]} \|\partial\Gamma^a u(s)\|_{L^\infty} &\le C\langle s \rangle^{-1} \sum_{|a| \le [\frac{k}{2}]+2} \|\partial\Gamma^a u(s)\|_{L^2} \\ &\le C\langle s \rangle^{-1} \sum_{|a| \le k-1} \|\partial\Gamma^a u(s)\|_{L^2}, \quad (k \ge 5) \\ &\le C\langle s \rangle^{-1} \tilde{E}_k[u(s)]. \end{aligned}$$

Inserting this into the energy inequality, we obtain

$$\tilde{E}_k[u(t)] \le \tilde{E}_k[u(0)] \exp C \log(1+t) \sup_{0 \le s \le t} \tilde{E}_k[u(s)].$$

Because the factor $\langle s\rangle^{-1}$ just fails to be integrable at infinity, this leads to an existence interval of the order $\exp C\varepsilon^{-1}$. In general, this is sharp because examples show that arbitrarily small solutions develop singularities in a finite time of the same order, [1], [7], [20].

Cubic and higher nonlinearities (in 3d) lead to a decay factor of $\langle s\rangle^{-2}$ which is enough to ensure global existence of small solutions. In the next sections, we will see how the null condition can be exploited to improve the dispersion rate sufficiently for global existence. In higher dimensions, the inherent decay rate of $\langle s\rangle^{-\frac{n-1}{2}}$ is sufficient for global existence of small solutions for quadratically nonlinear equations without further structural assumptions.

5.4. **The null condition and global existence.** Adopting the point of view already taken in Section 2.12, we note that the linear wave equation

$$\Box u = (\partial_t^2 - c^2\Delta)u = 0,$$

has plane wave solutions of the form

$$u(t,x) = f(ct + \langle \xi, x\rangle), \quad \xi \in S^2,$$

or with $y = (t,x)$ and $\eta = (c,\xi)$,

$$u(y) = f(\langle \eta, y\rangle).$$

The vector η belongs to the class of *null vectors*

$$\eta \in \mathcal{N} = \{\eta \in \mathbb{R}^4 : \eta_0^2 - c^2(\eta_1^2 + \eta_2^2 + \eta_3^2) = 0\}.$$

We denote the form coming from nonlinearity by

$$N(u,v) = B_{\lambda\mu\nu}\partial_\lambda(\partial_\mu u \partial_\nu v).$$

The null condition or nonresonance condition is $N(u,v) = 0$ for all pairs of plane wave solutions u and v. Since we are dealing with the scalar case (with only one wave family), there is no need to project N as we did for the case of a system. An example is given by

$$N(u,v) = \partial_\lambda[(\partial_0 u \partial_0 v) - c^2\langle \nabla u, \nabla v\rangle].$$

Written out explicitly, the null condition takes the form

$$B_{\lambda\mu\nu}\eta_\lambda\eta_\mu\eta_\nu = 0, \quad \eta \in \mathcal{N}.$$

We now explain briefly how the null condition is used to improve the dispersive behavior of the nonlinear terms. Using the vector fields Γ, we can decompose derivatives in the form

$$\partial = (\partial_t, \nabla) = \Big(-c, \frac{x}{|x|}\Big)\partial_r + \mathcal{R}, \tag{5.5}$$

in which $(-c, \frac{x}{|x|}) \in \mathcal{N}$ and $\mathcal{R}$ is tangent to the light cone $c^2t^2 - |x|^2 = 0$. In particular,

$$\langle t + |x| \rangle |\mathcal{R}u(t,x)| \leq C|\Gamma u(t,x)|.$$

As with the previous energy identity (5.2), we have that $\partial_t \tilde{E}_k[u(t)]$ is a sum of terms of the form

$$B'_{\lambda\mu\nu} \int \partial_\lambda \Gamma^a u \partial_\mu \Gamma^a u \partial_0 \partial_\nu u dx,$$

and

$$B'_{\lambda\mu\nu} \int \partial_\lambda (\partial_\mu \Gamma^b u \partial_\nu \Gamma^b u) \partial_0 \Gamma^a u dx.$$

The coefficients $B'_{\lambda\mu\nu}$ are related to $B_{\lambda\mu\nu}$, the modification coming from the commutation of the derivatives ∂ with Γ. The important point however is that the new coefficients $B'_{\lambda\mu\nu}$ still satisfy the null condition. Therefore, when the derivatives ∂ are replaced using the decomposition above, the leading term has the form $B'_{\lambda\mu\nu}\eta_\lambda\eta_\mu\eta_\nu$ which vanishes by the null condition. The remaining terms have an extra measure of decay sufficient to give global existence in 3D, in the quadratic case. We point out that certain technical obstacles arise in the execution of this argument because of the appearance of derivatives of the form $\Gamma^a u$ (instead of $\partial\Gamma^a u$). The interested reader can consult [12] for details. A completely different approach to this problem using conformal compactification was given in [3]. The book [5] gives a systematic treatment of these topics.

6. Nonrelativistic model problem

6.1. **The initial value problem.** In contrast to the scalar case, the equations of elastodynamics have multiple propagation speeds. This spoils the Lorentz invariance, and so the method outlined in the previous sections can not be directly applied. As further preparation for this case, we first consider a model problem that illustrates many of the difficulties while still retaining some similarities with the scalar case.

Let $0 < c_1 < \ldots < c_m$ be distinct propagation speeds and suppose that the coefficients $B^{ijk}_{\lambda\mu\nu}$ are symmetric with respect to permutations of the upper indices and also with respect to permutations of the lower indices. We start with a formal variational problem for

$$\mathcal{L}[u] = \int_{\mathbb{R}^4} e(\partial u(y)) dy,$$

in which $u : \mathbb{R}^4 \to \mathbb{R}^m$ and the density is

$$e(p) = \sum_i \tfrac{1}{2}[(p_0^i)^2 - c_i^2((p_1^i)^2 + (p_2^i)^2 + (p_3^i)^2)] - \tfrac{1}{6}B^{ijk}_{\lambda\mu\nu}p^i_\lambda p^j_\mu p^k_\nu.$$

This leads to the following initial value problem

$$\Box_i u^i = (\partial_0^2 - c_i^2\Delta)u^i = \tfrac{1}{2}B^{ijk}_{\lambda\mu\nu}\partial_\lambda(\partial_\mu u^j \partial_\nu u^k), \quad i = 1, \dots, m \tag{6.1}$$

$$u(0) = u_0, \quad \partial_t u(0) = u_1. \tag{6.2}$$

Note that summation in the index i is not performed on the left of (6.1).

6.2. **The energy norm.** Since the equations are only invariant under spatial rotations and scaling, but not Lorentz rotations, the list of commuting vector fields is smaller. We shall only use

$$\Gamma = (\partial, \Omega, S),$$

in the definition of the energy norm.

For a solution u, we define

$$\bar{e}(y,p) = \frac{1}{2}\frac{\partial^2 e}{\partial p^i_\lambda \partial p^j_\mu}(\partial u(y))p^i_\lambda p^j_\mu,$$

and using e and $\bar{e}$ to define T and $\bar{T}$ in (4.3) and (4.6), the energy $\tilde{E}_k[u(t)]$ is then defined as in (5.4) with the smaller set of vector fields. As before, we have that

$$\tilde{E}_k[u(t)]^{1/2} \sim \sum_{|a|\le k-1} \|\partial\Gamma^a u(t)\|_{L^2}, \quad |\partial u| \ll 1.$$

6.3. **Plane waves and the null condition.** Plane wave solutions for the diagonal operator

$$\Box = \text{diag }(\Box_1, \dots, \Box_m)$$

are of the form

$$u(t,x) = f(c_i t + \langle \xi, x\rangle)e_i, \quad i = 1, \dots, m.$$

Here $f : \mathbb{R} \to \mathbb{R}$, $\xi \in S^2$, and $e_i \in \mathbb{R}^m$ is the standard unit vector. As before, define the form

$$N^i(u,v) = B^{ijk}_{\lambda\mu\nu}\partial_\lambda(\partial_\mu u^j \partial_\nu v^k).$$

The nonresonance condition is

$$\langle N(u,v), w\rangle_{\mathbb{R}^m} = 0,$$

for all triples of plane waves (u, v, w) belonging to the same family. This is equivalent to the null condition

$$B^{iii}_{\lambda\mu\nu}\eta_\lambda\eta_\mu\eta_\nu = 0, \quad i = 1, \dots, m$$
$$\eta \in \mathcal{N}_i = \{\eta \in \mathbb{R}^4 : c_i^2\eta_0^2 - (\eta_1^2 + \eta_2^2 + \eta_3^2) = 0\}. \tag{6.3}$$

6.4. **Global existence.** Now that we have posed the null condition in this setting, we can state the global existence result:

Theorem 5 ([23], [28])**.** *If the null condition* (6.3) *holds and the initial data is such that $\tilde{E}_k[u(0)] < \varepsilon$, with ε sufficiently small, then the initial value problem* (6.1) *has a unique global classical solution defined on $\mathbb{R}^+ \times \mathbb{R}^3$ with $\tilde{E}_k[u(t)] < 2\varepsilon$, for all $t > 0$.*

6.5. **Sobolev lemma.** The smaller set of available vector fields weakens the Sobolev lemma.

Lemma 2 ([14])**.** *For any function $u(t, x)$ on $\mathbb{R}^+ \times \mathbb{R}^3$,*

$$\langle|x|\rangle\langle ct - |x|\rangle^{1/2}|\partial u(t, x)| \le C\Big\{\sum_{|a|\le 2}\|\partial\Omega^a u(t)\|_{L^2} + \sum_{|a|\le 1}\|\langle ct - |x|\rangle\nabla\partial\Omega^a u(t)\|_{L^2}\Big\},$$

provided the norm on the right is finite.

Another series of weighted L^2 estimates will be necessary to control the quantity on the right hand side of this inequality.

6.6. **Weighted L^2 estimates.** We are now faced with estimating the weighted L^2 quantity of the right in Lemma 2. We outline an easy method for getting the bound

$$\sum_{|a|\le k-2}\sum_{i=1}^{m}\|\langle c_i t - |x|\rangle\partial^2\Gamma^a u^i(t)\|_{L^2} \le C\tilde{E}_k[u(t)]^{1/2}, \quad k \ge 2, \tag{6.4}$$

based on elementary properties of the vector fields Γ and the linear operators $\Box_i$.

Recalling the definitions of S and Ω in (5.3), we have the following identity (see [14])

$$(c_i^2t^2 - r^2)\Delta u = t\partial_t Su - r\partial_r Su - t\partial_t u - r\partial_r u - \Omega^2 u - t^2\Box_i u.$$

Divide this $c_i t + r$ and use the facts that $t(c_i t + r)^{-1}$ and $r(c_i t + r)^{-1}$ are uniformly bounded and that fact that $|\Omega^2 u| \le |x||\nabla\Omega u|$ to get the pointwise bound

$$|c_i t - r||\Delta u| \le C[|\partial\Gamma u| + |\partial u| + t|\Box_i u|].$$

Thus, we obtain

$$\|\langle c_i t - |x|\rangle \Delta u\|_{L^2} \leq C[\|\partial\Gamma u\|_{L^2}\|\partial u\|_{L^2} + t\|\Box_i u\|_{L^2}].$$

The ellipticity of Δ gives the estimate for all second order spatial derivatives. Similar computations lead to bounds for $\partial_t \nabla u$ and $\partial_t^2 u$. This can equally be applied to derivatives $\Gamma^a u$, resulting in the estimate

$$\|\langle c_i t - |x|\rangle \partial^2(\Gamma^a u)\|_{L^2} \leq C[\|\partial\Gamma(\Gamma^a u)\|_{L^2} + \|\langle t + |x|\rangle \Box_i(\Gamma^a u)\|_{L^2}].$$

Using the commutation properties of $\Box_i$ and Γ and the pde's, the last norm on the right involves only nonlinear terms. Moreover, we have that

$$\langle t + |x|\rangle \leq C\langle |x|\rangle\langle c_i t - |x|\rangle,$$

so by Lemma 2, these terms can be controlled by $\tilde{E}_k[u(t)]^{1/2}$ times the weighted L^2 quantity. Thus, for small solutions, these terms can be absorbed on the left by a bootstrap argument, resulting in the estimate (6.4).

6.7. **Nonlinear interactions.** From the energy identity, we must estimate integrals of the form

$$B^{ijk}_{\lambda\mu\nu}\int \partial_\lambda \Gamma^a u^i \partial_\mu \Gamma^a u^j \partial_\nu \partial_0 u^k dx \tag{6.5}$$

$$B^{ijk}_{\lambda\mu\nu}\int \partial_\lambda(\partial_\mu \Gamma^b u^i \partial_\nu \Gamma^c u^j)\partial_0 \Gamma^a u^k dx,$$

for $|b| + |c| \leq |a| \leq k-1$, $|b|, |c| \neq k-1$.

The combination of the Sobolev inequality of Lemma 2 and the weighted L^2 estimate (6.4) gives the bounds

$$\|\langle x\rangle\langle c_i t - |x|\rangle \partial\Gamma^a u^i\|_{L^\infty} \leq C\tilde{E}_k[u(t)]^{1/2}, \qquad |a| \leq k-3,$$

and

$$\|\langle c_i t - |x|\rangle \partial^2\Gamma^a u^i\|_{L^2}, \leq C\tilde{E}_k[u(t)]^{1/2}, \qquad |a| \leq k-2.$$

.

The interaction of waves in different families is of lower order because these waves concentrate along their respective light cones. Therefore, if in (6.5) $(i,j,k) \neq (i,i,i)$, (say $i \neq j$), then we can absorb one of the weights

$$\langle x\rangle\langle c_i t - |x|\rangle^{1/2}\langle c_j t - |x|\rangle \quad \text{or} \quad \langle x\rangle\langle c_i t - |x|\rangle\langle c_j t - |x|\rangle^{1/2},$$

either of which is larger than $C\langle t\rangle^{3/2}$. This provides sufficient decay for these nonresonant terms.

In the resonant case, $(i, j, k) = (i, i, i)$, we must treat the region near the light cone differently from the region far from it. If we are away from the cone, $\langle c_i t - |x|\rangle \geq \langle t\rangle$, then we can use the weight

$$\langle x\rangle\langle c_i t - |x|\rangle^{3/2}$$

to gain $\langle t\rangle^{3/2}$ decay, which is sufficient.

The null condition is used to obtain decay for the resonant terms near the light cone, through cancellation of the nontangential derivatives along the cone. Since our collection of vector fields is smaller, the analog of the decomposition (5.5) is weaker. We have

$$\partial u = X(\partial_t - c_i\partial_r)u + \mathcal{R}u,$$

with $X \in \mathcal{N}_i$ and

$$\langle t\rangle|\mathcal{R}u| \leq C[|\Gamma u| + \langle c_i t - |x|\rangle|\partial u|], \quad \langle c_i t - |x|\rangle \leq \langle t\rangle.$$

Upon insertion of this decomposition into the remaining integrals, the null condition eliminates the leading terms. The remainder has improved decay which again is combined with our other estimates to complete the argument as described previously.

7. Global existence in elastodynamics

7.1. **The truncated initial value problem.**

We shall be looking for solutions of the equations of motion

$$D_t^2 x - D \cdot S(Dx) = 0,$$

which represent small displacements from the reference configuration. We set

$$u(t, X) = x(t, X) - X, \quad G(t, X) = Du(t, X) = F(t, X) - I.$$

For simplicity we will truncate the nonlinearities at the quadratic level because cubic and higher order nonlinearities present no difficulty in the construction of global small solutions in 3D. So we write

$$\begin{aligned} W(F) &= W(I + G) \\ &= W(I) + S_\ell^i(I)G_\ell^i + \tfrac{1}{2}A_{\ell m}^{ij}(I)G_\ell^i G_m^j + \tfrac{1}{6}B_{\ell mn}^{ijk}(I)G_\ell^i G_m^j G_n^k, \end{aligned}$$

ignoring higher order terms. Recall that $W(I) = 0$ and $S(I) = 0$.

The pde's then are

$$Lu^i = D_t^2 u^i - A_{\ell m}^{ij}(I)D_\ell D_m u^j = \tfrac{1}{2}B_{\ell mn}^{ijk}(I)D_\ell(D_m u^j D_n u^k), \tag{7.1}$$

with

$$A_{\ell m}^{ij}(I)D_\ell D_m = c_2^2\delta_{ij}\Delta + (c_1^2 - c_2^2)D_iD_j, \quad c_1^2 > \tfrac{4}{3}c_2^2 > 0.$$

7.2. **Invariance properties.** From the Galilean invariance and isotropy we have that

$$A^{ij}_{\ell m}(I) = A^{IJ}_{LM}(I) Q^{iI} Q^{jJ} Q^{L\ell} Q^{Mm},$$
$$B^{ijk}_{\ell mn}(I) = B^{IJK}_{LMN}(I) Q^{iI} Q^{jJ} Q^{kK} Q^{L\ell} Q^{Mm} Q^{Nn},$$

for all $Q \in \mathbf{O}^3_+$. This means that if u solves (7.1), then so does

$$u^*(t,X) = Q^T u(t, QX), \quad Q \in \mathbf{O}^3_+.$$

More generally, define the antisymmetric matrices

$$A_1 = -e_2 \otimes e_3 + e_3 \otimes e_2, \quad A_2 = -e_3 \otimes e_1 + e_1 \otimes e_3, \quad A_3 = -e_1 \otimes e_2 + e_2 \otimes e_1.$$

The solutions $U_j(s)$ of

$$U_j'(s) = A_j U(s), \quad U(0) = I,$$

are one-parameter families in $\mathbf{O}^3_+$. Thus, the transformation

$$R_j(s) u(t,X) = U_j^T(s) u(t, U_j(s) X)$$

preserves solutions.

Note that

$$D_s R_j(s) u|_{s=0} = DuA_j X - A_j u = \Omega_j u - A_j \equiv \tilde{\Omega}_j u,$$

where $\Omega = X \wedge D$, as before.

Similarly, the scaling

$$\Sigma(s) u(t,X) = (1+s)^{-1} u((1+s)t, (1+s)X)$$

preserves solutions for all $s > -1$, and

$$D_s \Sigma(s) u|_{s=0} = Su - u \equiv \tilde{S} u,$$

where $S = tD_t + rD_r$, as before.

Finally, the translations

$$\tau_0(s) u(t,X) = u(t+s, X), \quad \tau_j(s) u(t,X) = u(t, X + se_j),$$

preserve solutions, and

$$D_s \tau_0(s) u|_{s=0} = D_t u, \quad \text{and} \quad D_s \tau_j(s) u|_{s=0} = D_j u.$$

Relabel these families as

$$\Delta(s) = (\Delta_1(s), \dots, \Delta_8(s)) = (\tau_0(s), \dots, R_1(s), \dots, \Sigma(s)),$$

and let

$$\Gamma = (\Gamma_1, \dots, \Gamma_8) = (D_t, D, \tilde{\Omega}, \tilde{S}),$$

be the corresponding vector fields.

For $a = (a_1, \dots, a_p) \in \mathbb{Z}^p_8$, write

$$\Delta^a(s) = \Delta_{a_1}(s_1) \cdots \Delta_{a_p}(s_p), \quad \Gamma^a = \Gamma_{a_1} \cdots \Gamma_{a_p}.$$

Then

$$D_{s_1}\cdots D_{s_p}\Delta^a(s)u|_{s=0}=\Gamma^a u.$$

Note that this differs from the usual multi-index notation. For $a\in\mathbb{Z}_8^p$, we denote $|a|=p$, and we write $b+c=a$ whenever b and c are partitions of a.

Now if u is a solution of (7.1), then so is $\Delta^a(s)u$, for any $a\in\mathbb{Z}_8^p$, so that

$$L\Delta^a(s)u^i=\tfrac{1}{2}B^{ijk}_{\ell mn}(I)D_\ell(D_m\Delta^a(s)u^jD_n\Delta^a(s)u^k).$$

Take the derivative in s at $s=0$ to obtain

$$L\Gamma^a u^i-B^{ijk}_{\ell mn}(I)D_\ell(D_m\Gamma^a u^jD_nu^k) = \sum_{\substack{b+c=a\\ b,c\neq a}}\tfrac{1}{2}B^{ijk}_{\ell mn}(I)D_\ell(D_m\Gamma^b u^jD_n\Gamma^c u^k). \tag{7.2}$$

7.3. **Energy identity.** The truncated equations (7.1) correspond to (4.1) with the Lagrangian density

$$e(p)=\tfrac{1}{2}[p_0^2-A^{ij}_{\ell m}(I)p^i_\ell p^j_m]-\tfrac{1}{6}B^{ijk}_{\ell mn}(I)p^i_\ell p^j_m p^k_n.$$

From (7.2), we see that $v=\Gamma^a u$ solves (4.5) with Lagrangian density

$$\bar{e}(y,p)=\frac{1}{2}\frac{\partial^2 e}{\partial p_\lambda\partial p_\mu}(\partial u(y))p_\lambda p_\mu,$$

and $f^i(y)$ given by the right hand side of (7.2).

For $k>1$, we define the energy norm

$$\tilde{E}_k[u(t)]=\int T_{00}(\partial u(t,X)dX+\sum_{1\le|a|\le k-1}\bar{T}_{00}(y,\partial\Gamma^a u(t,X))dX,$$

with $T_{00}(p)$ and $\bar{T}_{00}$ defined in (4.3) and (4.6). Once again we have

$$\tilde{E}_k[u(t)]^{1/2}\sim\sum_{|a|\le k-1}\|\partial\Gamma^a u(t)\|_{L^2},\quad |\partial u(y)|\ll 1.$$

Then, (4.2) and (4.3) combine to give

$$\partial_t\tilde{E}_k[u(t)]=\sum_{1\le|a|\le k-1}\Big\{\int\tfrac{1}{2}B^{ijk}_{\ell mn}(I)D_\ell\Gamma^a u^iD_m\Gamma^a u^jD_0D_nu^kdX + \sum_{\substack{b+c=a\\ b,c\neq a}}\int\tfrac{1}{2}B^{ijk}_{\ell mn}(I)D_\ell(D_m\Gamma^b u^jD_n\Gamma^c u^k)dX\Big\}. \tag{7.3}$$

The smallness condition of Theorem 1 is expressed in terms of this energy:

$$\tilde{E}_k[u(t)]^{1/2}<\varepsilon,\quad \text{for } k \text{ sufficiently large.}$$

7.4. **Weighted L^2 estimates.** The weighted L^2 estimates from section 6.6 must be redone for the operator of linearized elasticity L. Although it can be diagonalized, this would involve using the L^2 projections onto divergence free and curl free vectors, and these nonlocal operators are not compatible with the L^∞ estimates. Thus, we use a local approximation.

We start with the simple identity

$$(tD_t - rD_r)\tilde{S}u = t^2 Lu + \frac{1}{c_i^2}(c_i^2 t^2 - r^2)A(D)u + \frac{r^2}{c_i^2}(A(D) - c_i^2 D_r^2)u. \tag{7.4}$$

Then we introduce the decomposition

$$D = \frac{X}{r}D_r - \frac{X}{r^2} \wedge \Omega \tag{7.5}$$

to write

$$A(D) = A\Big(\frac{X}{r}D_r\Big) + \mathcal{R} = c_2^2\Big(I - \frac{X}{r} \otimes \frac{X}{r}\Big)D_r^2 + c_1^2 \frac{X}{r} \otimes \frac{X}{r} D_r^2 + \mathcal{R},$$

with $\mathcal{R}u(t,X)| \leq Cr^{-1}|D\Gamma u(t,X)|$. Set

$$P_1(X) = \frac{X}{r} \otimes \frac{X}{r} \quad \text{and} \quad P_2(X) = I - P_1(X).$$

Then $P_\alpha^2(X) = P_\alpha(X)$, $P_1(X)P_2(X) = 0$ and

$$P_\alpha(X)(A(D) - c_\alpha^2 D_r^2) = P_\alpha(X)\mathcal{R}, \quad \alpha = 1, 2.$$

From (7.4), we get

$$|(c_\alpha t - r)P_\alpha(X)A(D)u(t,X)| \leq C[|D\Gamma u(t,X)| + t|Lu(t,X)|], \quad \alpha = 1, 2.$$

Using the ellipticity of $A(D)$, it follows that

$$\|\langle c_\alpha t - r\rangle P_\alpha(X)D^2 u(t)\|_{L^2} \leq C[\tilde{E}_2[u(t)]^{1/2} + t\|Lu(t)\|_{L^2}].$$

The notation $P_\alpha(X)D^2u$ denotes the projection matrix $P_\alpha(X)$ applied to any second spatial derivative of u. The estimate also holds for derivatives $\Gamma^a u$.

Using now the pde's and a bootstrap argument, we get for small solutions

$$\|\langle c_\alpha t - r\rangle P_\alpha(X)D^2\Gamma^a u(t)\|_{L^2} \leq C\tilde{E}_k[u(t)]^{1/2}, \quad |a| \leq k - 2.$$

7.5. **Sobolev inequality.** Lemma 2 can be modified to include the projections P_α:

$$\begin{aligned}&\langle r\rangle\langle c_\alpha t-r\rangle^{1/2}|P_\alpha(X)D\Gamma^a u(t,X)|\\&\quad\le C[\tilde{E}_k[u(t)]^{1/2}+\sum_{\beta=1,2}\sum_{|b|\le|a|+1}\|\langle c_\beta t-r\rangle P_\beta(X)D^2\Gamma^b u(t)\|_{L^2}],\\&\qquad\qquad |a|\le k-3,\end{aligned}$$

see [21].

7.6. **Nonlinear interactions.** Continuing from the energy (7.3), we must estimate integrals of the form

$$\int B^{ijk}_{\ell mn}(I)D_\ell\Gamma^a u^i D_m\Gamma^a u^j D_0 D_n u^k dX \tag{7.6}$$

$$\int B^{ijk}_{\ell mn}(I)D_\ell(D_m\Gamma^b u^j D_n\Gamma^c u^k)dX,\quad b+c=a,\ b,c\ne a.$$

We control

$$\begin{aligned}&\|\langle r\rangle\langle c_\alpha t-r\rangle^{1/2}P_\alpha D\Gamma^a u(t)\|_{L^\infty},\quad |a|\le k-3\\&\|\langle r\rangle\langle c_\alpha t-r\rangle P_\alpha D^2\Gamma^a u(t)\|_{L^2},\quad |a|\le k-2,\end{aligned}$$

by the energy $\tilde{E}_k[u(t)]^{1/2}$.

We separate the two wave families by inserting the projections. For example, we can write (7.6) as

$$\sum_{\alpha,\beta,\gamma}\int B^{ijk}_{\ell mn}(I)P_\alpha D_\ell\Gamma^a u^i P_\beta D_m\Gamma^a u^j P_\gamma D_0 D_n u^k dX.$$

In the nonresonant case, $(\alpha,\beta,\gamma)\ne(1,1,1)$ or $(2,2,2)$, then we can absorb either

$$\langle r\rangle\langle c_1 t-r\rangle^{1/2}\langle c_2 t-r\rangle\quad\text{or}\quad\langle r\rangle\langle c_1 t-r\rangle\langle c_2 t-r\rangle^{1/2},$$

which gives $\langle t\rangle^{-3/2}$ decay.

If $(\alpha,\beta,\gamma)=(1,1,1)$, then away from the cone $\langle c_1 t-r\rangle\ge\langle t\rangle$, we absorb the weight

$$\langle r\rangle\langle c_1 t-r\rangle^{3/2},$$

which again gives sufficient decay in this region. Near the cone, $\langle c_1 t-r\rangle\le\langle t\rangle$, we use the decomposition (7.5) to rewrite our integral as

$$\int B^{ijk}_{\ell mn}(I)\xi^i\xi^j\xi^k\xi^\ell\xi^m\xi^n\langle\xi,D_r\Gamma^a u\rangle^2\langle\xi,D_r D_0 u\rangle dX+\int\mathcal{R}dX,$$

in which $\xi=X/r$ and

$$|\mathcal{R}|\le Cr^{-1}|\Gamma^{a+1}u|^2|D\Gamma u|.$$

The first term vanishes, thanks to the null condition, and the additional decay in the remainder is sufficient to complete the estimates near the cone. The other resonant case, $(\alpha, \beta, \gamma) = (2, 2, 2)$ is similar.

8. Remarks on Theorems 2 and 3

The penalization term in the strain energy function of Theorem 2 is designed to drive the motion toward incompressibility, $\rho = \rho_0$, as the fast speed $\lambda \to \infty$. To measure this, it is natural to use the conservation of mass equation, but then this leads to the consideration of the first order version of the system in Eulerian coordinates (3.1) with constraints. The advantage is that, using the variable $\lambda(\rho - \rho_0)$, the equations can be written in such a way that the singular parameter only appears in linear terms. The disadvantage is that, in Eulerian coordinates, the null condition for the pressure waves is lost. We therefore settle for long time local existence on a time scale proportional to the parameter λ.

A simple model for this situation is given by the two speed system

$$\partial_t^2 u^1 - \lambda^2 \Delta u^1 = B^{1jk}_{\ell mn} D_\ell (D_m u^j D_n u^k)$$
$$\partial_t^2 u^2 - \Delta u^2 = B^{2jk}_{\ell mn} D_\ell (D_m u^j D_n u^k),$$

$(\lambda \gg 1)$, where the null condition for the slow family holds:

$$B^{222}_{\ell mn} \xi_\ell \xi_m \xi_n = 0, \quad \xi \in S^2.$$

Here, u^1 plays the role of $\lambda(\rho - \rho_0)$ and u^2 measures the shear waves. The obstruction to global existence for small solutions is the lack of the null condition for the fast waves. However, the dispersive estimate says that their amplitude is roughly

$$[\langle x \rangle \langle \lambda t - |x| \rangle]^{-1} \leq C \langle \lambda t \rangle^{-1}.$$

This is enough to obtain an extended lifespan as $\lambda \to \infty$.

The proof of Theorem 2 uses the same basic techniques as outlined above for Theorem 1, except that the first order system is more difficult technically. In particular, the dispersive estimate can only be obtained when the solutions satisfy the natural constraints (2.9) which link back to the original second order problem.

Theorem 3 is a corollary of the stability result which gives uniform estimates for the solution family and their time derivatives. These bounds provide enough compactness to be able to pass the the limit in the pde's locally uniformly. The interested reader can consult [22] for details.

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