Duhamel vs. Variation of Parameters

Consider the Cauchy problem for the 1D wave eqn.

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

D'Alembert's formula says

$$u(x,t) = [T'(t)f](x) + [T(t)g](x)$$

where $T(t)$ is the operator

$$[T(t)g](x) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.$$

Rewrite the problem in 1^{st} order form as follows:

Let $\quad v = u_t \qquad$ Then

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ c^2 u_{xx} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c^2\partial_x^2 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = A\begin{bmatrix} u \\ v \end{bmatrix}$$

$$u(0) = f \qquad v(0) = g$$

Formally, the solution is

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = e^{At}\begin{bmatrix} f \\ g \end{bmatrix}.$$

Comparing with d'Alembert, we have

$$e^{At} = \begin{bmatrix} T'(t) & T(t) \\ T''(t) & T'(t) \end{bmatrix}$$

{ Exercise: Verify that $e^{A(t+s)} = e^{At} e^{As}$. }

Now suppose that we want to solve the inhomogeneous problem

$$u_{tt} - c^2 u_{xx} = F(x,t)$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x).$$

This is equivalent to the 1st order system

$$\partial_t \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix}$$

$$u(0) = f \qquad v(0) = g.$$

Variation of parameters says

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = e^{At} \cdot \begin{bmatrix} f \\ g \end{bmatrix} + \int_0^t e^{A(t-s)} \begin{bmatrix} 0 \\ F(s) \end{bmatrix} ds$$

Reading off the first line gives Duhamel:

$$u(t) = T'(t) f + T(t) g + \int_0^t T(t-s) F(s)\, ds.$$