

**Ordinary Differential Equations  
and  
Dynamical Systems**

Thomas C. Sideris

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA,  
SANTA BARBARA, CA 93106

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## CHAPTER 1

### Linear Systems

#### 1.1. Exponential of a Linear Transformation

Let  $V$  be a finite dimensional normed vector space over  $\mathbb{R}$  or  $\mathbb{C}$ .  $L(V)$  will denote the set of linear transformations from  $V$  into  $V$ .

DEFINITION 1.1.1. Let  $A \in L(V)$ . Define the operator norm

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|.$$

Properties:

- $\|A\| < \infty$ , for every  $A \in L(V)$ .
- $L(V)$  with the operator norm is a finite dimensional normed vector space.
- $\|Ax\| \leq \|A\|\|x\|$ , for every  $A \in L(V)$  and  $x \in V$ .
- $\|AB\| \leq \|A\|\|B\|$ , for every  $A, B \in L(V)$ .

DEFINITION 1.1.2. A sequence  $\{A_n\}$  in  $L(V)$  converges to  $A$  if and only if

$$\lim_{n \rightarrow \infty} \|A_n - A\| = 0.$$

With this notion of convergence,  $L(V)$  is complete.

All norms on a finite dimensional space are equivalent, so  $A_n \rightarrow A$  in the operator norm implies componentwise convergence in any coordinate system.

DEFINITION 1.1.3. Given  $A \in L(V)$ , define  $\exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$ .

The exponential is well-defined in the sense that the sequence of partial sums

$$S_n = \sum_{k=0}^n \frac{1}{k!} A^k$$

has a limit. This can be seen by showing that  $S_n$  is a Cauchy sequence. Let  $m < n$ . Then,

$$\begin{aligned}
\|S_n - S_m\| &= \left\| \sum_{k=m+1}^n \frac{1}{k!} A^k \right\| \\
&\leq \sum_{k=m+1}^n \frac{1}{k!} \|A^k\| \\
&\leq \sum_{k=m+1}^n \frac{1}{k!} \|A\|^k \\
&= \frac{1}{(m+1)!} \|A\|^{m+1} \sum_{k=0}^{n-m-1} \frac{(m+1)!}{(k+m+1)!} \|A\|^k \\
&\leq \frac{1}{(m+1)!} \|A\|^{m+1} \sum_{k=0}^{\infty} \frac{1}{k!} \|A\|^k \\
&= \frac{1}{(m+1)!} \|A\|^{m+1} \exp \|A\|.
\end{aligned}$$

From this, we see that  $S_n$  is Cauchy.

It also follows that  $\|\exp A\| \leq \exp \|A\|$ .

LEMMA 1.1.1. *Given  $A, B \in L(V)$ , we have the follow properties:*

- (1)  $\exp At$  exists for all  $t \in \mathbb{R}$ .
- (2)  $\exp A(t+s) = \exp At \exp As = \exp As \exp At$ , for all  $t, s \in \mathbb{R}$ .
- (3)  $\exp(A+B) = \exp A \exp B = \exp B \exp A$ , provided  $AB = BA$ .
- (4)  $\exp At$  is invertible for every  $t \in \mathbb{R}$ , and  $(\exp At)^{-1} = \exp(-At)$ .
- (5)  $\frac{d}{dt} \exp At = A \exp At = \exp At A$ .

PROOF. (1) was shown in the preceding paragraph.

(2) is a consequence of (3).

To prove (3), we first note that when  $AB = BA$  the binomial expansion is valid:

$$(A+B)^k = \sum_{j=0}^k \binom{k}{j} A^j B^{k-j}.$$

Thus, by definition

$$\begin{aligned}
 \exp(A + B) &= \sum_{k=0}^{\infty} \frac{1}{k!} (A + B)^k \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} A^j B^{k-j} \\
 &= \sum_{j=0}^{\infty} \frac{1}{j!} A^j \sum_{k=j}^{\infty} \frac{1}{(k-j)!} B^{k-j} \\
 &= \sum_{j=0}^{\infty} \frac{1}{j!} A^j \sum_{\ell=0}^{\infty} \frac{1}{\ell!} B^{\ell} \\
 &= \exp A \exp B.
 \end{aligned}$$

The rearrangements are justified by the absolute convergence of all series.

(4) is an immediate consequence of (2).

(5) is proven as follows. We have

$$\begin{aligned}
 &\|(\Delta t)^{-1}[\exp A(t + \Delta t) \exp At] - \exp At A\| \\
 &= \|\exp At\{(\Delta t)^{-1}[\exp A\Delta t - I] - A\}\| \\
 &= \left\| \exp At \sum_{k=2}^{\infty} \frac{(\Delta t)^{k-1}}{k!} A^k \right\| \\
 &\leq \|\exp At\| \left\| A^2 \Delta t \sum_{k=2}^{\infty} \frac{(\Delta t)^{k-2}}{k!} A^{k-2} \right\| \\
 &\leq |\Delta t| \|A\|^2 \exp \|A\| (|t| + |\Delta t|).
 \end{aligned}$$

This last expression tends to 0 as  $\Delta t \rightarrow 0$ . Thus, we have shown that  $\frac{d}{dt} \exp At = \exp At A$ . This also equals  $A \exp At$  because  $A$  commutes with the partial sums for  $\exp At$  and hence with  $\exp At$  itself.  $\square$

## 1.2. Solution of the Initial Value Problem for Linear Homogeneous Systems

**THEOREM 1.2.1.** *Let  $A$  be an  $n \times n$  matrix over  $\mathbb{R}$ , and let  $x_0 \in \mathbb{R}^n$ . The initial value problem*

$$(1.2.1) \quad x'(t) = Ax(t), \quad x(t_0) = x_0$$

*has a unique solution defined for all  $t \in \mathbb{R}$  given by*

$$(1.2.2) \quad x(t) = \exp A(t - t_0) x_0.$$

PROOF. We use the method of the integrating factor. Multiplying the system (1.2.1) by  $\exp(-At)$  and using Lemma 1.1.1, we see that  $x(t)$  is a solution of the IVP if and only if

$$\frac{d}{dt}[\exp(-At)x(t)] = 0, \quad x(t_0) = x_0.$$

Integration of this identity yields the equivalent statement

$$\exp(-At)x(t) - \exp(-At_0)x_0 = 0,$$

which in turn is equivalent to (1.2.2). This establishes existence, and uniqueness.  $\square$

### 1.3. Computation of the Exponential

The main computational tool will be reduction to an elementary case by similarity transformation.

LEMMA 1.3.1. *Let  $A, S \in L(V)$  with  $S$  invertible. Then*

$$\exp(SAS^{-1}) = S(\exp A)S^{-1}.$$

PROOF. This follows immediately from the definition of the exponential together with the fact that  $(SAS^{-1})^k = SA^kS^{-1}$ , for every  $k \in \mathbb{N}$ .  $\square$

The simplest case is that of a diagonal matrix  $D = \text{diag} [\lambda_1, \dots, \lambda_n]$ . Since  $D^k = \text{diag} [\lambda_1^k, \dots, \lambda_n^k]$ , we immediately obtain

$$\exp Dt = \text{diag} [\exp \lambda_1 t, \dots, \exp \lambda_n t].$$

Now if  $A$  is diagonalizable, i.e.  $A = SDS^{-1}$ , then we can use Lemma 1.3.1 to compute

$$\exp At = S \exp Dt S^{-1}.$$

An  $n \times n$  matrix  $A$  is diagonalizable if and only if there is a basis of eigenvectors  $\{v_j\}_{j=1}^n$ . If such a basis exists, let  $\{\lambda_j\}_{j=1}^n$  be the corresponding set of eigenvalues. Then

$$A = SDS^{-1},$$

where  $D = \text{diag} [\lambda_1, \dots, \lambda_n]$  and  $S = [v_1 \cdots v_n]$  is the matrix whose columns are formed by the eigenvectors. Even if  $A$  has real entries, it can have complex eigenvalues, in which case the matrices  $D$  and  $S$  will have complex entries. However, if  $A$  is real, complex eigenvectors and eigenvalues occur in conjugate pairs.

In the diagonalizable case, the solution of the initial value problem (1.2.1) is

$$x(t) = \exp At x_0 = S \exp Dt S^{-1} x_0 = \sum_{j=1}^n c_j \exp \lambda_j t v_j,$$

where the coefficients  $c_j$  are the coordinates of the vector  $c = S^{-1}x_0$ . Thus, the solution space is spanned by the elementary solutions  $\exp \lambda_j t v_j$ .

There are two important situations where an  $n \times n$  matrix can be diagonalized.

- $A$  is real and symmetric, i.e.  $A = A^T$ . Then  $A$  has real eigenvalues and there exists an orthonormal basis of real eigenvectors. Using this basis yields an orthogonal diagonalizing matrix  $S$ , i.e.  $S^T = S^{-1}$ .
- $A$  has distinct eigenvalues. For each eigenvalue there is always at least one eigenvector, and eigenvectors corresponding to distinct eigenvalues are independent. Thus, there is a basis of eigenvectors.

An  $n \times n$  matrix over  $\mathbb{C}$  may not be diagonalizable, but it can always be reduced to Jordan canonical (or normal) form. A matrix  $J$  is in Jordan canonical form if it is block diagonal

$$J = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_p \end{bmatrix}$$

and each Jordan block has the form

$$B = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

Since  $B$  is upper triangular, it has the single eigenvalue  $\lambda$  with multiplicity equal to the size of the block  $b$ .

Computing the exponential of a Jordan block is easy. Write

$$B = \lambda I + N,$$

where  $N$  has 1's along the superdiagonal and 0's everywhere else. The matrix  $N$  is nilpotent. If the block size is  $d \times d$ , then  $N^d = 0$ . We also

clearly have that  $\lambda I$  and  $N$  commute. Therefore,

$$\exp Bt = \exp(\lambda I + N)t = \exp \lambda It \exp Nt = \exp(\lambda t) \sum_{j=1}^{d-1} \frac{t^j}{j!} N^j.$$

The entries of  $\exp Nt$  are polynomials in  $t$  of degree at most  $d - 1$ .

Again using the definition of the exponential, we have that the exponential of a matrix in Jordan canonical form is the block diagonal matrix

$$\exp Jt = \begin{bmatrix} \exp B_1 t & & \\ & \ddots & \\ & & \exp B_p t \end{bmatrix}.$$

The following central theorem in linear algebra will enable us to understand the form of  $\exp At$  for a general matrix  $A$ .

**THEOREM 1.3.1.** *Let  $A$  be an  $n \times n$  matrix over  $\mathbb{C}$ . There exists a basis  $\{v_j\}_{j=1}^n$  for  $\mathbb{C}^n$  which reduces  $A$  to Jordan normal form  $J$ . That is, if  $S = [v_1 \cdots v_n]$  is the matrix whose columns are formed from the basis vectors, then*

$$A = SJS^{-1}.$$

*The Jordan normal form of  $A$  is unique up to the permutation of its blocks.*

When  $A$  is diagonalizable, the basis  $\{v_j\}_{j=1}^n$  consists of eigenvectors of  $A$ . In this case, the Jordan blocks are  $1 \times 1$ . Thus, each vector  $v_j$  lies in the kernel of  $A - \lambda_j I$  for the corresponding eigenvalue  $\lambda_j$ .

In the general case, the basis  $\{v_j\}_{j=1}^n$  consists of appropriately chosen *generalized eigenvectors* of  $A$ . A vector  $v$  is a generalized eigenvector of  $A$  corresponding to an eigenvalue  $\lambda_j$  if it lies in the kernel of  $(A - \lambda_j I)^k$  for some  $k \in \mathbb{N}$ . The set of generalized eigenvectors of  $A$  corresponding to a given eigenvalue  $\lambda_j$  is a subspace,  $E(\lambda_j)$ , of  $\mathbb{C}^n$ , called the generalized eigenspace of  $\lambda_j$ . If  $\{\lambda_j\}_{j=1}^d$  are the distinct eigenvalues of  $A$ , then

$$\mathbb{C}^n = E(\lambda_1) \oplus \cdots \oplus E(\lambda_d),$$

as a direct sum.

We arrive at the following algorithm for computing  $\exp At$ . Given an  $n \times n$  matrix  $A$ , reduce it to Jordan canonical form  $A = SJS^{-1}$ , and then write

$$\exp At = S \exp Jt S^{-1}.$$

Even if  $A$  (and hence also  $\exp At$ ) has real entries, the matrices  $J$  and  $S$  may have complex entries. However, if  $A$  is real, then any complex eigenvalues and generalized eigenvectors occur in conjugate pairs.

### 1.4. Asymptotic Behavior of Linear Systems

DEFINITION 1.4.1. Let  $A$  be an  $n \times n$  matrix over  $\mathbb{R}$ . Define the complex stable, unstable, and center subspaces of  $A$ , denoted  $E_s^{\mathbb{C}}$ ,  $E_u^{\mathbb{C}}$ , and  $E_c^{\mathbb{C}}$ , respectively, to be the linear span over  $\mathbb{C}$  of the generalized eigenvectors of  $A$  corresponding to eigenvalues with negative, positive, and zero real parts, respectively.

Arrange the eigenvalues of  $A$  so that  $\operatorname{Re} \lambda_1 \leq \dots \leq \operatorname{Re} \lambda_n$ . Partition the set  $\{1, \dots, n\} = J_s \cup J_c \cup J_u$  so that

$$\begin{aligned} \operatorname{Re} \lambda_j < 0, & & j \in J_s \\ \operatorname{Re} \lambda_j = 0, & & j \in J_c \\ \operatorname{Re} \lambda_j > 0, & & j \in J_u. \end{aligned}$$

Let  $x_1, \dots, x_n \in \mathbb{C}^n$  be a basis of generalized eigenvectors corresponding to the eigenvalues  $\lambda_1, \dots, \lambda_n$ . Then

$$\begin{aligned} \operatorname{span} \{x_j : j \in J_s\} &= E_s^{\mathbb{C}} \\ \operatorname{span} \{x_j : j \in J_c\} &= E_c^{\mathbb{C}} \\ \operatorname{span} \{x_j : j \in J_u\} &= E_u^{\mathbb{C}}. \end{aligned}$$

It follows that  $\mathbb{C}^n = E_s^{\mathbb{C}} + E_c^{\mathbb{C}} + E_u^{\mathbb{C}}$  is a direct sum. Thus, any vector  $x \in \mathbb{C}^n$  is uniquely represented as

$$x = P_s x + P_c x + P_u x \in E_s^{\mathbb{C}} + E_c^{\mathbb{C}} + E_u^{\mathbb{C}}.$$

The maps  $P_s, P_c, P_u$  are linear projections onto the complex stable, center, and unstable subspaces. Thus, we have

$$P_s^2 = P_s, \quad P_c^2 = P_c, \quad P_u^2 = P_u.$$

Since these subspaces are independent of each other, we have that

$$P_s P_c = P_c P_s = 0, \dots$$

Since these subspaces are invariant under  $A$ , the projections commute with  $A$ , and thus also any function of  $A$ , including  $\exp At$ .

Since  $A$  is real, if  $v \in \mathbb{C}^n$  is a generalized eigenvector with eigenvalue  $\lambda \in \mathbb{C}$ , then its complex conjugate  $\bar{v}$  is a generalized eigenvector with eigenvalue  $\bar{\lambda}$ . It follows that the subspaces  $E_s^{\mathbb{C}}$ ,  $E_c^{\mathbb{C}}$ , and  $E_u^{\mathbb{C}}$  are closed under complex conjugation. For any vector  $x \in \mathbb{C}^n$ , we have

$$\overline{P_s x} + \overline{P_c x} + \overline{P_u x} = \bar{x} = P_s \bar{x} + P_c \bar{x} + P_u \bar{x}.$$

This gives two representations of  $\bar{x}$  in  $E_s^{\mathbb{C}} + E_c^{\mathbb{C}} + E_u^{\mathbb{C}}$ . By uniqueness of representations, we must have

$$P_s \bar{x} = \overline{P_s x}, \quad P_c \bar{x} = \overline{P_c x}, \quad P_u \bar{x} = \overline{P_u x}.$$

So if  $x \in \mathbb{R}^n$ , we have that

$$P_s x = \overline{P_s x}, \quad P_c x = \overline{P_c x}, \quad P_u x = \overline{P_u x}.$$

Therefore, the projections leave  $\mathbb{R}^n$  invariant:

$$P_s : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad P_c : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad P_u : \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

DEFINITION 1.4.2. *Let  $A$  be an  $n \times n$  matrix over  $\mathbb{R}$ . Define the real stable, unstable, and center subspaces of  $A$ , denoted  $E_s$ ,  $E_u$ , and  $E_c$ , to be the images of  $\mathbb{R}^n$  under the corresponding projections:*

$$E_s = P_s \mathbb{R}^n, \quad E_c = P_c \mathbb{R}^n, \quad E_u = P_u \mathbb{R}^n.$$

We have that  $\mathbb{R}^n = E_s + E_c + E_u$  is a direct sum. When restricted to  $\mathbb{R}^n$ , the projections possess the same properties as on  $\mathbb{C}^n$ .

The real stable subspace can also be characterized as the linear span over  $\mathbb{R}$  of the real and imaginary parts of all generalized eigenvectors of  $A$  corresponding to an eigenvalue with negative real part. Similar statements hold for  $E_c$  and  $E_u$ .

We are now ready for the main result of this section, which estimates the norm of  $\exp At$  on the invariant subspaces. These estimates will be used many times.

THEOREM 1.4.1. *Let  $A$  an  $n \times n$  matrix over  $\mathbb{R}$ . Define*

$$-\lambda_s = \max\{\lambda_j : j \in J_s\} \quad \text{and} \quad \lambda_u = \min\{\lambda_j : j \in J_u\}.$$

*There is a constant  $C > 0$  and an integer  $0 \leq p < n$ , depending on  $A$ , such that for all  $x \in \mathbb{C}^n$ ,*

$$\begin{aligned} \|\exp At P_s x\| &\leq C(1+t)^p e^{-\lambda_s t} \|P_s x\|, & t > 0 \\ \|\exp At P_c x\| &\leq C(1+|t|)^p \|P_c x\|, & t \in \mathbb{R} \\ \|\exp At P_u x\| &\leq C(1-t)^p e^{\lambda_u t} \|P_u x\|, & t < 0. \end{aligned}$$

Remark: The exponent  $p$  in these inequalities has the property that  $p+1$  is the size of the largest Jordan block corresponding to eigenvalues  $\lambda_j$  with  $j \in J_s, J_c, J_u$ , respectively.

PROOF. We will prove the first of these inequalities. The other two are similar.

Let  $\{x_j\}_{j=1}^n$  be a basis generalized eigenvectors with indices ordered as above. For any  $x \in \mathbb{C}^n$ , we have

$$x = \sum_{j=1}^n c_j x_j, \quad \text{and} \quad P_s x = \sum_{j \in J_s} c_j x_j.$$

Let  $S$  be the matrix whose columns are the vectors  $x_j$ . Then  $S$  reduces  $A$  to Jordan canonical form:  $A = S(D + N)S^{-1}$ , where  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $N^{p+1} = 0$ , for some  $p < n$ .

If  $\{e_j\}_{j=1}^n$  is the standard basis, the  $Se_j = x_j$ , and so,  $e_j = S^{-1}x_j$ . We may write

$$\begin{aligned} \exp At P_s x &= S \exp Nt \exp Dt S^{-1} P_s x \\ &= S \exp Nt \exp Dt \sum_{j \in J_s} c_j e_j \\ &= S \exp Nt \sum_{j \in J_s} c_j \exp(\lambda_j t) e_j \\ &= S \exp Nt y. \end{aligned}$$

Taking the norm, we have

$$\|\exp At P_s x\| \leq \|S\| \|\exp Nt y\|.$$

Now,  $y \in \text{span} \{e_j : j \in J_s\}$ , and so if  $p + 1$  is the size of the largest Jordan block corresponding to eigenvalues  $\{\lambda_j : j \in J_s\}$ , then  $N^{p+1}y = 0$ . Thus, we have that

$$\exp Nt y = \sum_{k=0}^p \frac{t^k}{k!} N^k y,$$

and so, for  $t > 0$ ,

$$\|\exp Nt y\| \leq \sum_{k=0}^p \frac{t^k}{k!} \|N\|^k \leq C_1 (1 + t^p) \|y\|.$$

Next, we have, for  $t > 0$ ,

$$\begin{aligned} \|y\|^2 &= \left\| \sum_{j \in J_s} c_j \exp(\lambda_j t) e_j \right\|^2 \\ &= \sum_{j \in J_s} |c_j|^2 \exp(2\text{Re } \lambda_j t) \\ &\leq \exp(-2\lambda_s t) \sum_{j \in J_s} |c_j|^2 \\ &= \exp(-2\lambda_s t) \|S^{-1} P_s x\|^2 \\ &\leq \exp(-2\lambda_s t) \|S^{-1}\|^2 \|P_s x\|^2, \end{aligned}$$

and so  $\|y\| \leq C_2 \exp(-\lambda_s t) \|P_s x\|$ .

The result follows with  $C = C_1 C_2$ .

□

All of the results in this section hold for complex matrices  $A$ , except for the remarks concerning the projections on  $\mathbb{R}^n$  and the ensuing definitions of real invariant subspaces. We will not need this, however.

Notice that for any  $\varepsilon > 0$ , the function  $(1 + t^p) \exp(-\varepsilon t)$  is bounded on the interval  $t > 0$ . Thus, for any constant  $0 < \alpha < \lambda_s$ , we have that  $(1 + t^p) \exp(-\lambda_s t) = (1 + t^p) \exp[-(\lambda_s - \alpha)t] \exp(-\alpha t) \leq C \exp(-\alpha t)$ . It will be convenient to use this slightly weaker version.

## CHAPTER 2

### Existence Theory

#### 2.1. The Initial Value Problem

Let  $\Omega \subset \mathbb{R}^{n+1}$  be an open connected set. We will denote points in  $\Omega$  by  $(t, x)$  where  $t \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ . Let  $f : \Omega \rightarrow \mathbb{R}^n$  be a continuous map. In this context,  $f(t, x)$  is called a vector field on  $\Omega$ . Given any initial point  $(t_0, x_0) \in \Omega$ , we wish to construct a unique solution to the initial value problem

$$(2.1.1) \quad x'(t) = f(t, x(t)) \quad x(t_0) = x_0.$$

In order for this to make sense,  $x(t)$  must be a  $C^1$  function from some interval  $I \subset \mathbb{R}$  containing the initial time  $t_0$  into  $\mathbb{R}^n$  such that the solution curve satisfies

$$\{(t, x(t)) : t \in I\} \subset \Omega.$$

Such a solution is referred to as a local solution when  $I \neq \mathbb{R}$ . When  $I = \mathbb{R}$ , the solution is called global.

#### 2.2. The Cauchy-Peano Existence Theorem

**THEOREM 2.2.1 (Cauchy-Peano).** *If  $f : \Omega \rightarrow \mathbb{R}^n$  is continuous, then for every point  $(t_0, x_0) \in \Omega$  the initial value problem (2.1.1) has local solution.*

The problem with this theorem is that it does not guarantee uniqueness. We will skip the proof, except to mention that it uses a compactness argument based on the Arzela-Ascoli Theorem.

Here is a simple example that demonstrates that uniqueness can indeed fail. Let  $\Omega = \mathbb{R}^2$  and consider the autonomous vector field  $f(t, x) = |x|^{1/2}$ . When  $(t_0, x_0) = (0, 0)$ , the initial value problem has infinitely many solutions. In addition to the zero solution  $x(t) = 0$ , for any  $\alpha, \beta \geq 0$ , the following is a family of solutions.

$$x(t) = \begin{cases} -\frac{1}{4}(t + \alpha)^2, & t \leq -\alpha \\ 0, & -\alpha \leq t \leq \beta \\ \frac{1}{4}(t - \beta)^2, & \beta \leq t. \end{cases}$$

This can be verified by direct substitution.

### 2.3. The Picard Existence Theorem

The failure of uniqueness can be rectified by placing an additional restriction on the vector field. The next definition introduces this key property.

**DEFINITION 2.3.1.** *Let  $\Omega \subset \mathbb{R}^{n+1}$  be an open set. A function  $f : \Omega \rightarrow \mathbb{R}^n$  is said to be locally Lipschitz continuous in  $x$  if for every compact set  $K \subset \Omega$ , there is a constant  $C_K > 0$  such that*

$$\|f(t, x_1) - f(t, x_2)\| \leq C_K \|x_1 - x_2\|,$$

*for every  $(t, x_1), (t, x_2) \in K$ . If there is a constant for which the inequality holds for all  $(t, x_1), (t, x_2) \in \Omega$ , then  $f$  is said to be Lipschitz continuous in  $x$ .*

The function  $\|x\|^\alpha$  is Lipschitz continuous for  $\alpha = 1$ , locally Lipschitz continuous for  $\alpha > 1$ , and not Lipschitz continuous (on any neighborhood of 0) when  $\alpha < 1$ .

**LEMMA 2.3.1.** *If  $f : \Omega \rightarrow \mathbb{R}^n$  is  $C^1$ , then it is locally Lipschitz continuous in  $x$ .*

**THEOREM 2.3.1 (Picard).** *Let  $\Omega \subset \mathbb{R}^{n+1}$  be open. Assume that  $f : \Omega \rightarrow \mathbb{R}^n$  is continuous and that  $f(t, x)$  is locally Lipschitz continuous in  $x$ . Let  $K \subset \Omega$  be any compact set. Then there is a  $\delta > 0$  such that for every  $(t_0, x_0) \in K$ , the initial value problem (2.1.1) has a unique local solution defined on the interval  $|t - t_0| < \delta$ .*

Before proving this important theorem, it is convenient to have the following technical ‘‘Covering Lemma’’.

First, some notation: Given a point  $(t, x) \in \mathbb{R}^{n+1}$  and positive numbers  $r$  and  $a$ , define the cylinder

$$\mathcal{C}(t, x) \equiv \{(t', x') \in \mathbb{R}^{n+1} : \|x - x'\| \leq r, |t - t'| \leq a\}.$$

**LEMMA 2.3.2 (Covering Lemma).** *Let  $K \subset \Omega \subset \mathbb{R}^n \times \mathbb{R}$  with  $\Omega$  an open set and  $K$  a compact set. There exists a compact set  $K'$  and positive numbers  $r$  and  $a$  such that  $K \subset K' \subset \Omega$  and  $\mathcal{C}(t, x) \subset K'$ , for all  $(t, x) \in K$ .*

**PROOF.** For every point  $p = (t, x) \in K$ , choose positive numbers  $a(p)$  and  $r(p)$  such that

$$D(p) = \{(t', x') \in \mathbb{R}^{n+1} : \|x - x'\| \leq 2r(p), |t - t'| \leq 2a(p)\} \subset \Omega.$$

This is possible because  $\Omega$  is open.

Define the cylinders

$$C(p) = \{(t', x') \in \mathbb{R}^{n+1} : \|x - x'\| < r(p), |t - t'| < a(p)\}.$$

The collection of open sets  $\{C(p) : p \in K\}$  forms an open cover of the set  $K$ .  $K$  is compact, therefore there is a finite number of cylinders  $C(p_1), \dots, C(p_N)$  whose union contains  $K$ . Set

$$K' = \cup_{i=1}^N D(p_i).$$

Then  $K'$  is compact, and

$$K \subset \cup_{i=1}^N C(p_i) \subset \cup_{i=1}^N D(p_i) = K' \subset \Omega.$$

Define

$$a = \min\{a(p_i) : i = 1, \dots, N\} \quad \text{and} \quad r = \min\{r(p_i) : i = 1, \dots, N\}.$$

The claim is that, for this uniform choice of  $a$  and  $r$ ,  $\mathcal{C}(t, x) \subset K'$ , for all  $(t, x) \in K$ .

If  $(t, x) \in K$ , then  $(t, x) \in C(p_i)$  for some  $i = 1, \dots, N$ . Let  $(t', x') \in \mathcal{C}(t, x)$ . Then

$$\|x' - x_i\| \leq \|x' - x\| + \|x - x_i\| \leq a + a(p_i) \leq 2a(p_i)$$

and

$$|t' - t_i| \leq |t' - t| + |t - t_i| \leq r + r(p_i) \leq 2r(p_i).$$

This shows that  $(t', x') \in D(p_i)$ , from which follows the conclusion  $\mathcal{C}(t, x) \subset D(p_i) \subset K'$ .  $\square$

**PROOF OF THE PICARD THEOREM.** The first step of the proof is to reformulate the problem. If  $x(t)$  is a  $C^1$  solution of the initial value problem (2.1.1) for  $|t - t_0| \leq \delta$ , then by integration we find that

$$(2.3.1) \quad x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds,$$

for  $|t - t_0| \leq \delta$ . Conversely, if  $x(t)$  is a  $C^0$  solution of the integral equation, then it is  $C^1$  and it solves the initial value problem (2.1.1).

Given a compact subset  $K \subset \Omega$ , choose  $a, r, K'$  as in the covering lemma.

Choose  $(t_0, x_0) \in K$ . Let  $\delta < a$  and set

$$I_\delta = \{|t - t_0| \leq \delta\}, \quad B_r = \{\|x - x_0\| \leq r\}, \quad X_\delta = C^0(I_\delta; B_r).$$

Note that  $X_\delta$  is a complete metric space with the sup norm metric.

By definition, if  $x \in X_\delta$ , then

$$(s, x(s)) \in \mathcal{C}(t_0, x_0) \subset K' \subset \Omega,$$

for  $s \in I_\delta$ . Thus, the operator

$$Tx(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds$$

is well-defined on  $X_\delta$  and the function  $Tx(t)$  is continuous for  $t \in I_\delta$ .

Define  $M_1 = \max_{K'} |f(t, x)|$ . This claim is that if  $\delta$  is chosen small enough so that  $M_1\delta \leq r$ , then  $T : X_\delta \rightarrow X_\delta$ . If  $x \in X_\delta$ , we have from (2.3.1)

$$\sup_{I_\delta} \|Tx(t) - x_0\| \leq M_1\delta \leq r,$$

for  $t \in I_\delta$ .

Next, let  $M_2$  be a Lipschitz constant for  $f(t, x)$  on  $K'$ . If  $\delta$  is further restricted so that  $M_2\delta < 1/2$ , then we claim that  $T : X_\delta \rightarrow X_\delta$  is a contraction. Let  $x_1, x_2 \in X_\delta$ . Then from (2.3.1), we have

$$\begin{aligned} \sup_{I_\delta} \|Tx_1(t) - Tx_2(t)\| &\leq M_2\delta \sup_{I_\delta} \|x_1(t) - x_2(t)\| \\ &\leq 1/2 \sup_{I_\delta} \|x_1(t) - x_2(t)\|. \end{aligned}$$

So by the Contraction Mapping Principle, there exists a unique function  $x \in X_\delta$  such that  $Tx = x$ . In other words,  $x$  solves (2.3.1).  $\square$

Note that the final choice of  $\delta$  is  $\min\{a, r/M_1, 1/M_2\}$  which depends only on the set  $K$  and on  $f$ .

The alert reader will notice that the solution constructed above is unique within the metric space  $X_\delta$ , but it is not necessarily unique in  $C^0(I_\delta, B_r)$ . The next result fills in this gap.

**THEOREM 2.3.2 (Uniqueness).** *Suppose that  $f : \Omega \rightarrow \mathbb{R}^n$  satisfies the hypotheses of the Picard Theorem. For  $j = 1, 2$ , let  $x_j(t)$  be solutions of  $x'(t) = f(t, x(t))$  on the interval  $I_j$ . If there is a point  $t_0 \in I_1 \cap I_2$  such that  $x_1(t_0) = x_2(t_0)$ , then  $x_1(t) = x_2(t)$  on the interval  $I_1 \cap I_2$ . Moreover, the function*

$$x(t) = \begin{cases} x_1(t), & t \in I_1 \\ x_2(t), & t \in I_2 \end{cases}$$

*defines a solution on the interval  $I_1 \cup I_2$ .*

**PROOF.** Let  $J \subset I_1 \cap I_2$  be any closed interval with  $t_0 \in J$ . Let  $M$  be a Lipschitz constant for  $f(t, x)$  on the compact set

$$\{(t, x_1(t)) : t \in J\} \cup \{(t, x_2(t)) : t \in J\}.$$

The solutions  $x_j(t)$ ,  $j = 1, 2$ , satisfy the integral equation (2.3.1) on the interval  $J$ . Thus, estimating as before

$$\|x_1(t) - x_2(t)\| \leq \left| \int_{t_0}^t M \|x_1(s) - x_2(s)\| ds \right|,$$

for  $t \in J$ . It follows from Gronwall's Lemma (below) that

$$\|x_1(t) - x_2(t)\| = 0$$

for  $t \in J$ .

From this it follows that  $x(t)$  is well-defined, is  $C^1$ , and is a solution.  $\square$

LEMMA 2.3.3 (Gronwall). *Let  $f(t)$ ,  $\varphi(t)$  be nonnegative continuous function on an open interval  $J = (\alpha, \beta)$  containing the point  $t_0$ . Let  $c_0 \geq 0$ . If*

$$f(t) \leq c_0 + \left| \int_{t_0}^t \varphi(s)f(s)ds \right|,$$

for all  $t \in J$ , then

$$f(t) \leq c_0 \exp \left| \int_{t_0}^t \varphi(s)ds \right|,$$

for  $t \in J$ .

PROOF. Suppose first that  $t \in [t_0, \beta)$ . Define

$$F(t) = c_0 + \int_{t_0}^t \varphi(s)f(s)ds.$$

Then  $F$  is  $C^1$  and

$$F'(t) = \varphi(t)f(t) \leq \varphi(t)F(t),$$

for  $t \in [t_0, \beta)$ , since  $f(t) \leq F(t)$ . This implies that

$$\frac{d}{dt} \left[ \exp \left( - \int_{t_0}^t \varphi(s)ds \right) F(t) \right] \leq 0,$$

for  $t \in [t_0, \beta)$ . Integrate this over the interval  $[t_0, \tau)$  to get

$$f(\tau) \leq F(\tau) \leq c_0 \exp \int_{t_0}^{\tau} \varphi(s)ds,$$

for  $\tau \in [t_0, \beta)$ .

On the interval  $(\alpha, t_0]$ , perform the analogous argument to the function

$$G(t) = c_0 + \int_t^{t_0} \varphi(s)f(s)ds.$$

$\square$

## 2.4. Extension of Solutions

THEOREM 2.4.1. *For every  $(t_0, x_0) \in \Omega$  the solution to the initial value problem (2.1.1) extends to a maximal existence interval  $I = (\alpha, \beta)$ . Furthermore, if  $K \subset \Omega$  is any compact set containing the point  $(t_0, x_0)$ , then there exist times  $\alpha(K)$ ,  $\beta(K)$  such that  $(t, x(t)) \notin K$ , for  $t \in (\alpha, \alpha(K)) \cup (\beta(K), \beta)$ .*

PROOF. Define the sets

$$A = \{a < t_0 : \text{there exists a solution of the IVP on } [a, t_0]\}$$

$$B = \{b > t_0 : \text{there exists a solution of the IVP on } [t_0, b]\}.$$

The existence theorem guarantees that these sets are nonempty, so we may define

$$\alpha = \inf A \quad \text{and} \quad \beta = \sup B.$$

Note that  $\alpha$  and/or  $\beta$  could be infinite – that's ok.

Choose sequences  $\alpha_j \in A$  and  $\beta_j \in B$  with  $\alpha_j \downarrow \alpha$  and  $\beta_j \uparrow \beta$ . On each interval  $[\alpha_j, \beta_j]$  we have a solution. By the uniqueness theorem (2.3.2), we obtain a unique solution on  $(\alpha, \beta)$ .

If  $(\alpha', \beta')$  is another interval on which a solution exists, we have that  $a \in A$  for all  $\alpha' < a < t_0$ . Taking the infimum of all such  $a$  we have that  $\alpha' \geq \alpha$ . Likewise, we have that  $\beta' \leq \beta$ . Thus, we obtain  $(\alpha', \beta') \subset (\alpha, \beta)$ , which proves maximality.

Let  $K \subset \Omega$  be compact with  $(t_0, x_0) \in K$ . Let  $|t - t_0| < \delta$  be the uniform existence interval given by the existence theorem (2.3.1).

If  $\beta = +\infty$ , then choose  $T > 0$  so that  $K \subset [-T, T] \times \mathbb{R}^n$ . Set  $\beta(K) = T$ . Then  $\beta(K) < \beta$  and  $(t, x(t)) \notin K$  for  $t \geq \beta(K)$ .

So we may now assume that  $\beta < +\infty$ . Define  $\beta(K) = \beta - \delta/2$ . Suppose that  $(\bar{t}, x(\bar{t})) \in K$  for some  $\bar{t} \in (\beta(K), \beta)$ . Let  $\bar{x}(t)$  solve the initial value problem with  $\bar{x}(\bar{t}) = x(\bar{t})$ . Then we have that  $\bar{x}(t)$  is defined at least for  $|t - \bar{t}| < \delta$ . By the uniqueness theorem (2.3.2), the function

$$\hat{x}(t) = \begin{cases} x(t), & t \in (\alpha, \beta) \\ \bar{x}(t), & t \in (\bar{t} - \delta, \bar{t} + \delta) \end{cases}$$

is a well-defined solution passing through the point  $(t_0, x_0)$  on an interval which properly contains the maximal interval  $(\alpha, \beta)$ , since  $\bar{t} + \delta > \beta$ . From this contradiction, we conclude that  $(t, x(t)) \notin K$  for all  $t \in (\beta(K), \beta)$ . □

As an example, consider the IVP

$$x' = x^2, \quad x(0) = x_0,$$

the solution of which is

$$x(t) = \frac{x_0}{1 - x_0 t}.$$

We see that the maximal interval of existence depends on the initial value  $x_0$ :

$$I = (\alpha, \beta) = \begin{cases} (-\infty, \infty), & \text{if } x_0 = 0 \\ (-\infty, 1/x_0), & \text{if } x_0 > 0 \\ (1/x_0, \infty), & \text{if } x_0 < 0. \end{cases}$$

## 2.5. Continuous Dependence on Initial Conditions

**DEFINITION 2.5.1.** *A function  $g$  from  $\mathbb{R}^m$  into  $\mathbb{R} \cup \{\infty\}$  is lower semi-continuous at a point  $y_0$  provided  $\liminf_{y \rightarrow y_0} g(y) \geq g(y_0)$ .*

*Equivalently, a function  $g$  into  $\mathbb{R} \cup \{\infty\}$  is lower semi-continuous at a point  $y_0$  provided for every  $L < g(y_0)$  there is a neighborhood  $V$  of  $y_0$  such that  $L \leq g(y)$  for  $y \in V$ .*

Let  $\Omega \subset \mathbb{R}^{n+1}$  be an open set. Let  $f : (t, x) \in \Omega \rightarrow \mathbb{R}^n$  be continuous and locally Lipschitz continuous in  $x$ . Given  $(t_0, x_0) \in \Omega$ , let  $x(t, t_0, x_0)$  denote the unique solution of the IVP

$$x' = f(t, x), \quad x(t_0) = x_0,$$

with maximal existence interval  $I(t_0, x_0) = (\alpha(t_0, x_0), \beta(t_0, x_0))$ .

**THEOREM 2.5.1.** *The domain of  $x(t, t_0, x_0)$ , namely*

$$D = \{(t, t_0, x_0) : (t_0, x_0) \in \Omega, t \in I(t_0, x_0)\},$$

*is an open set in  $\mathbb{R}^{n+2}$ .*

*The function  $x(t, t_0, x_0)$  is continuous on  $D$ .*

*The function  $\beta(t_0, x_0)$  is lower semi-continuous on  $\Omega$ , and the function  $\alpha(t_0, x_0)$  is upper semi-continuous on  $\Omega$ .*

**PROOF.** Fix  $(t_0, x_0) \in \Omega$ . We will use the abbreviations  $x_0(t) = x(t, t_0, x_0)$ ,  $\alpha_0 = \alpha(t_0, x_0)$ , and  $\beta_0 = \beta(t_0, x_0)$ .

**CLAIM.** Choose any pair  $\bar{\alpha}, \bar{\beta}$  such that  $\beta_0 < \bar{\beta} < \bar{\alpha} < \alpha_0$ . Given any  $\varepsilon > 0$ , there exists a neighborhood  $V \subset \Omega$  containing the point  $(t_0, x_0)$  such that for any  $(t_1, x_1) \in V$ , the solution  $x_1(t) = x(t, t_1, x_1)$  is defined for  $t \in [\bar{\alpha}, \bar{\beta}]$ , and

$$\|x_1(t) - x_0(t)\| < \varepsilon,$$

for  $t \in [\bar{\alpha}, \bar{\beta}]$ .

Let's assume that the CLAIM holds and use it to establish the theorem.

To show that  $x(t, t_0, x_0)$  is continuous, fix  $(t', t_0, x_0) \in D$  and let  $\varepsilon > 0$  be given. Choose  $\bar{\alpha}, \bar{\beta}$  such that  $\alpha_0 < \bar{\alpha} < t' < \bar{\beta} < \beta_0$ .

By the CLAIM, there is a neighborhood  $(t_0, x_0) \in V \subset \Omega$  such that  $x_1(t) = x(t, t_1, x_1)$  is defined for  $(t_1, x_1) \in V$ ,  $t \in [\bar{\alpha}, \bar{\beta}]$ , and

$$\|x_1(t) - x_0(t)\| < \varepsilon/2,$$

for  $t \in [\bar{\alpha}, \bar{\beta}]$ .

Now  $x_0(t)$  is continuous as a function of  $t$  on  $[\bar{\alpha}, \bar{\beta}]$ , since it's a solution of the IVP. So there is a  $\delta > 0$  with  $\{|t - t'| < \delta\} \subset [\bar{\alpha}, \bar{\beta}]$ , such that

$$\|x_0(t) - x_0(t')\| < \varepsilon/2,$$

provided  $|t - t'| < \delta$ .

For any  $(t, t_1, x_1) \in \{|t - t'| < \delta\} \times V$ , we have that

$$\begin{aligned} \|x(t, t_1, x_1) - x(t', t_0, x_0)\| &\leq \|x(t, t_1, x_1) - x(t, t_0, x_0)\| \\ &\quad + \|x(t, t_0, x_0) - x(t', t_0, x_0)\| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves continuity.

Let  $(t', t_0, x_0) \in D$ . In the preceding, we have seen that the set  $\{|t - t'| < \delta\} \times V$  is a neighborhood of  $(t', t_0, x_0)$  which is contained in  $D$ . This shows that  $D$  is open.

Using the CLAIM again, we have that for any  $\bar{\beta} < \beta_0$ , there is a neighborhood  $V \subset \Omega$  of  $(t_0, x_0)$  such that  $(t_1, x_1) \in V$  implies that  $x(t, t_1, x_1)$  is defined for  $t \in [t_1, \bar{\beta}]$ . This says that  $\beta(t_1, x_1) \geq \bar{\beta}$ , which in turn means that  $\beta$  is lower semi-continuous at  $(t_0, x_0)$ . The proof that  $\alpha$  is upper semi-continuous is similar.

It remains to prove the CLAIM.

Consider the compact set  $K = \{(s, x_0(s)) : s \in [\bar{\alpha}, \bar{\beta}]\}$ . By the covering lemma, there exist a compact set  $K'$  and numbers  $a, r > 0$  such that  $K \subset K' \subset \Omega$  and

$$\mathcal{C}(s, x_0(s)) = \{(s', x') : |s' - s| < a, \|x' - x_0(s)\| < r\} \subset K',$$

for all  $s \in [\bar{\alpha}, \bar{\beta}]$ . Define  $M_1 = \max_{K'} \|f(t, x)\|$  and let  $M_2$  be a Lipschitz constant for  $f$  on  $K'$ .

Given  $0 < \varepsilon < r$ , choose  $\delta$  small enough so that  $\delta < a$ ,  $\delta < r$ ,  $\{|t - t_0| < \delta\} \subset [\bar{\alpha}, \bar{\beta}]$ , and

$$\delta(M_1 + 1) \exp M_2(\bar{\beta} - \bar{\alpha}) < \varepsilon < r.$$

Set  $V = \{(t, x) : |t - t_0| < \delta, \|x - x_0\| < \delta\}$ . Notice that  $V \subset \mathcal{C}(t_0, x_0) \subset K'$ .

For  $|t_1 - t_0| < \delta$ , we have

$$\begin{aligned} \|x_0(t_1) - x_0\| &= \left\| \int_{t_0}^{t_1} f(s, x_0(s)) ds \right\| \\ &\leq M_1 |t_1 - t_0| \\ &\leq M_1 \delta. \end{aligned}$$

Now let  $x_1(t) = x(t, t_1, x_1)$  which is defined on the maximal interval  $(\alpha(t_1, x_1), \beta(t_1, x_1)) = (\alpha_1, \beta_1)$ . Define

$$t^* = \sup\{t : (s, x_1(s)) \in K', s \in [t_1, t]\}.$$

Since  $(t, x_1(t))$  must eventually exit  $K'$ , we have that  $t^* < \beta_1$ . For  $t < \min(t^*, \bar{\beta})$ , we have that  $(t, x_i(t)) \in K', i = 1, 2$ .

Since

$$x_i(t) = x_i + \int_{t_i}^t f(s, x_i(s)) ds,$$

$i = 1, 2$ , we have

$$\begin{aligned} x_1(t) - x_0(t) &= x_1 - x_0 + \int_{t_1}^t f(s, x_1(s)) ds - \int_{t_0}^t f(s, x_0(s)) ds \\ &= x_1 - x_0 + x_0 - x_0(t_1) + \int_{t_1}^t [f(s, x_1(s)) - f(s, x_0(s))] ds, \end{aligned}$$

For  $t_1 < t < \min(t^*, \bar{\beta})$ , we now have the following estimate:

$$\begin{aligned} \|x_1(t) - x_0(t)\| &\leq \|x_1 - x_0\| + \|x_0 - x_0(t_1)\| \\ &\quad + \int_{t_1}^t \|f(s, x_1(s)) - f(s, x_0(s))\| ds \\ &\leq \delta(1 + M_1) + \int_{t_1}^t M_2 \|x_1(s) - x_0(s)\| ds. \end{aligned}$$

By Gronwall's inequality and our choice of  $\delta$ , we obtain

$$\|x_1(t) - x_0(t)\| \leq \delta(1 + M_1) \exp M_2[t - t_1] \leq \varepsilon < r,$$

for  $t_1 < t < \min(t^*, \bar{\beta})$ . Throughout this time interval, we see that  $(t, x_1(t)) \in \mathcal{C}(t, x_0(t)) \subset K'$ . Thus, we have shown that  $\beta_1 > t^* \geq \bar{\beta}$ , and that  $x_1(t)$  remains within  $\varepsilon$  of  $x_0(t)$ . This completes the proof of the CLAIM. □

### 2.6. Flow of a Nonautonomous System

Let  $f : \Omega \rightarrow \mathbb{R}^n$  be a vector field which satisfies the hypotheses of the Picard Theorem 2.3.1. Given  $(t_0, x_0) \in \Omega$ , let  $x(t, t_0, x_0)$  be the corresponding solution of the initial value problem defined on the maximal existence interval  $I(t_0, x_0) = (\alpha(t_0, x_0), \beta(t_0, x_0))$ . Recall that the domain of  $x(t, t_0, x_0)$  is

$$D = \{(t, t_0, x_0) \in \mathbb{R}^{n+2} : (t_0, x_0) \in \Omega, t \in I(t_0, x_0)\}.$$

LEMMA 2.6.1. *Let  $(s, t_0, x_0) \in D$ . Then*

$$I(t_0, x_0) = I(s, x(s, t_0, x_0))$$

and

$$x(t, t_0, x_0) = x(t, s, x(s, t_0, x_0))$$

for all  $t \in I(t_0, x_0)$ .

PROOF. This is a consequence of the uniqueness theorem (2.3.2). Both solutions pass through the point  $(s, x(s, t_0, x_0))$ , and so they share the same maximal existence interval and they agree on that interval.  $\square$

LEMMA 2.6.2. *If  $(s, t_0, x_0) \in D$ , then  $(t_0, s, x(s, t_0, x_0)) \in D$  and*

$$x(t_0, s, x(s, t_0, x_0)) = x_0.$$

PROOF. Let  $(s, t_0, x_0) \in D$ . Then by Lemma (2.6.1), we have

$$t_0 \in I(t_0, x_0) = I(s, x(s, t_0, x_0)),$$

and we may substitute  $t_0$  for  $t$  to get the result:

$$x_0 = x(t_0, t_0, x_0) = x(t_0, s, x(s, t_0, x_0)).$$

$\square$

DEFINITION 2.6.1. *Let  $t, s \in \mathbb{R}$ . The flow of the vector field  $f$  from time  $s$  to time  $t$  is the map  $\Phi_{t,s}(y) \equiv x(t, s, y)$ . The domain of the flow map is therefore the set*

$$U(t, s) \equiv \{y \in \mathbb{R}^n : t \in I(s, y)\}.$$

Notice that  $U(t, s) \subset \mathbb{R}^n$  is open because the domain  $D$  is open. It is possible that  $U(t, s)$  is empty for some pairs  $t, s$ .

Lemma 2.6.1 says that

$$\Phi_{s,t_0} : U(t, t_0) \cap U(s, t_0) \rightarrow U(t, s)$$

and

$$\Phi_{t,t_0}(y) = \Phi_{t,s} \circ \Phi_{s,t_0}(y), \quad y \in U(t, t_0) \cap U(s, t_0).$$

Lemma 2.6.2 says that

$$\Phi_{s,t_0} : U(s, t_0) \rightarrow U(t_0, s)$$

and

$$\Phi_{t_0,s} \circ \Phi_{s,t_0}(y) = y, \quad y \in U(s, t_0).$$

It follows that  $\Phi_{s,t_0}$  is a homeomorphism from  $U(s, t_0)$  onto  $U(t_0, s)$ .

## 2.7. Flow of Autonomous Systems

Suppose now that the vector field  $f(t, x) = f(x)$  is autonomous. Then we may assume that its domain has the form  $\Omega = \mathbb{R} \times \mathcal{O}$  for an open set  $\mathcal{O} \subset \mathbb{R}^n$ .

LEMMA 2.7.1. *Let  $x_0 \in \mathcal{O}$ ,  $t, \tau \in \mathbb{R}$ . Then*

$$t + \tau \in I(t_0, x_0) \quad \text{if and only if} \quad t \in I(t_0 - \tau, x_0),$$

and

$$x(t + \tau, t_0, x_0) = x(t, t_0 - \tau, x_0), \quad \text{for } t \in I(t_0 - \tau, x_0).$$

PROOF. Let  $y(t) = x(t + \tau, t_0, x_0)$ . The function  $y(t)$  is defined for all  $t \in J = \{t : t + \tau \in I(t_0, x_0)\}$ . Since the system is autonomous,  $y(t)$  solves the equation  $y' = f(y)$  on the interval  $J$ . Since  $y(t_0 - \tau) = x_0$ , it follows by the uniqueness theorem 2.3.2 that

$$x(t + \tau, t_0, x_0) = x(t, t_0 - \tau, x_0),$$

and  $I(t_0, x_0) = J$ . □

Lemma 2.7.1 says that

$$U(t + \tau, t_0) = U(t, t_0 - \tau)$$

and

$$\Phi_{t+\tau,t_0} = \Phi_{t,t_0-\tau}.$$

If we combine this fact with the general result, we have that

$$\Phi_{t+s,0} = \Phi_{t+s,s} \circ \Phi_{s,0} = \Phi_{t,0} \circ \Phi_{s,0},$$

on the domain  $U(t + s, 0) \cap U(s, 0)$ .

DEFINITION 2.7.1. *Given  $x_0 \in \mathcal{O}$ , define the orbit of  $x_0$  to be the curve*

$$\gamma(x_0) = \{x(t, 0, x_0) : t \in I(0, x_0)\}.$$

Notice that the orbit is a curve in the phase space  $\mathcal{O} \subset \mathbb{R}^n$ , as opposed to the solution trajectory  $\{(t, x(t, t_0, x_0)) : t \in I(t_0, x_0)\}$  which is a curve in the space-time domain  $\Omega \subset \mathbb{R}^{n+1}$ .

For autonomous flow, we have the following strengthening of the Uniqueness Theorem 2.3.2.

**THEOREM 2.7.1.** *If  $z \in \gamma(x_0)$ , then  $\gamma(x_0) = \gamma(z)$ . Thus, if two orbits intersect, then they are identical.*

**PROOF.** Suppose that  $z \in \gamma(x_0)$ . This means that  $z = x(t_0, 0, x_0)$  for  $t_0 \in I(0, x_0)$ . Or in terms of the flow, this says that  $z = \Phi_{t_0, 0}(x_0)$ .

Using the property of autonomous flow, we have

$$\Phi_{t, 0}(x_0) = \Phi_{t-t_0, 0} \circ \Phi_{t_0, 0}(x_0) = \Phi_{t-t_0, 0}(z).$$

This shows that an arbitrary point  $\Phi_{t, 0}(x_0) \in \gamma(x_0)$  belongs to  $\gamma(z)$ .

Replacing  $t$  with  $t + t_0$  shows that an arbitrary point  $\Phi_{t, 0}(z) \in \gamma(z)$  belongs to  $\gamma(x_0)$ .

Thus,  $\gamma(x_0) = \gamma(z)$ .  $\square$

From the existence and uniqueness theory for general systems, we have that the domain  $\Omega$  is foliated by the solution trajectories

$$\{(t, x(t, t_0, x_0)) : t \in I(t_0, x_0)\}.$$

That is, every point  $(t_0, x_0) \in \Omega$  has a unique trajectory passing through it. This result says that, for autonomous systems, the phase space  $\mathcal{O}$  is foliated by the orbits.

Since  $x' = f(x)$ , the orbits are curves in  $\mathcal{O}$  everywhere tangent to the vector field  $f(x)$ . They are sometimes also referred to as integral curves. They can be obtained by solving the system

$$\frac{dx_1}{f_1(x)} = \cdots = \frac{dx_n}{f_n(x)}.$$

For example, consider the harmonic oscillator

$$x_1' = x_2, \quad x_2' = -x_1.$$

The system for the integral curves is

$$\frac{dx_1}{x_2} = \frac{dx_2}{-x_1}.$$

Solutions satisfy

$$x_1^2 + x_2^2 = c,$$

and so we confirm that the orbits are concentric circles centered at the origin.

## 2.8. Global Solutions

As usual, we assume that  $f : \Omega \rightarrow \mathbb{R}^n$  satisfies the hypotheses of the Picard Existence Theorem 2.3.1.

Recall that a global solution of the initial value problem is one whose maximal interval of existence is  $\mathbb{R}$ . For this to be possible, it is necessary for the domain  $\Omega$  to be unbounded in the time direction. So let's assume that  $\Omega = \mathbb{R} \times \mathcal{O}$ , where  $\mathcal{O} \subset \mathbb{R}^n$  is open.

**THEOREM 2.8.1.** *Let  $I = (\alpha, \beta)$  be the maximal interval of existence of some solution  $x(t)$  of the initial value problem. Then either  $\beta = +\infty$  or for every compact set  $K \subset \mathcal{O}$ , there exists a time  $\beta(K) < \beta$  such that  $x(t) \notin K$  for all  $t \in (\beta(K), \beta)$ .*

**PROOF.** Assume that  $\beta < +\infty$ . Suppose that  $K \subset \mathcal{O}$  is compact. Then  $K' = [t_0, \beta] \times K \subset \Omega$  is compact. By Theorem 2.4.1, there exists a time  $\beta(K') < \beta$  such that  $(t, x(t)) \notin K'$  for  $t \in (\beta(K'), \beta)$ . Since  $t < \beta$ , this implies that  $x(t) \notin K$  for  $t \in (\beta(K'), \beta)$ .  $\square$

Of course, the analogous result holds for  $\alpha$ , the left endpoint of  $I$ .

**THEOREM 2.8.2.** *If  $\Omega = \mathbb{R}^{n+1}$ , then either  $\beta = +\infty$  or*

$$\lim_{t \rightarrow \beta^-} \|x(t)\| = +\infty.$$

**PROOF.** If  $\beta < +\infty$ , then by the previous result, for every  $R > 0$ , there is a time  $\beta(R)$  such that  $x(t) \notin \{x : \|x\| \leq R\}$  for  $t \in (\beta(R), \beta)$ . In other words,  $\|x(t)\| > R$  for  $t \in (\beta(R), \beta)$ , i.e. the desired conclusion holds.  $\square$

**COROLLARY 2.8.1.** *Assume that  $\Omega = \mathbb{R}^{n+1}$ . If there exists a nonnegative continuous function  $\psi(t)$  defined for all  $t \in \mathbb{R}$  such that*

$$\|x(t)\| \leq \psi(t) \quad \text{for all } t \in [t_0, \beta),$$

*then  $\beta = +\infty$ .*

**PROOF.** The assumed estimate implies that  $\|x(t)\|$  remains bounded on any bounded time interval. By the previous result, we must have that  $\beta = +\infty$ .  $\square$

**THEOREM 2.8.3.** *Let  $\Omega = \mathbb{R}^{n+1}$ . Suppose that there exist continuous nonnegative functions  $c_1(t)$ ,  $c_2(t)$  defined on  $\mathbb{R}$  such that the vector field  $f$  satisfies*

$$\|f(t, x)\| \leq c_1(t)\|x\| + c_2(t), \quad \text{for all } (t, x) \in \mathbb{R}^{n+1}.$$

*Then the solution to the initial value problem is global, for every  $(t_0, x_0) \in \mathbb{R}^{n+1}$ .*

PROOF. We know that a solution of the initial value problem also solves the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds,$$

for all  $t \in (\alpha, \beta)$ . Applying the estimate for  $\|f(t, x)\|$ , we find that

$$\begin{aligned} \|x(t)\| &\leq \|x_0\| + \left| \int_{t_0}^t [c_1(s)\|x(s)\| + c_2(s)] ds \right| \\ &\leq \|x_0\| + \left| \int_{t_0}^t c_2(s) ds \right| + \left| \int_{t_0}^t c_1(s)\|x(s)\| ds \right|. \end{aligned}$$

Our version of Gronwall's inequality 2.3.3 can easily be adapted to this slightly more general situation. Fix  $T \in \mathbb{R}$ , and define

$$C_T = \max \left\{ \|x_0\| + \left| \int_{t_0}^t c_2(s) ds \right| : |t - t_0| \leq |T| \right\}.$$

Then for  $t \in \{|t - t_0| \leq |T|\}$ , we have by Gronwall's inequality 2.3.3

$$\|x(t)\| \leq C_T \exp \left| \int_{t_0}^t c_1(s)\|x(s)\| ds \right|.$$

In particular, this holds for  $t = T$ . Thus, by Corollary 2.8.1, we have that  $\beta = +\infty$  (and  $\alpha = -\infty$ ).  $\square$

COROLLARY 2.8.2. *Let  $A(t)$  be an  $n \times n$  matrix and let  $F(t)$  be an  $n$ -vector which depend continuously on  $t \in \mathbb{R}$ . Then the linear initial value problem*

$$x'(t) = A(t)x(t) + F(t), \quad x(t_0) = x_0$$

*have a unique global solution for every  $(t_0, x_0) \in \mathbb{R}^{n+1}$ .*

PROOF. This is an immediate corollary of the preceding result, since the vector field

$$f(t, x) = A(t)x + F(t)$$

satisfies

$$\|f(t, x)\| \leq \|A(t)\|\|x\| + \|F(t)\|.$$

$\square$

## 2.9. Stability

Let  $\Omega = \mathbb{R} \times \mathcal{O}$  for some open set  $\mathcal{O} \subset \mathbb{R}^n$ , and suppose that  $f : \Omega \rightarrow \mathbb{R}^n$  satisfies the hypotheses of the Picard Theorem.

**DEFINITION 2.9.1.** *A point  $\bar{x} \in \mathcal{O}$  is called an equilibrium point (singular point, critical point) if  $f(t, \bar{x}) = 0$ , for all  $t \in \mathbb{R}$ .*

**DEFINITION 2.9.2.**

*An equilibrium point  $\bar{x}$  is stable if given any  $\varepsilon > 0$  and  $t_0 > 0$  there exists a  $\delta = \delta(\varepsilon, t_0)$  such that for all  $\|x_0 - \bar{x}\| < \varepsilon$ , the solution of the initial value problem  $x(t, t_0, x_0)$  exists for all  $t \geq t_0$  and*

$$\|x(t, t_0, x_0) - \bar{x}\| < \varepsilon, \quad t \geq t_0.$$

*An equilibrium point  $\bar{x}$  is asymptotically stable if it is stable and there exists a  $b(t_0) > 0$  such that if  $\|x_0 - \bar{x}\| < b(t_0)$ , then*

$$\lim_{t \rightarrow \infty} \|x(t, t_0, x_0) - \bar{x}\| = 0.$$

*An equilibrium point  $\bar{x}$  is unstable if it is not stable.*

Examples:

- A center in  $\mathbb{R}^2$  is stable, but not asymptotically stable.
- A sink or a spiral sink is asymptotically stable.
- A saddle is unstable.

**THEOREM 2.9.1.** *Let  $A$  be an  $n \times n$  matrix over  $\mathbb{R}$ , and define the linear vector field  $f(x) = Ax$ .*

*If  $\operatorname{Re} \lambda < 0$  for all eigenvalues of  $A$ , then  $\bar{x} = 0$  is asymptotically stable.*

*If  $\operatorname{Re} \lambda \leq 0$  for all eigenvalues of  $A$ , and  $A$  has no generalized eigenvectors corresponding to eigenvalues with  $\operatorname{Re} \lambda = 0$ , then  $\bar{x} = 0$  is stable.*

*If  $\operatorname{Re} \lambda > 0$  for at least one eigenvalue of  $A$ , then  $\bar{x} = 0$  is unstable.*

**PROOF.** Recall that the solution of the initial value problem is  $x(t, t_0, x_0) = \exp A(t - t_0) x_0$ .

If  $\operatorname{Re} \lambda < 0$  for all eigenvalues, then  $E_s = \mathbb{R}^n$  and by Theorem 1.4.1,

$$\|x(t, t_0, x_0)\| \leq C(1 + (t - t_0)^p) \exp[-\lambda_s(t - t_0)] \|x_0\|,$$

for all  $t > t_0$ . Asymptotic stability follows from this estimate.

If  $\operatorname{Re} \lambda \leq 0$  for all eigenvalues, then  $E_s + E_c = \mathbb{R}^n$  and by Theorem 1.4.1,

$$\begin{aligned} \|x(t, t_0, x_0)\| &= \|\exp A(t - t_0) (P_s + P_u)x_0\| \\ &\leq C(1 + (t - t_0)^{p_1}) \exp[-\lambda_s(t - t_0)] \|P_s x_0\| \\ &\quad + C(1 + (t - t_0)^{p_2}) \|P_c x_0\|, \end{aligned}$$

for all  $t > t_0$ .

Now if  $A$  has no generalized eigenvectors corresponding to eigenvalues with  $\operatorname{Re} \lambda = 0$ , then all Jordan blocks corresponding to eigenvalues with zero real part are  $1 \times 1$ . This means that we can take  $p_2 = 0$ , above. Thus, the right-hand side is bounded by  $C\|x_0\|$ . Stability follows from this estimate.

Suppose that  $x_0$  is an eigenvector corresponding to an eigenvalue with  $\operatorname{Re} \lambda > 0$ . For any  $\delta > 0$ ,  $\operatorname{Re} [\exp \lambda(t - t_0) \delta x_0]$  is a solution whose initial data is arbitrarily small and whose norm is unbounded as  $t \rightarrow \infty$ . This proves that the origin is unstable.  $\square$

**THEOREM 2.9.2.** *Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open set, and let  $f : \mathcal{O} \rightarrow \mathbb{R}^n$  be  $C^1$ . Suppose that  $\bar{x} \in \mathcal{O}$  is an equilibrium point of  $f$  and that the eigenvalues of  $A = Df(\bar{x})$  all satisfy  $\operatorname{Re} \lambda < 0$ . Then  $\bar{x}$  is asymptotically stable.*

**PROOF.** Let  $g(y) = f(\bar{x} + y) - Ay$ . Then  $g \in C^1(\mathcal{O})$ ,  $g(0) = 0$ , and  $Dg(0) = 0$ .

Asymptotic stability of the equilibrium  $x = \bar{x}$  for the system  $x' = f(x)$  is equivalent to asymptotic stability of the equilibrium  $y = 0$  for the system  $y' = Ay + g(y)$ .

Since  $\operatorname{Re} \lambda < 0$  for all eigenvalues of  $A$ , know that from Theorem lin-asym-est that there exists  $\lambda_s > 0$  such that

$$\|\exp A(t - t_0)\| \leq C_0(1 + (t - t_0)^p) \exp[-\lambda_s(t - t_0)], \quad t \geq t_0.$$

Thus, for any  $0 < \alpha < \lambda_s$ , we have

$$\|\exp A(t - t_0)\| \leq C_1 \exp[-\alpha(t - t_0)], \quad t \geq t_0,$$

for an appropriate constant  $C_1 > C_0$ .

Since  $Dg(y)$  is continuous and  $Dg(0) = 0$ , given  $\rho < \alpha$ , there is a  $\delta > 0$  such that

$$\|Dg(y)\| \leq \rho, \quad \text{for } \|y\| \leq \delta.$$

using the elementary Taylor expansion, we see that

$$g(y) = \int_0^1 \frac{d}{ds} [g(sy)] ds = \int_0^1 Dg(sy)y ds.$$

Thus, for  $\|y\| \leq \delta$ , we have

$$\|g(y)\| \leq \int_0^1 \|Dg(sy)\| \|y\| ds \leq \rho \|y\|.$$

Choose  $\delta < \rho$  as above, and assume that  $\|y_0\| < \delta$ . Let  $y(t) = y(t, 0, y_0)$  be the solution of the initial value problem

$$(2.9.1) \quad y'(t) = Ay(t) + g(y(t)), \quad y(0) = y_0,$$

defined for  $t \in (\alpha, \beta)$ . Define

$$T = \sup\{t : \|y(s)\| \leq \rho \text{ for } 0 \leq s \leq t\}.$$

Then  $T > 0$ , by continuity, and of course,  $T \leq \beta$ .

If we treat  $g(y(t))$  as an inhomogeneous term, then by the variation of parameters formula

$$y(t) = \exp At y_0 + \int_0^t \exp A(t-s)g(y(s))ds.$$

Thus, on the interval  $[0, T]$ , we have

$$\|y(t)\| \leq C_1 e^{-\alpha t} \|y_0\| + \int_0^t e^{-\alpha(t-s)} \rho \|y(s)\| ds.$$

Set  $z(t) = e^{\alpha t} \|y(t)\|$ . Then

$$z(t) \leq C_1 \|y_0\| + \int_0^t \rho z(s) ds.$$

By the Gronwall inequality 2.3.3, we obtain

$$z(t) \leq C_1 \|y_0\| e^{C_1 \rho t}, \quad 0 \leq t \leq T.$$

In other words, we have

$$\|y(t)\| \leq C_1 \|y_0\| e^{(C_1 \rho - \alpha)t} \leq C_1 \delta e^{(C_1 \rho - \alpha)t}, \quad 0 \leq t \leq T.$$

So now, choose  $\rho$  small enough so that  $C_1 \rho - \alpha < 0$ , and choose  $\delta$  small enough so that  $\delta < \rho$  and  $C_1 \delta < \rho$ . The preceding estimate shows that  $\|y(t)\| < \rho$  throughout its interval of existence, and thus, by Corollary 2.8.1, we have that  $\beta = +\infty$ . Thus, we have shown that the origin is stable for the initial value problem (2.9.1). The estimate also shows that  $\|y(t)\| \rightarrow 0$ , as  $t \rightarrow \infty$ , which establishes asymptotic stability of the origin. □

## 2.10. Liapunov Stability

Let  $f(x)$  be a locally Lipschitz continuous vector field on an open set  $\mathcal{O} \subset \mathbb{R}^n$ . Assume that  $f$  has an equilibrium point at  $\bar{x} \in \mathcal{O}$ .

**DEFINITION 2.10.1.** *Let  $U \subset \mathcal{O}$  be a neighborhood of  $\bar{x}$ . A Liapunov function for an equilibrium point  $\bar{x}$  of a vector field  $f$  is a function  $E : U \rightarrow \mathbb{R}$  such that*

- (i)  $E \in C(U) \cap C^1(U \setminus \{\bar{x}\})$ ,

- (ii)  $E(x) > 0$  for  $x \in U \setminus \{\bar{x}\}$  and  $E(\bar{x}) = 0$ ,
- (iii)  $DE(x) f(x) \leq 0$  for  $x \in U \setminus \{\bar{x}\}$ .

If strict inequality holds in (iii), then  $E$  is called a strict Liapunov function.

**THEOREM 2.10.1.** *If an equilibrium point  $\bar{x}$  of  $f$  has a Liapunov function, then it is stable.*

*If  $\bar{x}$  has a strict Liapunov function, then it is asymptotically stable.*

**PROOF.** Suppose that  $E$  is a Liapunov function for  $\bar{x}$ .

Choose any  $\varepsilon > 0$  such that  $\overline{B_\varepsilon(\bar{x})} \subset U$ . Define

$$m = \min\{E(x) : \|x\| = \varepsilon\} \quad \text{and} \quad U_\varepsilon = \{x \in U : E(x) < m\} \cap B_\varepsilon(\bar{x}).$$

Notice that  $U_\varepsilon \subset U$  is a neighborhood of  $\bar{x}$ .

The claim is that for any  $x_0 \in U_\varepsilon$ , the solution  $x(t) = x(t, 0, x_0)$  of the IVP  $x' = f(x)$ ,  $x(0) = x_0$  is defined for all  $t \geq 0$  and remains in  $U_\varepsilon$ .

By the local existence theorem and continuity of  $x(t)$ , we have that  $x(t) \in U_\varepsilon$  on some nonempty interval of the form  $[0, \tau)$ . Let  $[0, T)$  be the maximal such interval. The claim amounts to showing that  $T = \infty$ .

On the interval  $[0, T)$ , we have that  $x(t) \in U_\varepsilon \subset U$  and since  $E$  is a Liapunov function,

$$\frac{d}{dt}E(x(t)) = DE(x(t)) \cdot x'(t) = DE(x(t)) \cdot f(x(t)) \leq 0.$$

From this it follows that

$$(2.10.1) \quad E(x(t)) \leq E(x(0)) = E(x_0) < m,$$

on  $[0, T)$ .

Suppose that  $T < \infty$ . Then since  $\|x(t)\| < \varepsilon$  for  $t < T$ , we have that the solution  $x(t)$  is defined on the interval  $[0, T]$  and  $\|x(T)\| \leq \varepsilon$ . From the inequality (2.10.1), we have that  $E(x(T)) < m$ . Thus, it cannot be that  $\|x(T)\| = \varepsilon$ . Thus,  $x(T) \in U_\varepsilon$ , but this contradicts the maximality of the interval  $[0, T)$ . It follows that  $T = \infty$ .

We now use the claim to establish stability. Let  $\varepsilon > 0$  be given. Without loss of generality, we may assume that  $\overline{B_\varepsilon(\bar{x})} \subset U$ . Choose  $\delta > 0$  so that  $B_\delta(\bar{x}) \subset U_\varepsilon$ . Then for every  $x_0 \in B_\delta(\bar{x})$ , we have that  $x(t) \in U_\varepsilon \subset B_\varepsilon(\bar{x})$ , for all  $t > 0$ .

To prove the second statement of the theorem, suppose now that  $E$  is a strict Liapunov function.

The equilibrium  $\bar{x}$  is stable, so given  $\varepsilon > 0$  with  $\overline{B_\varepsilon(\bar{x})} \subset U$ , there is a  $\delta > 0$  so that  $x_0 \in B_\delta(\bar{x})$  implies  $x(t) \in B_\varepsilon(\bar{x})$ , for all  $t > 0$ .

Let  $x_0 \in B_\delta(\bar{x})$ . We must show that  $x(t) = x(t, 0, x_0)$  satisfies  $\lim_{t \rightarrow \infty} x(t) = \bar{x}$ . We may assume that  $x_0 \neq \bar{x}$ .

Since  $E$  is strict and  $x(t) \neq \bar{x}$ , we have that

$$\frac{d}{dt}E(x(t)) = DE(x(t)) \cdot x'(t) = DE(x(t)) \cdot f(x(t)) < 0.$$

Thus,  $E(x(t))$  is a monotonically decreasing function bounded below by 0. Set  $E^* = \inf\{E(x(t)) : t > 0\}$ . Then  $E(x(t)) \downarrow E^*$ .

Since the solution  $x(t)$  remains in the bounded set  $U_\varepsilon$ , it has a limit point. That is, there exist a point  $z \in \bar{U}_\varepsilon$  and a sequence of times  $t_k \rightarrow \infty$  such that  $x(t_k) \rightarrow z$ .

Let  $s > 0$ . By the properties of autonomous flow, we have that

$$\begin{aligned} x(s + t_k) &= x(s + t_k, 0, x_0) \\ &= \Phi_{s+t_k, 0}(x_0) \\ &= \Phi_{s, 0} \circ \Phi_{t_k, 0}(x_0) \\ &= x(s, 0, x(t_k, 0, x_0)) \\ &= x(s, 0, x(t_k)). \end{aligned}$$

By continuous dependence on initial conditions, we have that

$$\lim_{k \rightarrow \infty} x(s + t_k) = \lim_{k \rightarrow \infty} x(s, 0, x(t_k)) = x(s, 0, z).$$

From this it follows that

$$E^* = \lim_{k \rightarrow \infty} E(x(s + t_k)) = E(x(s, 0, z)).$$

Thus,  $x(s, 0, z)$  is a solution along which  $E$  is constant. On the other hand, we know that if  $z \neq x_0$ , then  $E(x(s, 0, z))$  is monotonically decreasing. The conclusion therefore is that  $z = \bar{x}$ .

We have shown that the unique limit point of  $x(t)$  is  $\bar{x}$  which equivalent to  $\lim_{t \rightarrow \infty} x(t) = \bar{x}$ .  $\square$

Examples.

- Hamiltonian Systems

Let  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a  $C^1$  function. A system of the form

$$\begin{aligned} \dot{p} &= H_q(p, q) \\ \dot{q} &= -H_p(p, q) \end{aligned}$$

is referred to as Hamiltonian, the function  $H$  being called the Hamiltonian. Suppose that  $H(0, 0) = 0$  and  $H(p, q) > 0$ , for  $(p, q) \neq (0, 0)$ . Then since the origin is a minimum for  $H$ , we have that  $H_p(0, 0) = H_q(0, 0) = 0$ . In other words, the origin is an equilibrium for the system. Moreover, it is easy to verify that the Hamiltonian  $H$  satisfies the criteria of a Liapunov function. Therefore, the origin is stable.

More generally, we may take  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  in  $C^1$ . Again with  $H(0,0) = 0$  and  $H(p,q) > 0$ , for  $(p,q) \neq (0,0)$ , the origin is stable.

- Newton's equation

Let  $G : \mathbb{R} \rightarrow \mathbb{R}$  be  $C^1$ , with  $G(0) = 0$  and  $G(u) > 0$ , for  $u \neq 0$ . Consider the second order equation

$$\ddot{u} + G'(u) = 0.$$

With  $x_1 = u$  and  $x_2 = \dot{u}$ , this is equivalent to the first order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -G'(x_1).$$

Notice that the origin is an equilibrium, since  $G'(0) = 0$ . The system is, in fact, Hamiltonian with  $H(x_1, x_2) = \frac{1}{2}x_2^2 + G(x_1)$ . Since  $H$  is positive away from the equilibrium, we have that the origin is stable.

The nonlinear pendulum arises when  $G(u) = 1 - \cos u$ .

- Van der Pol's equation

Let  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  be  $C^1$ , with  $\Phi(0) = 0$  and  $\Phi'(u) \geq 0$ . Van der Pol's equation is

$$\ddot{u} + \Phi'(u) + u = 0.$$

We rewrite this as a first order system in a nonstandard way. Let

$$x_1 = u, \quad x_2 = \dot{u} + \Phi(u).$$

Then

$$\dot{x}_1 = x_2 - \Phi(x_1), \quad \dot{x}_2 = -x_1,$$

and the origin is an equilibrium.

The function  $E(x) = \frac{1}{2}[x_1^2 + x_2^2]$  serves as a Liapunov function at the origin, since

$$DE(x)f(x) = -x_1\Phi(x_1) \leq 0,$$

by our assumptions on  $\Phi$ . If  $\Phi'(u) > 0$ , for  $u \neq 0$ , then the origin is actually asymptotically stable, although this does not follow from Theorem 2.10.1.

## CHAPTER 3

### Nonautonomous Linear Systems

#### 3.1. Fundamental Matrices

Let  $t \mapsto A(t)$  be a continuous map from  $\mathbb{R}$  into the set of  $n \times n$  matrices over  $\mathbb{R}$ . Recall that, according to Corollary 2.8.2, the initial value problem

$$x'(t) = A(t)x(t), \quad x(t_0) = x_0,$$

has a unique global solution  $x(t, t_0, x_0)$  for all  $(t_0, x_0) \in \mathbb{R}^{n+1}$ .

By linearity and the Uniqueness Theorem 2.3.2, it follows that

$$x(t, t_0, c_1x_1 + c_2x_2) = c_1x(t, t_0, x_1) + c_2x(t, t_0, x_2).$$

In other words, the flow map  $\Phi_{t,t_0}(x_0)$  is linear in  $x_0$ . Thus, there is a continuous  $n \times n$  matrix  $X(t, t_0)$  such that

$$\Phi_{t,t_0}(x_0) = x(t, t_0, x_0) = X(t, t_0)x_0.$$

The matrix  $X(t, t_0)$  is called the fundamental matrix for  $A(t)$ .

By the general properties of the flow map, we have that

$$\Phi_{t,s} \circ \Phi_{s,t_0} = \Phi_{t,t_0}, \quad \Phi_{t,t} = id, \quad \Phi_{t,t_0}^{-1} = \Phi_{t_0,t}.$$

In the linear context, this implies that

$$X(t, s)X(s, t_0) = X(t, t_0), \quad X(t, t) = I, \quad X(t, t_0)^{-1} = X(t_0, t).$$

Of course, if  $A(t) = A$  is constant, then  $X(t, t_0) = \exp A(t-t_0)$ , and the preceding relations reflect the familiar properties of the exponential matrix for  $A$ .

Since

$$\frac{d}{dt}X(t, t_0)x_0 = \frac{d}{dt}x(t, t_0, x_0) = A(t)x(t, t_0, x_0) = A(t)X(t, t_0)x_0,$$

for every  $x_0 \in \mathbb{R}^n$ , we see that  $X(t, t_0)$  is a matrix solution of

$$\frac{d}{dt}X(t, t_0) = A(t)X(t, t_0), \quad X(t_0, t_0) = I.$$

The  $i^{\text{th}}$  column of  $X(t, t_0)$ , namely  $y_i(t) = X(t, t_0)e_i$ , satisfies

$$\frac{d}{dt}y_i(t) = A(t)y_i(t), \quad y_i(t_0) = e_i.$$

Since this problem has a unique solution for each  $i = 1, \dots, n$ , it follows that  $X(t, t_0)$  is unique.

From the formula  $x(t, t_0, x_0) = X(t, t_0)x_0$ , we have that every solution is a linear combination of the solutions  $y_i(t)$ . We say that the  $y_i(t)$  span the solution space.

The next result generalizes our earlier Variation of Parameters formula to the case where the coefficient matrix is not constant.

**THEOREM 3.1.1** (Variation of Parameters). *Let  $t \mapsto A(t)$  be a continuous map from  $\mathbb{R}$  into the set of  $n \times n$  matrices over  $\mathbb{R}$ . Let  $t \mapsto F(t)$  be a continuous map from  $\mathbb{R}$  into  $\mathbb{R}^n$ . Then for every  $(t_0, x_0) \in \mathbb{R}^n$ , the initial value problem*

$$x'(t) = A(t)x(t) + F(t), \quad x(t_0) = x_0,$$

has a unique global solution  $x(t, t_0, x_0)$ , given by the formula

$$x(t, t_0, x_0) = X(t, t_0)x_0 + \int_{t_0}^t X(t, s)F(s)ds.$$

**PROOF.** Global existence and uniqueness was shown in Corollary 2.8.2. So we need only verify that the solution  $x(t) = x(t, t_0, x_0)$  satisfies the given formula.

Let  $Z(t, t_0)$  be the fundamental matrix for  $-A(t)^T$ . Set  $Y(t, t_0) = Z(t, t_0)^T$ . Then

$$\frac{d}{dt}Y(t, t_0) = \frac{d}{dt}Z(t, t_0)^T = [-A(t)Z(t, t_0)]^T = -Y(t, t_0)A(t),$$

and so

$$\begin{aligned} \frac{d}{dt}[Y(t, t_0)X(t, t_0)] &= \left[\frac{d}{dt}Y(t, t_0)\right]X(t, t_0) + Y(t, t_0)\left[\frac{d}{dt}X(t, t_0)\right] \\ &= -Y(t, t_0)A(t)X(t, t_0) + Y(t, t_0)A(t)X(t, t_0) \\ &= 0. \end{aligned}$$

Therefore,  $Y(t, t_0)X(t, t_0) = Y(t, t_0)X(t, t_0)|_{t=t_0} = I$ . In other words, we have that  $Y(t, t_0) = X(t, t_0)^{-1}$ .

Now we mimic our previous derivation based on the integrating factor.

$$\begin{aligned} \frac{d}{dt}[Y(t, t_0)x(t)] &= \left[\frac{d}{dt}Y(t, t_0)\right]x(t) + Y(t, t_0)\left[\frac{d}{dt}x(t)\right] \\ &= -Y(t, t_0)A(t)x(t) + Y(t, t_0)[A(t)x(t) + F(t)] \\ &= Y(t, t_0)F(t). \end{aligned}$$

Upon integration, we find

$$\begin{aligned}
 x(t) &= Y(t, t_0)^{-1}x_0 + \int_{t_0}^t Y(t, t_0)^{-1}Y(s, t_0)F(s)ds \\
 &= X(t, t_0)x_0 + \int_{t_0}^t X(t, t_0)X(s, t_0)^{-1}F(s)ds \\
 &= X(t, t_0)x_0 + \int_{t_0}^t X(t, t_0)X(t_0, s)F(s)ds \\
 &= X(t, t_0)x_0 + \int_{t_0}^t X(t, s)F(s)ds.
 \end{aligned}$$

□

Although in general, there is no formula for the fundamental matrix, there is a formula for its determinant.

**THEOREM 3.1.2.** *If  $X(t, t_0)$  is the fundamental matrix for  $A(t)$ , then*

$$\det X(t, t_0) = \exp \int_{t_0}^t \operatorname{tr} A(s)ds.$$

**PROOF.** Regard the determinant of as a multi-linear function  $\Delta$  of the rows  $X_i(t, t_0)$  of  $X(t, t_0)$ . Then using multi-linearity, we have

$$\begin{aligned}
 \frac{d}{dt} \det X(t, t_0) &= \frac{d}{dt} \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} \\
 &= \Delta \begin{bmatrix} X'_1(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} + \dots + \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X'_n(t, t_0) \end{bmatrix}.
 \end{aligned}$$

From the differential equation,  $X'(t, t_0) = A(t)X(t, t_0)$ , we get for each row

$$X'_i(t, t_0) = \sum_{j=1}^n A_{ij}(t)X_j(t, t_0).$$

Thus, using multi-linearity and the property that  $\Delta = 0$  if two rows are equal, we obtain

$$\begin{aligned} \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X'_i(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} &= \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ \sum_{j=1}^n A_{ij}(t)X_j(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} = \sum_{j=1}^n A_{ij}(t) \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X_j(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} \\ &= A_{ii}(t) \Delta \begin{bmatrix} X_1(t, t_0) \\ \vdots \\ X_i(t, t_0) \\ \vdots \\ X_n(t, t_0) \end{bmatrix} = A_{ii}(t) \det X(t, t_0). \end{aligned}$$

The result now follows if we substitute this above.  $\square$

### 3.2. Floquet Theory

Let  $A(t)$  be a real  $n \times n$  defined and continuous for  $t \in \mathbb{R}$ . Assume that  $A(t)$  is  $T$ -periodic for some  $T > 0$ . Let  $X(t) = X(t, 0)$  be the fundamental matrix for  $A(t)$ . We shall examine the form of  $X(t)$ .

**THEOREM 3.2.1.** *There  $n \times n$  matrices  $P(t)$ ,  $L$  such that*

$$X(t) = P(t) \exp Lt,$$

where  $P(t)$  is  $C^1$  and  $T$ -periodic and  $L$  is constant.

Remarks:

- Theorem 3.2.1 also holds for complex  $A(t)$ .
- Even if  $A(t)$  is real,  $P(t)$  and  $L$  need not be real.
- $P(t)$  and  $L$  are not unique.

For example, let  $S$  be an invertible matrix which transforms  $L$  to Jordan normal form  $J = \text{diag}[B_1, \dots, B_p]$ . For each Jordan block  $B_j = \lambda_j I + N$ , let  $\tilde{B}_j = (\lambda_j + \frac{2\pi i k_j}{T})I + N$ , for some  $k_j \in \mathbb{Z}$ . Then  $B_j$  commutes with  $\tilde{B}_j$  and

$$\exp(B_j - \tilde{B}_j)T = \exp 2\pi i k_j I = I.$$

Define  $\tilde{J} = \text{diag}[\tilde{B}_1, \dots, \tilde{B}_p]$  and  $\tilde{L} = S\tilde{J}S^{-1}$ . It follows that  $L$  commutes with  $\tilde{L}$  and

$$\exp(L - \tilde{L})T = I.$$

Now take

$$\tilde{P}(t) = P(t) \exp(L - \tilde{L})t.$$

Notice that  $\tilde{P}(t)$  is  $T$ -periodic:

$$\begin{aligned} \tilde{P}(t + T) &= P(t + T) \exp[(L - \tilde{L})(t + T)] \\ &= P(t) \exp(L - \tilde{L})t \exp(L - \tilde{L})T \\ &= \tilde{P}(t). \end{aligned}$$

Since  $L$  commutes with  $\tilde{L}$  we have

$$\begin{aligned} \tilde{P}(t) \exp \tilde{L}t &= \tilde{P}(t) \exp(\tilde{L} - L)t \exp Lt \\ &= P(t) \exp Lt. \end{aligned}$$

- Since  $X(0) = I$ , we have that  $P(0) = I$ . Hence,

$$X(T) = P(T) \exp LT = \exp LT,$$

for any  $L$ .

- The eigenvalues of  $X(T) = \exp LT$  are called the Floquet multipliers. They are unique.
- The eigenvalues of  $L$  are called the Floquet exponents. They are not unique. (It can be shown that they are unique modulo  $2\pi i/T$ .)

**THEOREM 3.2.2.** *If the Floquet multipliers of  $X(t)$  all lie off the negative real axis, then the matrices  $P(t)$  and  $L$  in Theorem 3.2.1 may be chosen to be real.*

**THEOREM 3.2.3.** *The fundamental matrix  $X(t)$  can be written in the form*

$$X(t) = P(t) \exp Lt,$$

where  $P(t)$  and  $L$  are real,  $P(t + T) = P(t)R$ ,  $R^2 = I$ , and  $RL = LR$ .

**PROOF OF THEOREMS 3.2.1, 3.2.2, AND 3.2.3.** By periodicity of  $A(t)$ , we have that

$$X(t + T) = X(t)X(T), \quad t \in \mathbb{R},$$

since both sides satisfy the initial value problem  $Z'(t) = A(t)Z(t)$ ,  $Z(0) = X(T)$ . The matrix  $X(T)$  is nonsingular, so by Lemma 3.2.1 below, there exists a matrix  $L$  such that

$$X(T) = \exp LT.$$

Set  $P(t) = X(t) \exp(-Lt)$ . Then to prove Theorem 3.2.1, we need only verify the periodicity of  $P(t)$ :

$$\begin{aligned} P(t+T) &= X(t+T) \exp[-L(t+T)] \\ &= X(t)X(T) \exp(-LT) \exp(-Lt) \\ &= X(t)I \exp(-Lt) \\ &= P(t), \end{aligned}$$

since by the choice of  $L$ , we have  $X(T) \exp(-LT) = I$ .

Theorem 3.2.2 is obtained by using Lemma 3.2.2 in place of Lemma 3.2.1.

Now we prove Theorem 3.2.3. By Lemma 3.2.3, there exist real matrices  $L$  and  $R$  such that

$$X(T) = R \exp LT, \quad RL = LR, \quad R^2 = I.$$

Define  $P(t) = X(t) \exp(-Lt)$ . Then exactly as before

$$\begin{aligned} P(t+T) &= X(t+T) \exp[-L(t+T)] \\ &= X(t)X(T) \exp(-LT) \exp(-Lt) \\ &= X(t)R \exp(-Lt) \\ &= X(t) \exp(-Lt)R \\ &= P(t)R. \end{aligned}$$

Note we used the fact that  $R$  and  $L$  commute implies that  $R$  and  $\exp Lt$  commute.  $\square$

**LEMMA 3.2.1.** *If  $M$  is an  $n \times n$  invertible matrix, then there exists an  $n \times n$  matrix  $L$  such that  $M = \exp L$ . If  $\tilde{L}$  is another matrix such that  $M = \exp \tilde{L}$ , then the eigenvalues of  $L$  and  $\tilde{L}$  are equal modulo  $2\pi i$ .*

**PROOF.** Choose an invertible matrix  $S$  which transforms  $B$  to Jordan normal form

$$S^{-1}MS = J.$$

$J$  has the block structure

$$J = \text{diag} [B_1, \dots, B_p],$$

where the Jordan blocks have the form  $B_j = \lambda_j I - N$  for some eigenvalue  $\lambda_j \neq 0$ , since  $B$  is invertible. Here,  $N$  is the nilpotent matrix with  $-1$  above the main diagonal.

Suppose that for each block we can find  $L_j$  such that  $B_j = \exp L_j$ . Then if we set

$$L = \text{diag} [L_1, \dots, L_p],$$

we get

$$J = \exp L = \text{diag} [\exp L_1, \dots, \exp L_p].$$

Thus, we would have

$$B = SJS^{-1} = S \exp LS^{-1} = \exp SLS^{-1}.$$

We have therefore reduced the problem to that of finding the logarithm of an invertible Jordan block. Consider a  $d \times d$  Jordan block  $B = \lambda I - N = \lambda(I - \frac{1}{\lambda}N)$ , with  $N^d = 0$ . For any  $t \in \mathbb{R}$ ,

$$\begin{aligned} I &= I - \left(\frac{t}{\lambda}N\right)^d \\ &= \left(I - \frac{t}{\lambda}N\right) \left(I + \frac{t}{\lambda}N + \dots + \left(\frac{t}{\lambda}N\right)^{d-1}\right). \end{aligned}$$

Hence,  $I - \frac{t}{\lambda}N$  is invertible and  $\left(I - \frac{t}{\lambda}N\right)^{-1} = \sum_{k=0}^{d-1} \left(\frac{t}{\lambda}N\right)^k$ .

Set

$$L(t) = - \sum_{k=1}^{d-1} \frac{1}{k} \left(\frac{t}{\lambda}N\right)^k.$$

$L(t)$  will be shown to be  $\log(I - \frac{t}{\lambda}N)$ . We have

$$L'(t) = - \sum_{k=1}^{d-1} t^{k-1} \left(\frac{1}{\lambda}N\right)^k = \left(-\frac{1}{\lambda}N\right) \left(I - \frac{t}{\lambda}N\right)^{-1}.$$

Thus,

$$\left(I - \frac{t}{\lambda}N\right) L'(t) = -\frac{1}{\lambda}N.$$

Another differentiation gives

$$\left(-\frac{1}{\lambda}N\right) L'(t) + \left(I - \frac{t}{\lambda}N\right) L''(t) = 0.$$

This shows that

$$L''(t) = \left(I - \frac{t}{\lambda}N\right)^{-1} \left(-\frac{1}{\lambda}N\right) L'(t) = -L'(t)^2.$$

So now

$$\frac{d}{dt} \exp L(t) = \exp L(t) L'(t).$$

Of course, this does not hold in general. But, because  $L(t)$  is a polynomial in  $tN$ , we have that  $L(t)$  and  $L(s)$  commute for all  $t, s \in \mathbb{R}$ , and

thus, the formula is easily verified using the definition of the derivative. Next, we also have

$$\frac{d^2}{dt^2} \exp L(t) = \exp L(t) [L''(t) + L'(t)^2] = 0.$$

Hence,  $\exp L(t)$  is linear in  $t$ :

$$\exp L(t) = \exp L(0) + t \exp L(0) L'(0) = I - \frac{t}{\lambda} N,$$

and as a consequence,

$$\exp L(1) = I - \frac{1}{\lambda} N.$$

To finish, we simply write

$$B = \lambda I - N = \exp [\log \lambda I + L(1)],$$

where  $\log \lambda = \log |\lambda| + i\theta$  is any complex logarithm of  $\lambda = |\lambda|e^{i\theta}$ .

Suppose that  $M = \exp L$  for some matrix  $L$ . Transform  $L$  to Jordan canonical form  $L = SJS^{-1} = S[D + N]S^{-1}$ . Then  $M = \exp L = S \exp D \exp NS^{-1}$ . Now  $\exp D$  is diagonal and  $\exp N$  is upper triangular with the value 1 along the diagonal. It follows that the eigenvalues of  $M$  are the same as those of  $\exp D$ . Therefore, if  $\tilde{L}$  is another matrix such that  $M = \exp \tilde{L}$ , then  $L$  and  $\tilde{L}$  have the eigenvalues, modulo  $2\pi i$ .  $\square$

**LEMMA 3.2.2.** *If  $M$  is a real  $n \times n$  invertible matrix with no eigenvalues in  $(-\infty, 0)$ , then there exists a real  $n \times n$  matrix  $L$  such that  $M = \exp L$ .*

**PROOF.** If  $B_j$  is a Jordan block of  $M$  corresponding to a real eigenvalue  $\lambda_j > 0$ , then the construction of Lemma 1 gives a real matrix  $L_j$  such that  $B_j = \exp L_j$ .

Suppose that  $B_j = \lambda_j I + N$  is a  $d \times d$  Jordan block of  $M$  corresponding to a complex eigenvalue  $\lambda_j$  with a string of generalized eigenvectors  $x_1^{(j)}, \dots, x_d^{(j)}$ . Then since  $M$  is real,  $\bar{B}_j$  is also a Jordan block corresponding to the eigenvalue  $\bar{\lambda}$  and generalized eigenvectors  $\bar{x}_1^{(j)}, \dots, \bar{x}_d^{(j)}$ . If  $S_j$  denotes the  $n \times d$  matrix whose columns are formed by  $x_1^{(j)}, \dots, x_d^{(j)}$ , the preceding is expressed by the matrix equation

$$M[S_j \bar{S}_j] = [S_j \bar{S}_j] \begin{bmatrix} B_j & 0 \\ 0 & \bar{B}_j \end{bmatrix}.$$

By Lemma 1, there is a  $d \times d$  matrix  $L_j$  such that  $B_j = \exp L_j$ . Taking conjugates, it follows that  $\bar{B}_j = \exp \bar{L}_j$ . Thus, we have that

$$(3.2.1) \quad \begin{bmatrix} B_j & 0 \\ 0 & \bar{B}_j \end{bmatrix} = \begin{bmatrix} \exp L_j & 0 \\ 0 & \exp \bar{L}_j \end{bmatrix} = \exp \mathcal{L}_j,$$

with

$$\mathcal{L}_j = \begin{bmatrix} L_j & 0 \\ 0 & \bar{L}_j \end{bmatrix}.$$

Now define the  $2d \times 2d$  block matrix

$$U_j = \frac{1}{2} \begin{bmatrix} I & -iI \\ I & iI \end{bmatrix}.$$

It is straightforward to check that

$$U_j^{-1} = \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix},$$

$$[S_j \bar{S}_j]U_j = [\operatorname{Re} S_j \operatorname{Im} S_j] \equiv T_j,$$

and

$$U_j^{-1} \mathcal{L}_j U_j = \begin{bmatrix} \operatorname{Re} L_j & \operatorname{Im} L_j \\ -\operatorname{Im} L_j & \operatorname{Re} L_j \end{bmatrix} \equiv \mathcal{P}_j.$$

Returning to (3.2.1), we find

$$\begin{aligned} MT_j &= M[S_j \bar{S}_j]U_j = [S_j \bar{S}_j]U_j U_j^{-1} \exp \mathcal{L}_j U_j \\ &= T_j \exp U_j^{-1} \mathcal{L}_j U_j = T_j \exp \mathcal{P}_j. \end{aligned}$$

The  $2d$  columns of  $T_j$  are linearly independent because they have the same span as the set  $\{x_1^{(j)}, \dots, x_d^{(j)}, \bar{x}_1^{(j)}, \dots, \bar{x}_d^{(j)}\}$ .

Applying these procedures to each real block and each pair of complex conjugate blocks yields a real invertible matrix  $T$  and a real block diagonal matrix  $\mathcal{L}$  such that  $MT = T \exp \mathcal{L}$ . In other words, we have shown that  $M = \exp T \mathcal{L} T^{-1}$ , with  $T \mathcal{L} T^{-1}$  real.  $\square$

**LEMMA 3.2.3.** *If  $M$  is a real  $n \times n$  invertible matrix, then there exists real  $n \times n$  matrices  $L$  and  $R$  such that  $M = R \exp L$ ,  $RL = LR$ , and  $R^2 = I$ .*

**PROOF.** Retracing the steps in the proof of Lemma 2, we also have that the matrix  $T$  reduces  $M$  to real canonical form

$$T^{-1}MT = J = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_p \end{bmatrix},$$

in which  $B_j$  is either a Jordan block corresponding to a real eigenvalue of  $M$  or

$$B_j = \begin{bmatrix} \mu_j I + N & \nu_j I \\ -\nu_j I & \mu_j I + N \end{bmatrix}$$

corresponding to a pair of complex conjugate eigenvalues  $\mu_j \pm \nu_j$ .

Define the block matrix

$$R = \begin{bmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_p \end{bmatrix},$$

where  $R_j$  has the same size as  $B_j$  and  $R_j = -I$  if  $B_j$  corresponds to an eigenvalue in  $(-\infty, 0)$  and  $R_j = I$ , otherwise. Thus,  $R^2 = I$ .

Now the real matrix  $RJ$  has no eigenvalues in  $(-\infty, 0)$ , and so by Lemma 2, there exists a real matrix  $L$  such that  $RJ = \exp L$ . Clearly,  $RL = LR$  since these two matrices have the same block structure and the blocks of  $R$  are  $\pm I$ .

So now we have

$$M = TJT^{-1} = TR^2JT^{-1} = TR \exp LT^{-1} = \tilde{R} \exp \tilde{L},$$

in which  $\tilde{R} = TRT^{-1}$  and  $\tilde{L} = TLT^{-1}$ . We have that  $\tilde{R}^2 = I$  and  $\tilde{R}\tilde{L} = \tilde{L}\tilde{R}$  as a consequence of the analogous properties of  $R$  and  $L$ .  $\square$

**THEOREM 3.2.4.** *Let  $A(t)$  be a continuous  $T$ -periodic  $n \times n$  matrix over  $\mathbb{R}$ . Let  $\{\mu_j\}_{j=1}^n$  be the Floquet multipliers and let  $\{\lambda_j\}_{j=1}^n$  be a set of Floquet exponents. Then*

$$\prod_{j=1}^n \mu_j = \exp \int_0^T \operatorname{tr} A(t) dt,$$

and

$$\sum_{j=1}^n \lambda_j = \frac{1}{T} \int_0^T \operatorname{tr} A(t) dt, \quad (\text{mod } \frac{2\pi i}{T}).$$

**PROOF.** Let  $X(t)$  be the fundamental matrix for  $A(t)$ . By 3.1.2 we have that

$$\prod_{j=1}^n \mu_j = \det X(T) = \exp \int_0^T \operatorname{tr} A(t) dt.$$

By Theorem 3.2.1, we have  $X(T) = \exp LT$ . Since  $\{\lambda_j\}_{j=1}^n$  are the eigenvalues of  $L$ , we have

$$\det X(T) = \exp \left( T \sum_{j=1}^n \lambda_j \right).$$

Thus,

$$T \sum_{j=1}^n \lambda_j = \log \det X(T) = \int_0^T \operatorname{tr} A(t) dt, \quad (\text{mod } 2\pi i).$$

and so,

$$\sum_{j=1}^n \lambda_j = \frac{1}{T} \int_0^T \operatorname{tr} A(t) dt, \quad (\text{mod } \frac{2\pi i}{T}).$$

□

### 3.3. Stability of Linear Periodic Systems

**COROLLARY 3.3.1.** *Let  $A(t)$  be a real  $n \times n$  matrix which is continuous for  $t \in \mathbb{R}$  and  $T$ -periodic for  $T > 0$ . By Theorem 3.2.1, the fundamental matrix  $X(t)$  has the form*

$$X(t) = P(t) \exp Lt,$$

where  $P(t)$  is  $T$ -periodic.

The origin is stable for the system

$$(3.3.1) \quad x'(t) = A(t)x(t)$$

if and only if the Floquet multipliers  $\mu$  satisfy  $|\mu| \leq 1$  and there are a complete set of eigenvectors for any multipliers of modulus 1.

The origin is asymptotically stable if and only if  $|\mu| < 1$  for all Floquet multipliers.

The stability of the origin for the system

$$(3.3.2) \quad y'(t) = Ly(t)$$

is the same as for (3.3.1)

**PROOF.** The solutions of the system (3.3.1) are given by

$$x(t) = P(t) \exp Lt x_0,$$

whereas the solutions of the system (3.3.2) are of the form

$$y(t) = \exp Lt x_0,$$

for some  $x_0 \in \mathbb{R}^n$ .

Now since  $P(t)$  is continuous and periodic, there exists a constant such that  $\|P(t)\| \leq C$  for all  $t \in \mathbb{R}$ . This implies that

$$\|x(t)\| \leq C \|\exp Lt x_0\| = C \|y(t)\|,$$

and thus the stability or asymptotic stability of the origin for (3.3.2) implies the same for (3.3.1).

The Floquet multipliers have the form  $\mu = \exp \lambda T$ , where  $\lambda$  is an eigenvalue of  $L$ , i.e. a Floquet exponent. We see that  $|\mu| = \exp \operatorname{Re} \lambda T$ .

So if  $|\mu| < 1$ , for all Floquet multipliers, then  $\operatorname{Re} \lambda < 0$  for all Floquet exponents. In this case, the origin is asymptotically stable for (3.3.2) and hence for (3.3.1).

Next, suppose that  $|\mu| \leq 1$  for all Floquet multiplier and there are no generalized eigenvectors of  $\exp LT$  corresponding to Floquet multipliers with  $|\mu| = 1$ . Then  $\operatorname{Re} \lambda \leq 0$  for all eigenvalues of  $L$ , and there are no generalized eigenvectors of  $L$  corresponding to eigenvalues with  $\operatorname{Re} \lambda = 0$ . It follows that  $\|\exp Lt\|$  is uniformly bounded, and so the origin is stable for (3.3.2) and also then for (3.3.1).

If  $|\mu| > 1$  for some Floquet exponent or if there is a generalized eigenvector of  $\exp LT$  with  $|\mu| = 1$ , then either  $L$  has an eigenvalue with  $\operatorname{Re} \lambda > 1$  or  $L$  has a generalized eigenvector with  $\operatorname{Re} \lambda = 0$ . In either case, the system (3.3.2) has a solution  $y(t)$  with  $\|y(t)\| \rightarrow \infty$ , as  $t \rightarrow \infty$ . This says that the origin is unstable for (3.3.2). Since  $P(t)$  is periodic and  $P(0) = I$ , we see that

$$x(kT) = P(kT) \exp LkT x_0 = \exp LkT x_0 = y(kT),$$

and so  $\lim_{k \rightarrow \infty} \|x(kT)\| = \lim_{k \rightarrow \infty} \|y(kT)\| = \infty$ . Thus, the origin is also unstable for (3.3.1).  $\square$

Example. There is no simple relationship between  $A(t)$  and the Floquet multipliers. Consider the  $2\pi$ -periodic coefficient matrix

$$A(t) = \begin{bmatrix} -1 + \frac{3}{2} \cos^2 t & 1 - \frac{3}{2} \cos t \sin t \\ -1 - \frac{3}{2} \sin t \cos t & -1 + \frac{3}{2} \sin^2 t \end{bmatrix}.$$

By direct calculation it can be verified that

$$X(t) = \begin{bmatrix} e^{t/2} \cos t & e^{-t} \sin t \\ -e^{t/2} \sin t & e^{-t} \cos t \end{bmatrix}$$

is the fundamental matrix for  $A(t)$  with  $X(0) = I$ . Since

$$X(2\pi) = \operatorname{diag} [e^\pi, e^{-2\pi}],$$

we have that the Floquet multipliers are  $e^\pi, e^{-2\pi}$ , and so the origin is unstable. Indeed,  $X(t)e_1$  is an unbounded solution.

On the other hand, the eigenvalues of  $A(t)$  are

$$\lambda_1 = \frac{1}{4}[-1 + \sqrt{7}i], \quad \lambda_2 = \bar{\lambda}_1,$$

both of which have negative real parts. We see that the eigenvalues of  $A(t)$  have no influence on stability.

Notice that

$$\mu_1 \mu_2 = \exp \int_0^{2\pi} [\lambda_1 + \lambda_2] ds = \exp \int_0^{2\pi} \operatorname{tr} A(s) ds,$$

which confirms Theorem 3.2.4.

### 3.4. Parametric Resonance – The Mathieu Equation

The Mathieu equation is

$$u'' + (\omega^2 + \varepsilon \cos t)u = 0.$$

With  $x(t) = [u(t) \ u'(t)]^T$  and

$$A(t) = \begin{bmatrix} 0 & 1 \\ -(\omega^2 + \varepsilon \cos t) & 0 \end{bmatrix}$$

the equation can be written as a first order system

$$x'(t) = A(t)x(t).$$

Notice that  $A(t)$  is  $2\pi$ -periodic, and so Floquet theory applies.

Question: *For which values of  $\omega$  and  $\varepsilon$  is the zero solution stable?*

Corollary 3.3.1 tells us to look at the Floquet multipliers in order to answer this question. In this case, the Floquet multipliers are the eigenvalues of  $X(2\pi)$ , where  $X(t) = X(t, 0)$  is the fundamental matrix for  $A(t)$ .

Since  $\text{tr } A(t) = 0$  for every  $t \in \mathbb{R}$ , We have from Theorem 3.2.4 that the Floquet multipliers  $\mu_1, \mu_2$  satisfy

$$\mu_1\mu_2 = \det X(t) = 1, \quad t \in \mathbb{R}.$$

If  $\mu_1 \notin \mathbb{R}$ , then  $\mu_1 = \bar{\mu}_2$ . It follows that  $\mu_1, \mu_2$  are distinct points on the unit circle, and so there are no generalized eigenvectors. By Corollary 3.3.1, the origin is stable.

When  $\varepsilon = 0$ , the system reduces to a harmonic oscillator. The fundamental matrix for this constant coefficient system is

$$X(t) = \begin{bmatrix} \cos \omega t & -\omega^{-1} \sin \omega t \\ \omega \sin \omega t & \cos \omega t \end{bmatrix},$$

The Floquet multipliers are the eigenvalues of  $X(2\pi)$ . They satisfy  $\mu_1 + \mu_2 = \text{tr } X(2\pi) = 2 \cos 2\pi\omega$ . Since  $\mu_1\mu_2 = 1$ , we have that  $\mu_j \in \mathbb{R}$  if and only if  $\mu_1 + \mu_2 = \pm 2$ . This can happen if and only if  $2\omega \in \mathbb{Z}$ .

If  $2\omega \notin \mathbb{Z}$ , then by continuous dependence on parameters (to be discussed, see Theorem 5.2.1), the Floquet multipliers will not be real for  $\varepsilon$  small. Thus, for every  $2\pi\omega_0 \notin \mathbb{Z}$ , there is a small ball in the  $(\omega, \varepsilon)$  plane with center  $(\omega_0, 0)$  where the origin is stable for Mathieu's equation.

It can also be shown, although we will not do so now, that there are regions of instability branching off of the points  $(\omega_0, 0)$  when  $2\pi\omega_0 \in \mathbb{Z}$ . This is the so-called parametric resonance.

### 3.5. Existence of Periodic Solutions

**THEOREM 3.5.1.** *Let  $A(t)$  be  $T$ -periodic. The system  $x'(t) = A(t)x(t)$  has a nonzero  $T$ -periodic solution if and only if  $A(t)$  has the Floquet multiplier  $\mu = 1$ .*

**PROOF.** By the periodicity of  $A(t)$  and uniqueness, we have that a solution is  $T$ -periodic if and only if

$$(3.5.1) \quad x(T) = x(0).$$

Let  $X(t)$  be the fundamental matrix for  $A(t)$  with  $X(0) = I$ . Then every nonzero solution has the form  $x(t) = X(t)x_0$  for some  $x_0 \in \mathbb{R}^n$  with  $x_0 \neq 0$ . It follows that (3.5.1) holds if and only if  $X(T)x_0 = x_0$ . Thus,  $x_0$  is an eigenvector for  $X(T)$  with eigenvalue 1. But the eigenvalues of  $X(T)$  are the Floquet multipliers.  $\square$

**THEOREM 3.5.2.** *Let  $A(t)$  be a continuous  $n \times n$  matrix and let  $F(t)$  be a continuous vector in  $\mathbb{R}^n$ . Assume that  $A(t)$  and  $F(t)$  are  $T$ -periodic. The equation*

$$(3.5.2) \quad x'(t) = A(t)x(t) + F(t)$$

*has a  $T$ -periodic solution if and only if*

$$(3.5.3) \quad \int_0^T y(t) \cdot F(t) dt = 0,$$

*for all  $T$ -periodic solutions  $y(t)$  of the adjoint system*

$$(3.5.4) \quad y'(t) = -A(t)^T y(t).$$

**PROOF.** Let  $X(t) = X(t, 0)$  be the fundamental matrix for  $A(t)$  with  $X(0) = I$ . By variation of parameters, we have that the solution  $x(t) = x(t, 0, x_0)$  of (3.5.2) is given by

$$x(t) = X(t)x_0 + X(t) \int_0^t X(s)^{-1} F(s) ds.$$

By uniqueness and periodicity,  $x(t)$  is  $T$ -periodic if and only if  $x(T) = x(0) = x_0$ . This is equivalent to

$$[I - X(T)]x_0 = X(T) \int_0^T X(s)^{-1} F(s) ds,$$

and so, multiplying both sides by  $X(T)^{-1}$ , we obtain an equivalent linear system of equations

$$(3.5.5) \quad Bx_0 = g,$$

in which

$$B = X(T)^{-1} - I \quad \text{and} \quad g = \int_0^T X(s)^{-1}F(s)ds.$$

Thus,  $x(t, 0, x_0)$  is a  $T$ -periodic solution (3.5.2) if and only if  $x_0$  is a solution of (3.5.5).

By the Fredholm Alternative 4.2.1 (to follow), the system (3.5.5) has a solution if and only if  $g \cdot y_0 = 0$  for all  $y_0 \in N(B^T)$ .

We now characterize  $N(B^T)$ . Let  $Y(t) = Y(t, 0)$  be the fundamental matrix for  $-A(t)^T$ . Then  $Y(t) = [X(t)^{-1}]^T$ , so

$$B^T = Y(T) - I.$$

Thus,  $y_0 \in N(B^T)$  if and only if  $y_0 = Y(T)y_0$ . This, in turn, is equivalent to saying  $y_0 \in N(B^T)$  if and only if  $y(t) = Y(t)y_0$  is a  $T$ -periodic solution of the adjoint system (3.5.4).

Now we examine the orthogonality condition. We have

$$\begin{aligned} y_0 \cdot g &= \int_0^T y_0 \cdot X(s)^{-1}F(s)ds \\ &= \int_0^T y_0 \cdot Y(s)^T F(s)ds \\ &= \int_0^T Y(s)y_0 \cdot F(s)ds \\ &= \int_0^T y(s) \cdot F(s)ds. \end{aligned}$$

The result now follows from the following chain of equivalent statements: Equation (3.5.2) has a  $T$ -periodic solution iff the system (3.5.5) has a solution iff  $y_0 \cdot g = 0$  for every  $y_0 \in N(B^T)$  iff (3.5.3) holds for every  $T$ -periodic solution of (3.5.4).  $\square$

Remark: Theorem 3.5.2 is interesting only when  $F \neq 0$ , because the conditions (3.5.3), (3.5.4) hold trivially when  $F = 0$  and the trivial  $x = 0$  is  $T$ -periodic.

Example. Consider the periodically forced harmonic oscillator:

$$u'' + u = \cos \omega t, \quad \omega > 0.$$

This is equivalent to the first order system

$$(3.5.6) \quad x'(t) = Ax(t) + F(t)$$

with

$$x(t) = \begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 \\ \cos \omega t \end{bmatrix}.$$

Notice that  $F(t)$  (and  $A$ ) are  $T$ -periodic with  $T = 2\pi/\omega$ .

Since  $-A^T = A$ , the adjoint equation is

$$y' = Ay,$$

the solutions of which are

$$y(t) = \exp At y_0, \quad \text{with} \quad \exp At = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

All solutions of the adjoint equation are  $2\pi$ -periodic.

If the forcing frequency  $\omega$  satisfies  $\omega \neq 1$ , then the adjoint equation has no  $T$ -periodic solutions. Thus, the system (3.5.6) has a  $T$ -periodic solution. This solution is, in fact, unique because the system (3.5.5) which determines its initial data has a unique solution (the  $2 \times 2$  matrix  $B$  is onto and hence one-to-one).

If  $\omega = 1$ , then  $T = 2\pi$ , and there are no  $2\pi$ -periodic solutions of (3.5.6), since there exist  $2\pi$ -periodic solutions of the adjoint equation for which the orthogonality condition (3.5.3) does not hold. Here is the calculation:

$$\begin{aligned} \int_0^T y(s) \cdot F(s) ds &= \int_0^T \exp As y_0 \cdot F(s) ds \\ &= \int_0^T y_0 \cdot [\exp As]^T F(s) ds \\ &= \int_0^T y_0 \cdot [-\sin s \cos s e_1 + \cos^2 s e_2] ds \\ &= \pi y_0 \cdot e_2. \end{aligned}$$

This is nonzero for  $y_0 \cdot e_2 \neq 0$ . This is the case of resonance.

This overly simple example can be solved explicitly, since  $A$  is constant. However, the example illustrates the use of the Fredholm alternative in such problems.

## CHAPTER 4

### Results from Functional Analysis

#### 4.1. Operators on Banach Space

DEFINITION 4.1.1. A Banach space is a complete normed vector space over  $\mathbb{R}$  or  $\mathbb{C}$ .

A Hilbert space is a Banach space whose norm is induced by an inner product.

Here are some examples of Banach spaces that will be relevant for us:

- Let  $\mathcal{F} \subset \mathbb{R}^n$ .  $C^0(\mathcal{F}, \mathbb{R}^m)$  is set the of continuous functions from  $\mathcal{F}$  into  $\mathbb{R}^m$ . Define the norm

$$\|f\|_{C^0} = \sup_{x \in \mathcal{F}} \|f(x)\|.$$

Then

$$C_b^0(\mathcal{F}, \mathbb{R}^m) = \{f \in C^0(\mathcal{F}, \mathbb{R}^m) : \|f\|_{C^0} < \infty\},$$

is a Banach space. If  $\mathcal{F}$  is compact, then  $C_b^0(\mathcal{F}, \mathbb{R}^m) = C^0(\mathcal{F}, \mathbb{R}^m)$ .

- $C^1(\mathcal{F}, \mathbb{R}^m)$  is set the of functions  $f$  from  $\mathcal{F}$  into  $\mathbb{R}^m$  such that  $Df(x)$  exists and is continuous. Define the norm

$$\|f\|_{C^1} = \sup_{x \in \mathcal{F}} \|f(x)\| + \sup_{x \in \mathcal{F}} \|Df(x)\|.$$

Then

$$C_b^1(\mathcal{F}, \mathbb{R}^m) = \{f \in C^1(\mathcal{F}, \mathbb{R}^m) : \|f\|_{C^1} < \infty\},$$

is a Banach space.

- $\text{Lip}(\mathcal{F}, \mathbb{R}^m)$  is the set of Lipschitz continuous functions from  $\mathcal{F}$  into  $\mathbb{R}^m$  such that the norm

$$\|f\|_{\text{Lip}} = \sup_{x \in \mathcal{F}} \|f(x)\| + \sup_{\substack{x, y \in \mathcal{F} \\ x \neq y}} \frac{\|f(x) - f(y)\|}{\|x - y\|}$$

is finite.

Notice that  $C_b^1(\mathcal{F}, \mathbb{R}^m) \subset \text{Lip}(\mathcal{F}, \mathbb{R}^m) \subset C_b^0(\mathcal{F}, \mathbb{R}^m)$ .

DEFINITION 4.1.2. Let  $X$  and  $Y$  be Banach spaces. Let  $A : X \rightarrow Y$  be a linear operator.  $A$  is bounded if and only if

$$\sup_{\|x\|_X \neq 0} \frac{\|Ax\|_Y}{\|x\|_X} \equiv \|A\|_{X,Y} < \infty.$$

$\|A\|_{X,Y}$  is the operator norm.

The set of all bounded linear operators from  $X$  to  $Y$  is denoted by  $\mathcal{L}(X, Y)$ . It is a Banach space with the operator norm.

A linear operator from  $X$  to  $Y$  is bounded if and only if it is continuous.

DEFINITION 4.1.3. Let  $f : X \rightarrow Y$  be any mapping between the Banach spaces  $X$  and  $Y$ . ( $f$  is not necessarily linear.)  $f$  is Fréchet differentiable at a point  $x_0 \in X$  if and only if there exists a linear operator  $Df(x_0) \in \mathcal{L}(X, Y)$  such that

$$f(x) - f(x_0) - Df(x_0)(x - x_0) \equiv R(x, x_0)$$

satisfies

$$\lim_{x \rightarrow x_0} \frac{\|R(x, x_0)\|_Y}{\|x - x_0\|_X} = 0.$$

( $D_x f(x_0)$  is unique if it exists.)

Let  $U \subset X$  be an open set. We say that  $f : X \rightarrow Y$  is differentiable on  $U$  if and only if  $D_x f(x)$  exists for all  $x \in U$ .

$f$  is continuously differentiable on  $U$  if and only if it is differentiable on  $U$  and  $D_x f(x)$  is a continuous map from  $X$  into  $\mathcal{L}(X, Y)$ .

Remark: The Fréchet derivative can be computed as follows

$$Df(x_0)x = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} [f(x_0 + \varepsilon x) - f(x_0)] = D_\varepsilon f(x_0 + \varepsilon x)|_{\varepsilon=0}.$$

Example: Let  $X = C^0([0, 1], \mathbb{R})$ . Given  $f \in X$ , define the nonlinear mapping

$$T(f)(t) = \int_0^t [f(s)]^2 ds.$$

Then  $T : X \rightarrow X$ . In fact,  $T$  is differentiable, and given  $f_0, f \in X$ , we have that  $DT(f_0)f$  is the function whose value at a point  $t \in [0, 1]$  is

$$DT(f_0)f(t) = 2 \int_0^t f_0(s)f(s) ds.$$

### 4.2. The Fredholm Alternative

**THEOREM 4.2.1** (Fredholm Alternative). *Let  $H_i$ ,  $i = 1, 2$  be Hilbert spaces with inner product  $\langle \cdot, \cdot \rangle_i$ . Let  $A : H_1 \rightarrow H_2$  be a bound linear operator, and let  $A^* : H_2 \rightarrow H_1$  be its adjoint. Then  $R(A) = N(A^*)^\perp$ .*

**PROOF.** Let  $f \in R(A)$ . Then  $f = Ax$ , for some  $x \in H_1$ . For any  $y \in N(A^*)$ , we have

$$\langle f, y \rangle_2 = \langle Ax, y \rangle_2 = \langle x, A^*y \rangle_1 = 0.$$

Thus,  $f \in N(A^*)^\perp$ . This shows that

$$(4.2.1) \quad R(A) \subset N(A^*)^\perp.$$

Write  $H_2 = N(A^*) \oplus N(A^*)^\perp$  (orthogonal direct sum). The subspace  $N(A^*)$  is closed and

$$A^* : N(A^*)^\perp \rightarrow R(A^*)$$

is one-to-one and onto. The subspace  $R(A^*)$  is closed by the Closed Graph Theorem. But then also by the Closed Graph Theorem,

$$A^{*-1} : R(A^*) \rightarrow N(A^*)^\perp$$

is bounded.

Let  $g \in N(A^*)^\perp$ . Define the bounded linear functional

$$\phi(z) = \langle g, A^{*-1}z \rangle_2$$

on the Hilbert space  $R(A^*)$ . By the Riesz representation theorem, there is a unique  $x \in R(A^*)$  such that

$$\langle g, A^{*-1}z \rangle_2 = \langle x, z \rangle_1,$$

for all  $z \in R(A^*)$ . Thus, given any  $y \in N(A^*)^\perp$ , we may write  $y = A^*z$ , for some  $z \in R(A^*)$  and

$$\langle g, y \rangle_2 = \langle g, A^*z \rangle_2 = \langle x, z \rangle_1 = \langle x, A^*y \rangle_1.$$

Thus, we have that

$$\langle g - Ax, y \rangle_2 = 0,$$

for all  $y \in N(A^*)^\perp$ . Now by (4.2.1), we have that  $Ax \in N(A^*)^\perp$ . So since  $g \in N(A^*)^\perp$ , we have that  $g - Ax \in N(A^*)^\perp$ . Taking  $y = g - Ax$ , we see that  $g - Ax = 0$ , and thus,  $g \in R(A)$ . This proves that  $N(A^*)^\perp \subset R(A)$ .  $\square$

### 4.3. The Contraction Mapping Principle in Banach Space

Let  $X$  be a Banach space. If we define the distance between two points  $x$  and  $y$  in  $X$  to be  $d(x, y) = \|x - y\|_X$ , then  $X$  is a complete metric space. Any closed subset of  $X$  is also a complete metric space. In this context, the contraction mapping principle says:

**THEOREM 4.3.1** (Contraction mapping principle in Banach space). *Let  $V \subset X$  be a closed subset. Let  $T : V \rightarrow V$  be a contraction mapping, i.e. there exists a constant  $0 < \alpha < 1$  such that*

$$\|T(x) - T(y)\|_X \leq \alpha \|x - y\|_X,$$

*for all  $x, y \in V$ . Then  $T$  has a unique fixed point  $x \in V$ , i.e.  $T(x) = x$ .*

**PROOF.** The set  $V$  is closed, so  $(V, d)$  is a complete metric space. Apply the standard contraction mapping principle.  $\square$

We want to generalize this result to the situation where the mapping, and hence the corresponding fixed points, depends on parameters.

**DEFINITION 4.3.1.** *Let  $X$  and  $Y$  be Banach spaces. Let  $U \subset X$  and  $V \subset Y$  be open sets. A mapping  $T : U \times \overline{V} \rightarrow \overline{V}$  is called a uniform contraction if there is a constant  $0 < \alpha < 1$  such that*

$$\|T(x, y_1) - T(x, y_2)\|_Y \leq \alpha \|y_1 - y_2\|_Y,$$

*for all  $x \in U$  and  $y_1, y_2 \in \overline{V}$ . Notice that the contraction number  $\alpha$  is uniform for  $x$  throughout  $U$ .*

An application of the contraction mapping principle shows that if  $T : U \times \overline{V} \rightarrow \overline{V}$  is a uniform contraction, then for every  $x \in U$  there is a unique fixed point  $g(x) \in \overline{V}$ , i.e. a unique solution of the equation

$$T(x, g(x)) = g(x).$$

The following result shows that if the mapping  $T$  is continuous or differentiable then the fixed point  $g(x)$  depends continuously or differentially on  $x$ .

**THEOREM 4.3.2** (Uniform Contraction Principle). *Let  $T$  be a uniform contraction, and let  $g : U \rightarrow \overline{V}$  be the corresponding fixed point. If  $T \in C^k(U \times \overline{V}, Y)$  for  $k = 0$  or  $1$ , then  $g \in C^k(U, Y)$ .*

PROOF. The case  $k = 0$  is easy. By the definition of  $g$ , the triangle inequality, and the uniform contraction hypothesis, we have

$$\begin{aligned} \|g(x+h) - g(x)\| &= \|T(x+h, g(x+h)) - T(x, g(x))\| \\ &\leq \|T(x+h, g(x+h)) - T(x+h, g(x))\| \\ &\quad + \|T(x+h, g(x)) - T(x, g(x))\| \\ &\leq \alpha \|g(x+h) - g(x)\| \\ &\quad + \|T(x+h, g(x)) - T(x, g(x))\|. \end{aligned}$$

Thus, since  $\alpha < 1$ , we get

$$\|g(x+h) - g(x)\| \leq \frac{1}{1-\alpha} \|T(x+h, g(x)) - T(x, g(x))\|.$$

But  $T$  is assumed to be continuous, so

$$\lim_{h \rightarrow 0} \|g(x+h) - g(x)\| = 0,$$

i.e.  $g$  is continuous at  $x$ .

Let's first look at the strategy for the case  $k = 1$ . Since  $T(x, g(x)) = g(x)$ , we would have if  $g$  were  $C^1$  (remember that this is what we're trying to prove)

$$D_x T(x, g(x)) + D_y T(x, g(x)) D_x g(x) = D_x g(x).$$

Here,  $D_x T = DT|_{X \times \{0\}}$  and  $D_y T = DT|_{\{0\} \times Y}$ . This inspires us to consider the operator equation

$$(4.3.1) \quad D_x T(x, g(x)) + D_y T(x, g(x)) M(x) = M(x).$$

We will first show that we can solve this for  $M(x) \in \mathcal{L}(X, Y)$ , and then we will show that  $M(x) = D_x g(x)$ .

$T$  is assumed to be  $C^1$ , so for each  $(x, y) \in U \times \bar{V}$ ,  $D_y T(x, y) \in \mathcal{L}(Y, Y)$ . Since  $T$  is a uniform contraction, it can easily be shown that  $\|D_y T(x, y)\| \leq \alpha < 1$ . It follows that  $I - D_y T(x, g(x))$  is invertible for all  $x \in U$ , its inverse depends continuously on  $x$ , and the inverse is bounded by  $1/(1-\alpha)$ . Thus, the solution of (4.3.1) is

$$(4.3.2) \quad M(x) = [I - D_y T(x, g(x))]^{-1} D_x T(x, g(x)) \in \mathcal{L}(X, Y),$$

and  $M(x)$  depends continuously on  $x \in U$ .

Having constructed  $M(x)$ , it remains to show that  $M(x) = D_x g(x)$ . We are going to prove that

$$R(x, h) \equiv g(x+h) - g(x) - M(x)h$$

satisfies

$$\lim_{h \rightarrow 0} \frac{\|R(x, h)\|}{\|h\|} = 0.$$

Put  $\gamma(h) = g(x+h) - g(x)$ . Then

$$\begin{aligned}
 (4.3.3) \quad \gamma(h) &= T(x+h, g(x+h)) - T(x, g(x)) \\
 &= T(x+h, g(x) + \gamma(h)) - T(x, g(x)) \\
 &= D_x T(x, g(x))h + D_y T(x, g(x))\gamma(h) + \Delta(h, \gamma(h)),
 \end{aligned}$$

in which

$$\begin{aligned}
 \Delta(h, \gamma) &= T(x+h, g(x) + \gamma) \\
 &\quad - T(x, g(x)) - D_x T(x, g(x))h - D_y T(x, g(x))\gamma.
 \end{aligned}$$

Since  $T$  is  $C^1$ , for any  $0 < \varepsilon < (1 - \alpha)/2$ , there is a  $\delta > 0$  such that

$$\|\Delta(h, \gamma)\| < \varepsilon(\|h\| + \|\gamma\|),$$

for all  $(h, \gamma) \in U \times \bar{V}$  with  $\|h\| + \|\gamma\| < \delta$ .

Next, since  $\gamma(h)$  is continuous in  $h$  and  $\gamma(0) = 0$ , we can find  $0 < \delta' < \delta/2$  such that

$$\|\gamma(h)\| < \delta/2 \quad \text{provided that} \quad \|h\| < \delta'.$$

So for  $\|h\| < \delta'$ , we have  $\|h\| + \|\gamma(h)\| < \delta/2 + \delta/2 = \delta$ .

Combining the last two paragraphs, we see that if  $\|h\| < \delta'$ , then

$$(4.3.4) \quad \|\Delta(h, \gamma(h))\| < \varepsilon(\|h\| + \|\gamma(h)\|).$$

By (4.3.3), we get for all  $\|h\| < \delta'$ ,

$$\|\gamma(h)\| \leq \|D_x T(x, g(x))\| \|h\| + \alpha \|\gamma(h)\| + \varepsilon(\|h\| + \|\gamma(h)\|),$$

or upon rearrangement,

$$\|\gamma(h)\| \leq \frac{1}{1 - \alpha - \varepsilon} [\|D_x T(x, g(x))\| \|h\| + \varepsilon \|h\|] \leq k \|h\|,$$

with  $k$  independent of  $\varepsilon < (1 - \alpha)/2$ .

Therefore, going back to (4.3.4), we have that

$$\|\Delta(h, \gamma(h))\| < \varepsilon(1 + k)\|h\|,$$

for all  $\|h\| < \delta'$ .

Now return to (4.3.3). Solving for  $\gamma(h)$  and inserting the definition (4.3.2), the formula (4.3.3) can be rewritten as

$$\begin{aligned}
 \gamma(h) &= [I - D_y T(x, g(x))]^{-1} [D_x T(x, g(x))h + \Delta(h, \gamma(h))] \\
 &= M(x)h + [I - D_y T(x, g(x))]^{-1} \Delta(h, \gamma(h)).
 \end{aligned}$$

So finally, we get

$$\begin{aligned} R(x, h) &= \|\gamma(h) - M(x)h\| \\ &\leq \|[I - D_y T(x, g(x))]^{-1}\| \|\Delta(h, \gamma(h))\| \\ &\leq \frac{1}{1 - \alpha} \varepsilon (1 - k) \|h\|, \end{aligned}$$

for all  $\|h\| < \delta'$ . It follows that

$$\lim_{h \rightarrow 0} \frac{\|\gamma(h) - M(x)h\|}{\|h\|} = 0.$$

□

#### 4.4. The Implicit Function Theorem in Banach Space

**THEOREM 4.4.1.** *Suppose that  $X$ ,  $Y$ , and  $Z$  are Banach spaces. Let  $U \subset X$  and  $V \subset Y$  be open sets. Let  $F : U \times V \rightarrow Z$  be a  $C^1$  mapping. Assume that there exists a point  $(x_0, y_0) \in U \times V$  such that  $F(x_0, y_0) = 0$  and  $D_y F(x_0, y_0)$  has a bounded inverse in  $\mathcal{L}(Z, Y)$ . Then there are a neighborhood  $U_1 \times V_1 \subset U \times V$  of  $(x_0, y_0)$  and a  $C^1$  mapping  $g : U_1 \rightarrow V_1$  such that*

$$y_0 = g(x_0) \quad \text{and} \quad F(x, g(x)) = 0,$$

for all  $x \in U_1$ . If  $F(x, y) = 0$  for  $(x, y) \in U_1 \times V_1$ , then  $y = g(x)$ .

##### 4.4.1

**PROOF.** Let  $L = D_y F(x_0, y_0)^{-1}$ , and define the  $C^1$  map  $G : U \times V \rightarrow Y$  by

$$G(x, y) = y - LF(x, y).$$

Notice that  $G(x, y) = y$  if and only if  $F(x, y) = 0$ . We also have  $G(x_0, y_0) = y_0$  and  $D_y G(x_0, y_0) = 0$ .

By continuity, given  $\varepsilon = 1/4$  there exists a  $\delta_1 > 0$  such that

$$\|D_y G(x, y)\| = \|D_y G(x, y) - D_y G(x_0, y_0)\| < 1/4,$$

for  $(x, y) \in N = B_{\delta_1}(x_0) \times B_{\delta_1}(y_0)$ . We can also find a constant  $M > 0$  such that

$$\|D_x G(x, y)\| \leq M,$$

for  $(x, y) \in N$ .

Take a pair of points  $(x_i, y_i) \in N$ ,  $i = 1, 2$ . Set  $x(s) = sx_2 + (1-s)x_1$  and  $y(s) = sy_2 + (1-s)y_1$ . Then

$$\begin{aligned} G(x_2, y_2) - G(x_1, y_1) &= \int_0^1 \frac{d}{ds} G(x(s), y(s)) ds \\ &= \int_0^1 D_x G(x(s), y(s))(x_2 - x_1) ds \\ &\quad + \int_0^1 D_y G(x(s), y(s))(y_2 - y_1) ds. \end{aligned}$$

So by the preceding estimates, we obtain

$$\|G(x_2, y_2) - G(x_1, y_1)\| \leq M\|x_2 - x_1\| + (1/4)\|y_2 - y_1\|.$$

Set  $\delta_2 = \min\{\delta_1, \delta_1/4M\}$ . Define  $U_1 = B_{\delta_2}(x_0)$  and  $V_1 = B_{\delta_1}(y_0)$ . The claim is that  $G : U_1 \times \bar{V}_1 \rightarrow \bar{V}_1$  and that  $G$  is a uniform contraction.

Let  $(x, y) \in U_1 \times \bar{V}_1$ . Then

$$\begin{aligned} \|G(x, y) - y_0\| &= \|G(x, y) - G(x_0, y_0)\| \\ &\leq M\|x - x_0\| + (1/4)\|y - y_0\| \\ &\leq M\delta_2 + (1/4)\delta_1 \\ &\leq (1/4)\delta_1 + (1/4)\delta_1 = \delta_1/2. \end{aligned}$$

Thus  $G(x, y) \in \bar{V}_1$ .

We also have for all  $(x, y_1), (x, y_2) \in U_1 \times \bar{V}_1$  that

$$\|G(x, y_1) - G(x, y_2)\| \leq (1/4)\|y_1 - y_2\|,$$

which means that  $G$  is a uniform contraction.

Applying the uniform contraction principle, there is a  $C^1$  map  $g : U_1 \rightarrow V_1$  such that

$$G(x, g(x)) = g(x),$$

for all  $x \in U_1$ . Moreover, given  $x \in U_1$ ,  $y = g(x) \in V_1$  is the unique point such that  $G(x, y) = y$ .

□

Of course, this theorem includes the case of finite dimensions:

$$X = \mathbb{R}^m, \quad Y = Z = \mathbb{R}^n.$$

## CHAPTER 5

### Dependence on Initial Conditions and Parameters

#### 5.1. Smooth Dependence on Initial Conditions

We have seen in Theorem 2.5.1 that solutions of the initial value problem depend continuously on initial conditions. We will now show that this dependence is as smooth as the vector field.

**THEOREM 5.1.1.** *Let  $\Omega \subset \mathbb{R}^{n+1}$  be an open set, and suppose that  $f : \Omega \rightarrow \mathbb{R}^n$  is  $C^1$ .*

*For  $(s, p) \in \Omega$ , the unique local solution  $x(t, s, p)$  of the initial value problem*

$$(5.1.1) \quad \frac{d}{dt}x(t, s, p) = f(t, x(t, s, p)), \quad x(s, s, p) = p.$$

*is  $C^1$  in its open domain of definition*

$$D = \{(t, s, p) \in \mathbb{R}^{n+2} : \alpha(s, p) < t < \beta(s, p), (s, p) \in \Omega\}.$$

*The differential matrix  $D_p x(t, s, p)$  satisfies the so-called linear variational equation*

$$(5.1.2) \quad \frac{d}{dt}D_p x(t, s, p) = D_x f(t, x(t, s, p))D_p x(t, s, p),$$

$$D_p x(s, s, p) = I.$$

*Also,*

$$(5.1.3) \quad \frac{\partial x}{\partial s}(t, s, p) = -D_p x(t, s, p)f(s, p).$$

**PROOF.** Suppose, temporarily, that we have shown that  $x(t, s, p) \in C^1(D)$ .

Then (5.1.2) follows immediately by taking the derivative of (5.1.1) with respect to  $p$ .

Next, use the properties of the flow to write

$$x(t, s, p) = x(t, \tau, x(\tau, s, p)) \quad t, \tau \in I(s, p).$$

Take the derivative of this with respect to  $\tau$  to get

$$0 = \frac{\partial x}{\partial s}(t, \tau, x(\tau, s, p)) + D_p x(t, \tau, x(\tau, s, p)) \frac{\partial x}{\partial t}(\tau, s, p).$$

From this and the ODE (5.1.1), we obtain

$$\frac{\partial x}{\partial s}(t, \tau, x(\tau, s, p)) = -D_p x(t, \tau, x(\tau, s, p)) f(\tau, x(\tau, s, p)).$$

Equation (5.1.3) follows by letting  $\tau = s$ .

Since we already know that the solution is continuously differentiable in  $t$ , we must only establish continuous differentiability of  $x(t, s, p)$  in  $(s, p)$ . We are going to do this by a uniqueness argument. The flow  $x(t, s, p)$  satisfies the standard integral equation. We will show that the implicit function guarantees this equation has a unique  $C^1$  solution. We now proceed to set this up precisely.

For an arbitrary point  $(s_0, p_0) \in \Omega$ , let  $x_0(t) = x(t, s_0, p_0)$  be the corresponding solution to the initial value problem (5.1.1), defined on the maximal interval  $I(s_0, p_0) = (\alpha(s_0, p_0), \beta(s_0, p_0))$ . Choose an arbitrary closed interval  $J = [a, b]$  with

$$\alpha(s_0, p_0) < a < s_0 < b < \beta(s_0, p_0).$$

Define the compact set

$$K = \{(t, x(t, s_0, p_0)) \in \Omega : t \in J\}.$$

By the Covering Lemma 2.3.2, there exist numbers  $\delta, \rho > 0$  and a compact set  $K' \subset \Omega$  such that for every  $(s, p) \in K$ , the cylinder

$$\mathcal{C}(s, p) = \{(s', p') \in \mathbb{R}^{n+1} : |s' - s| \leq \delta, \|p' - p\| \leq \rho\}$$

satisfies

$$\mathcal{C}(s, p) \subset K'.$$

Define the Banach spaces  $X = \mathbb{R}^{n+1}$  and  $Y = Z = C(J, \mathbb{R}^n)$  with the sup norm. Let  $U = (a, b) \times \mathbb{R}^n$  and

$$V = \{x \in Y : \|x - x_0\| = \sup_J \|x(t) - x_0(t)\| < \rho\}.$$

We have that  $U \subset X$  and  $V \subset Y$  are open.

Suppose that  $x \in V$ . Then for any  $\sigma \in J$ , we have that

$$\|x(\sigma) - x_0(\sigma)\| < \rho,$$

and so,  $(\sigma, x(\sigma)) \in \mathcal{C}(\sigma, x_0(\sigma)) \subset \Omega$ , for any  $\sigma \in J$ . Therefore, the operator

$$T : U \times V \rightarrow Z$$

given by

$$(s, p, x) \mapsto T(s, p, x)(t) = x(t) - p - \int_s^t f(\sigma, x(\sigma))d\sigma, \quad t \in J,$$

is well-defined. It is straight-forward to verify that  $T$  is  $C^1$ . In particular, we have that  $D_x T(s, p, x)$  is linear map from  $Y$  to  $Z (= Y)$  which takes  $y \in Y$  to the function  $D_x T(s, p, x)[y] \in Z$  whose value at a point  $t \in J$  is

$$D_x T(s, p, x)[y](t) = y(t) - \int_s^t A(\sigma)y(\sigma)d\sigma \quad \text{with} \quad A(\sigma) = D_x f(\sigma, x(\sigma)).$$

(It is here that we are using the assumption  $f \in C^1(\Omega)$ .)

Since  $x_0(t) = x(t, s_0, p_0)$  solves (5.1.1) we have that

$$T(s_0, p_0, x_0) = 0.$$

Now we claim that  $D_x T(s_0, p_0, x_0)$  is invertible as a linear map from  $Y$  to  $Z$ . Let  $g \in Z$ . The equation  $D_x T(s_0, p_0, x_0)[y] = g$  can be written explicitly as

$$(5.1.4) \quad y(t) - \int_{s_0}^t A(\sigma)y(\sigma)d\sigma = g(t).$$

Letting  $u(t) = y(t) - g(t)$ , this is equivalent to

$$u(t) = \int_{s_0}^t A(\sigma)[u(\sigma) + g(\sigma)]d\sigma.$$

Notice that the right-hand side is  $C^1$  in  $t$ . So this is equivalent to

$$u'(t) = A(t)[u(t) + g(t)], \quad t \in J, \quad u(s_0) = 0.$$

We can represent the unique solution of this initial value problem using the variation of parameters formula. Let  $W(t, s)$  be the fundamental matrix for  $A(t)$ . Then

$$u(t) = \int_{s_0}^t W(t, \sigma)A(\sigma)g(\sigma)d\sigma.$$

Since  $u = y - g$ , this is equivalent to

$$(5.1.5) \quad y(t) = g(t) + \int_{s_0}^t W(t, \sigma)A(\sigma)g(\sigma)d\sigma.$$

We have shown that for every  $g \in Z$ , the equation (5.1.4) has a unique solution  $y \in Y$  given by (5.1.5). This proves the invertibility of  $D_x T(s_0, p_0, x_0)$ . Finally, from the formula (5.1.5), we see that the inverse is a bounded map from  $Z$  to  $Y$ .

By the Implicit Function Theorem 4.4.1, there are a neighborhood  $U_0 \subset U$  of  $(s_0, p_0)$ , a neighborhood  $V_0 \subset V$  of  $x_0$ , and a  $C^1$  map  $\phi : U_0 \rightarrow V_0$  such that  $\phi(s_0, p_0) = x_0$  and  $T(s, p, \phi(s, p)) = 0$  for all  $(s, p) \in U_0$ . Furthermore, if  $(s, p, x) \in U_0 \times V_0$  is a point such that  $T(s, p, x) = 0$ , then  $x = \phi(s, p)$ .

Since the map  $(s, p) \mapsto \phi(s, p)$  is  $C^1$  from  $U_0$  into  $V_0 \subset Y$ , it follows that the function  $\phi(s, p)(t)$  is  $C^1$  in  $(s, p)$  for all  $t \in J$ .

By continuous dependence on initial conditions, Theorem (2.5.1), there is a neighborhood  $N$  of  $(s_0, p_0)$  such that for all  $(s, p) \in N$ , we have that

$$J \subset I(s, p), \quad \text{and} \quad \|x(t, s, p) - x(t, s_0, p_0)\| < \rho, \quad \text{for all } t \in J.$$

Therefore, for all  $(s, p) \in N$ , we have that  $x(t, s, p) \in V_0$ . Since  $x(t, s, p)$  solves the initial value problem, we also have that  $T(s, p, x(\cdot, s, p)) = 0$ .

By the uniqueness portion of the implicit function theorem, we conclude that

$$x(t, s, p) = \phi(s, p)(t), \quad \text{for all } (s, p) \in U_0 \cap N, \quad t \in J.$$

Therefore,  $x(t, s, p)$  is  $C^1$  on  $J \times (U_0 \cap N)$ . Since,  $(s_0, p_0)$  is an arbitrary point in  $\Omega$ , and  $J$  is an arbitrary subinterval of  $I(s_0, p_0)$ , it follows that  $x(t, s, p)$  is  $C^1$  on all of  $D$ . □

**COROLLARY 5.1.1.** *If  $f(t, x)$  is continuous on  $\Omega \subset \mathbb{R}^{n+1}$  and  $C^k$  in  $x$ , then  $x(t, s, p)$  is  $C^k$  in  $(s, p)$  on its open domain of definition. (If  $f(t, x)$  is in  $C^k(\Omega)$ , then  $x(t, s, p)$  is  $C^k$  in  $(t, s, p)$  on its domain of definition.)*

This is proved by induction on  $k$ . The corollary can be proved for  $k = m + 1$ , by applying the result for  $k = m$  to the ODE's satisfied by the first derivatives of  $x(t, s, p)$ . The first derivatives of  $x(t, s, p)$  in  $C^m$  implies that  $x(t, s, p)$  is in  $C^{m+1}$ .

**COROLLARY 5.1.2.**

$$\det D_p x(t, s, p) = \exp \int_s^t \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}(\tau, x(\tau, s, p)) d\tau.$$

*Proof:* This follows by applying Theorem 3.1.2 to the variational equation:

$$\frac{d}{dt} D_p x(t, s, p) = D_x f(t, x(t, s, p)) D_p x(t, s, p),$$

and noting that

$$\text{tr } D_x f = \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}.$$

### 5.2. Continuous Dependence on Parameters

**THEOREM 5.2.1.** *Let  $f(t, x, \lambda)$  be continuous on an open set  $\Omega \subset \mathbb{R}^{n+1} \times \mathbb{R}^m$  with values in  $\mathbb{R}^n$ . Assume that  $f(t, x, \lambda)$  is locally Lipschitz with respect to  $(x, \lambda)$ . Then given  $(t_0, x_0, \lambda) \in \Omega$ , the initial value problem*

$$x' = f(t, x, \lambda), \quad x(t_0) = x_0$$

*has a unique local solution  $x(t, t_0, x_0, \lambda)$  on a maximal interval of definition*

$$I(t_0, x_0, \lambda) = (\alpha(t_0, x_0, \lambda), \beta(t_0, x_0, \lambda))$$

where

- (i)  $\alpha(t_0, x_0, \lambda)$  is upper semi-continuous.
- (ii)  $\beta(t_0, x_0, \lambda)$  is lower semi-continuous.
- (iii)  $x(t, t_0, x_0, \lambda)$  is continuous on its open domain of definition

$$\{(t, t_0, x_0, \lambda) : t \in I(t_0, x_0, \lambda); (t_0, x_0, \lambda) \in \Omega\}.$$

**Proof:** Here's the trick: turn the parameter  $\lambda$  into a dependent variable and use the old continuous dependence result, Theorem 2.5.1. Define a new vector

$$y = \begin{bmatrix} x \\ \lambda \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^m,$$

and a new vector field

$$F(t, y) = \begin{bmatrix} f(t, x, \lambda) \\ 0 \end{bmatrix},$$

on  $\Omega$ . Apply the result on continuous dependence to the so-called *suspended system*

$$y' = F(t, y), \quad y(t_0) = y_0 = \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}.$$

Since the vector field  $F$  is 0 in its last  $m$  components, the last  $m$  components of  $y$  are constant and, hence, equal to  $\lambda_0$ . Extraction of the first  $n$  components yields the desired result for  $x(t, t_0, x_0, \lambda_0)$ .

**Remark:** This result is still true even if  $f(t, x, \lambda)$  is not locally Lipschitz continuous in  $\lambda$ , however the easy proof no longer works.

**COROLLARY 5.2.1.** *If  $f(t, x, \lambda)$  is in  $C^k(\Omega)$ , then  $x(t, t_0, x_0, \lambda)$  is in  $C^k$  on its open domain of definition.*

**Proof:** Use the trick above, and then apply the result on smooth dependence.



## CHAPTER 6

### Linearization and Invariant Manifolds

#### 6.1. Autonomous Flow At Regular Points

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DEFINITION 6.1.1. For  $j = 1, 2$ , let

$$f_j : \mathcal{O}_j \rightarrow \mathbb{R}^n,$$

be  $C^1$  autonomous vector fields, with  $\mathcal{O}_j \subset \mathbb{R}^n$  open. Let  $\phi_t^{(j)}$ ,  $j = 1, 2$  be the associated flows. We say that the two flows  $\phi_t^{(1)}$  and  $\phi_t^{(2)}$  are topologically conjugate if there exists a homeomorphism  $\eta : \mathcal{O}_1 \rightarrow \mathcal{O}_2$  such that  $\eta \circ \phi_t^{(1)} = \phi_t^{(2)} \circ \eta$ .

If the map  $\eta$  can be chosen to be a diffeomorphism, then we will say that the flows are diffeomorphically conjugate.

If  $\phi_t^{(1)}$  and  $\phi_t^{(2)}$  are conjugate, then the maximal interval of existence of  $\phi_t^{(1)}(p)$  is the same as for  $\phi_t^{(2)}(\eta(p))$ , for every  $p \in \mathcal{O}_1$ .

Conjugacy is an equivalence relation.

THEOREM 6.1.1. Two flows  $\phi_t^{(1)}$  and  $\phi_t^{(2)}$  are diffeomorphically conjugate under  $\eta$ :

$$(6.1.1) \quad \eta(\phi_t^{(1)}(p)) = \phi_t^{(2)}(\eta(p)), \quad p \in \mathcal{O}_1,$$

if and only if the associated vector fields satisfy

$$(6.1.2) \quad D\eta(p)f_1(p) = f_2(\eta(p)), \quad p \in \mathcal{O}_1.$$

PROOF. We are given that

$$\frac{d}{dt}\phi_t^{(j)}(p) = f_j(\phi_t^{(j)}(p)), \quad \phi^{(j)}(p)|_{t=0} = p,$$

for every  $p \in \mathcal{O}_j$ ,  $j = 1, 2$ .

By the chain rule, we have that

$$(6.1.3) \quad \begin{aligned} \frac{d}{dt}\eta(\phi_t^{(1)}(p)) &= D\eta((\phi_t^{(1)}(p)))\frac{d}{dt}\phi_t^{(1)}(p) \\ &= D\eta((\phi_t^{(1)}(p)))f_1(\phi_t^{(1)}(p)). \end{aligned}$$

So if (6.1.2) holds, then

$$\frac{d}{dt}\eta(\phi_t^{(1)}(p)) = f_2(\eta(\phi_t^{(1)}(p))).$$

Since  $\eta(\phi_t^{(1)}(p))|_{t=0} = \eta(p)$ , we have by uniqueness that (6.1.1) is true.

On the other hand, if (6.1.1) holds, then take the derivative in  $t$  and apply (6.1.3) to get,

$$D\eta((\phi_t^{(1)}(p))f_1(\phi_t^{(1)}(p)) = \frac{d}{dt}\eta(\phi_t^{(1)}(p)) = \frac{d}{dt}\phi_t^{(2)}(\eta(p)) = f_2(\phi_t^{(2)}(\eta(p))).$$

Setting  $t = 0$ , we recover (6.1.2).  $\square$

**DEFINITION 6.1.2.** *A point  $x \in \mathcal{O}$  is a regular point if for a vector field  $f$  if  $f(x) \neq 0$ .*

**DEFINITION 6.1.3.** *An  $n - 1$  dimensional hyperplane in  $\mathbb{R}^n$  is a subset  $H \subset \mathbb{R}^n$  of the form  $H = x_0 + S$  where  $x_0 \in \mathbb{R}^n$  and  $S$  is an  $n - 1$  dimensional subspace of  $\mathbb{R}^n$ .*

Since an  $n - 1$  dimensional subspace of  $\mathbb{R}^n$  can always be expressed in the form  $S = \{x \in \mathbb{R}^n : x \cdot \xi = 0\}$  for some fixed nonzero vector  $\xi \in \mathbb{R}^n$ , we have  $H = \{x \in \mathbb{R}^n : (x - x_0) \cdot \xi = 0\}$ . The vector  $\xi$  is unique up to a scalar multiple. A hyperplane divides  $\mathbb{R}^n$  into two disjoint sets. Given a choice of the vector  $\xi$ , we can describe them as the positive side  $\mathcal{P} = \{x \in \mathbb{R}^n : x \cdot \xi > 0\}$  and the negative side  $\mathcal{N} = \{x \in \mathbb{R}^n : x \cdot \xi < 0\}$ .

Note that if  $H = x_0 + S$  is a hyperplane, then  $H = x + S$  for any  $x \in H$ .

**DEFINITION 6.1.4.** *Let  $H$  be a hyperplane. A transversal to  $f$  is a connected set of the form  $T = H \cap U$ ,  $U \subset \mathcal{O}$  open, with the property that  $f(x) \notin S$  for all  $x \in T$ .*

Note that if  $T$  is a transversal, then  $f(x) \neq 0$  on  $T$ , i.e.  $T$  contains only regular points. Conversely, if  $x_0$  is a regular point, then  $f$  has a transversal at  $x_0$ . (Exercise)

If  $T \subset H = x_0 + S$  and  $S = \{x \cdot \xi = 0\}$ , then  $f(x) \cdot \xi \neq 0$  on  $T$ . Since  $T$  is connected, either  $f(x) \cdot \xi > 0$  or  $f(x) \cdot \xi < 0$ , for all  $x \in T$ . Thus,  $f(x) \in \mathcal{P}$  or  $f(x) \in \mathcal{N}$ , for all  $x \in T$ . In words, along  $T$ ,  $f(x)$  points to the same side of  $T$ .

The next result shows that a smooth vector field can be transformed to a constant vector, at a regular point. The situation near a critical point is more complicated and will be taken up in the next section when we come to the Hartman-Grobman theorem.

**THEOREM 6.1.2.** *Let  $x_0 \in \mathcal{O}$  be a regular point for the vector field  $f(x)$ . Let  $T \subset H = x_0 + S$  be a transversal at  $x_0$ . Then there exists a neighborhood  $U \subset \mathcal{O}$  of  $x_0$  which is diffeomorphic to a neighborhood of  $(0, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$  of the form  $I \times V$ , in which  $I = (-\varepsilon, \varepsilon)$ . The diffeomorphism  $\eta$  has the following properties:*

$$\eta : I \times V \rightarrow U, \quad \eta(0, 0) = x_0, \quad \eta : \{0\} \times V = T \cap U,$$

and the image of the vector field  $f(x)$  under  $\psi = \eta^{-1}$  is a constant vector:

$$g(\psi(x)) = D_x \psi(x) f(x) = e_1.$$

**PROOF.** Choose vectors  $\xi_1, \dots, \xi_{n-1} \in \mathbb{R}^n$  which span the subspace  $S$ . For  $y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$ , define the linear combination

$$p(y) = x_0 + \sum_{i=1}^{n-1} y_i \xi_i.$$

Then  $p : \mathbb{R}^{n-1} \rightarrow H$ .

Now for any  $y \in \mathbb{R}^{n-1}$  such that  $p(y) \in \mathcal{O}$ , the existence theorem allows us to define

$$\eta(t, y) = x(t, p(y)) \quad \text{for} \quad \alpha(p(y)) < t < \beta(p(y)).$$

Since the vector field  $f(x)$  is  $C^1$ , the theorem on smooth dependence (Theorem 5.1.1) ensures us that  $\eta(t, y)$  is a  $C^1$  function on its open domain of definition. Notice that  $\eta(0, 0) = x(0, p(0)) = x(0, x_0) = x_0$  and that  $\eta(0, y) = x(0, p(y)) = p(y) \in H \cap \mathcal{O}$ .

We need to show that  $\eta$  is invertible in a neighborhood of  $(0, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ . This will be done using the inverse function theorem. So let's check the Jacobian:

$$\begin{aligned} D\eta(t, y) &= \left[ \frac{\partial}{\partial t} x(t, p(y)), \frac{\partial}{\partial y_1} x(t, p(y)), \dots, \frac{\partial}{\partial y_{n-1}} x(t, p(y)) \right] \\ (6.1.4) \quad &= [f(x(t, p(y))), D_p x(t, p(y))\xi_1, \dots, D_p x(t, p(y))\xi_{n-1}]. \end{aligned}$$

Remember that in general,  $D_p x(0, p) = I$ . Setting  $(t, y) = (0, 0)$ , we see that

$$D\eta(0, 0) = [f(x_0), \xi_1, \dots, \xi_{n-1}].$$

The columns of this matrix are independent since  $f(x_0) \notin S$ . By the inverse function theorem,  $\eta$  is a local diffeomorphism from a neighborhood of  $(0, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$  to a neighborhood of  $0 \in \mathbb{R}^n$ . We can restrict the size of the domain of  $\eta$  so that it has the convenient "rectangular" form  $I \times V$  where  $I = (-\varepsilon, \varepsilon)$  and  $0 \in V \subset \mathbb{R}^{n-1}$ . Set  $U = \eta(I \times V)$ .

By (6.1.4), the first column of  $D\eta$  is

$$D\eta(t, y)e_1 = f(x(t, p(y))) = f(\eta(t, y)).$$

Let  $\psi = \eta^{-1}$ , and write  $p = \eta(t, y)$  and  $(t, y) = \psi(p)$ . Then this is the same as

$$D\eta(\psi(p))e_1 = f(p).$$

Since  $\psi(\eta(p)) = p$ , the chain rule gives us that  $D\psi(p)D\eta(\psi(p)) = I$ , for all  $p \in U$ . Therefore, we see that

$$D\psi(p)f(p) = D\psi(p)D\eta(\psi(p))e_1 = e_1.$$

□

The pair  $(I \times V, \eta)$  is called a flow box.

**THEOREM 6.1.3.** *With the notation of Theorem 6.1.2, let  $\phi_t(p)$  be the flow of the vector field  $f(x)$  in  $U$  and  $\chi_t(q) = q + te_1$  be the flow of the constant vector field  $g(y) = e_1$  in  $I \times V$ . Then*

$$\phi_t(\eta(q)) = \eta(\chi_t(q)),$$

for all  $q \in I \times V$  and  $|t - q_1| < \varepsilon$ .

**PROOF.** This is an immediate consequence of (6.1.2) and Theorem 6.1.1. □

This result shows that the orbits of  $f(x)$  in  $U$  are in one-to-one correspondence with line segments orthogonal to  $V$ . More precisely, for every  $x \in U$ , there exists a unique  $p \in H \cap U$  and a unique time  $|t| < \varepsilon$  such that  $x = \phi_t(p)$ . From this we see that each orbit of  $f(x)$  which enters  $U$  remains there for a time interval of length  $2\varepsilon$  and then exits  $U$ , during which time it crosses  $T = H \cap U$  exactly once. All orbits cross  $T$  in the same direction.

## 6.2. The Hartman-Grobman Theorem

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**DEFINITION 6.2.1.** *Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open set. Let  $F : \mathcal{O} \rightarrow \mathbb{R}^n$  be a  $C^1$  autonomous vector field. We say a critical point  $x_0 \in \mathcal{O}$  is hyperbolic if the eigenvalues of  $D_x F(x_0)$  have nonzero real parts.*

Suppose that  $F$  has a critical point at  $x_0 = 0$ . Set  $A = D_x F(0)$ . Writing  $F(x) = Ax + [F(x) - Ax] = Ax + f(x)$ , we have that  $f(x)$  is  $C^1$ ,  $f(0) = 0$ , and  $D_x f(0) = 0$ .

We are going to consider the flow of the vector field  $Ax + f(x)$  in relation to the flow of the linear vector field  $Ax$ . The Hartman-Grobman theorem says that the two flows are topologically conjugate. Before coming to a precise statement of this theorem, we need to define some spaces and set down some notation.

Define the sets

$$X = \{g : \mathbb{R}^n \rightarrow \mathbb{R}^n : g \text{ is continuous, } g(0) = 0, \sup_x \|g(x)\| < \infty\},$$

and

$$Y = \left\{ f \in X : \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} < \infty \right\}.$$

These become Banach spaces with the following norms:

$$\|g\|_X = \sup_x \|g(x)\|,$$

and

$$\|f\|_Y = \|f\|_X + \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$

Notice that  $Y \subset X$  and  $\|f\|_X \leq \|f\|_Y$ .

We are going to assume that the nonlinear portion,  $f(x)$ , of the vector field  $Ax + f(x)$  lies in  $Y$ . This would appear to be a strong restriction. However, since ultimately we are only interested in the flow near the origin, the behavior of the vector field away from the origin is unimportant. We will return to this point later on. One advantage of having  $f \in Y$  is that the flow  $\psi_t(p) = x(t, p)$  of  $Ax + f(x)$  is then globally defined by Theorem 2.8.3, since  $f$  is Lipschitz and bounded. Of course, the flow of the vector field  $Ax$  is just  $\phi_t(p) = \exp At p$ .

**THEOREM 6.2.1 (Hartman-Grobman).** *Let  $A$  be hyperbolic, and suppose that  $f \in Y$ . Let  $\phi_t$  be the flow of  $Ax$ , and let  $\psi_t$  be the flow of  $Ax + f(x)$ .*

*There is a  $\delta > 0$  such that if  $\|f\|_Y < \delta$  then there exists a unique homeomorphism  $\Lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\Lambda - I \in X$  and*

$$\Lambda \circ \psi_t = \phi_t \circ \Lambda.$$

*Moreover, the map  $f \rightarrow \Lambda - I$  is continuous from  $Y$  into  $X$ .*

Note: Here and below  $I$  denotes the identity map on  $\mathbb{R}^n$ . Thus,  $\Lambda - I$  is the function whose value at a point  $x$  is  $\Lambda(x) - x$ .

Notice that this theorem only guarantees the existence of a homeomorphism  $\Lambda$  which conjugates the linear and nonlinear flows. In general,  $\Lambda$  will not be smooth, unless a certain nonresonance condition is satisfied. This is the content of Sternberg's Theorem, the proof of which is substantially more difficult. The limitation on the smoothness of the linearizing map  $\Lambda$  will be illustrated by an example at the end of this section.

The proof of the Theorem 6.2.1 will follow from the next two lemmas, both of which are applications of the contraction mapping principle.

**Warning.** We will be using four different norms: the Euclidean norm of vectors in  $\mathbb{R}^n$ , the operator norm of  $n \times n$  matrices, and the norms in the spaces  $X$  and  $Y$ . The first two will be written  $\|\cdot\|$  as usual, and the last two will be denoted  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ , respectively.

LEMMA 6.2.1. *If  $F \in Y$  with  $\|F\|_Y < 1$ , then  $I + F$  is a homeomorphism on  $\mathbb{R}^n$ . If  $S(F) = (I + F)^{-1} - I$ , then  $S(F) \in Y$  and  $\|S(F)\|_X = \|F\|_X$ .*

PROOF. Let  $F \in Y$ , with  $\|F\|_Y = \alpha < 1$ . Choose any point  $y \in \mathbb{R}^n$  and define the function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , by the formula  $T(x) = y - F(x)$ . Given any pair of points  $x_1, x_2 \in \mathbb{R}^n$ , we have that

$$\|T(x_1) - T(x_2)\| = \|F(x_1) - F(x_2)\| \leq \|F\|_Y \|x_1 - x_2\| = \alpha \|x_1 - x_2\|.$$

This shows that  $T$  is a contraction on  $\mathbb{R}^n$ . According to the contraction mapping principle,  $T$  has a unique fixed point  $\bar{x} \in \mathbb{R}^n$ , so that  $\bar{x} = T(\bar{x}) = y - F(\bar{x})$ . This proves that the map  $I + F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is one-to-one and onto. It follows that  $(I + F)^{-1}$  exists.

Set  $G = S(F) = (I + F)^{-1} - I$ . Then  $G(0) = 0$  and

$$\begin{aligned} \|G\|_X &= \sup_{y \in \mathbb{R}^n} \|G(y)\| \\ &= \sup_{x \in \mathbb{R}^n} \|G(x + F(x))\| \\ &= \sup_{x \in \mathbb{R}^n} \|x - (x + F(x))\| \\ &= \sup_{x \in \mathbb{R}^n} \|F(x)\| \\ &= \|F\|_X. \end{aligned}$$

Take any points  $y_1, y_2 \in \mathbb{R}^n$ . There are unique points  $x_1, x_2 \in \mathbb{R}^n$  such that  $y_i = x_i + F(x_i)$ ,  $i = 1, 2$ . Now

$$\begin{aligned} \|y_1 - y_2\| &= \|x_1 - x_2 + F(x_1) - F(x_2)\| \\ &\geq \|x_1 - x_2\| - \|F(x_1) - F(x_2)\| \\ &\geq \|x_1 - x_2\| - \|F\|_Y \|x_1 - x_2\| \\ &\geq (1 - \alpha) \|x_1 - x_2\|. \end{aligned}$$

Since

$$G(y_i) = (I + F)^{-1}(y_i) - y_i = x_i - (x_i + F(x_i)) = -F(x_i),$$

we have that

$$\|G(y_1) - G(y_2)\| = \|F(x_1) - F(x_2)\| \leq \alpha \|x_1 - x_2\| \leq \frac{\alpha}{1 - \alpha} \|y_1 - y_2\|.$$

The last inequality shows that  $G$  is Lipschitz continuous.

We have therefore shown that  $S(F) = G \in Y$ .

Finally, we see that  $I + F$  is a homeomorphism because  $G$  is Lipschitz continuous and  $(I + F)^{-1} = I + G$  therefore is continuous.  $\square$

Recall that if  $A$  is hyperbolic, i.e. its eigenvalues all have nonzero real part, then  $\mathbb{R}^n = E_s + E_u$ . The stable and unstable subspaces  $E_s$  and  $E_u$  are the linear span of the real and imaginary parts of the generalized eigenvectors corresponding to eigenvalues with negative and positive real part, respectively. Thus, each  $x \in \mathbb{R}^n$  is uniquely written as  $x = x_s + x_u$ , with  $x_s \in E_s$  and  $x_u \in E_u$ . The projections  $P_s x = x_s$  and  $P_u x = x_u$  are bounded, and they commute with  $A$ , and hence, also with  $\exp At$ . Recall that by Theorem 1.4.1, there are positive constants  $C_0, \lambda$  such that

$$\|P_s \exp At\| \leq C_0 e^{-\lambda t}, \quad t \geq 0$$

and

$$\|P_u \exp At\| \leq C_0 e^{\lambda t}, \quad t \leq 0.$$

Therefore, given any  $\varepsilon > 0$ , we can find a  $T > 0$  such that  $L = \exp AT$  satisfies

$$\|P_s L\| \leq \varepsilon \quad \text{and} \quad \|P_u L^{-1}\| \leq \varepsilon.$$

LEMMA 6.2.2. *Suppose that  $\|P_s L\|, \|P_u L^{-1}\| \leq \varepsilon < 1/2$ . Let  $F, G \in Y$  with  $\|F\|_Y, \|G\|_Y \leq \mu < (1 - 2\varepsilon)/(1 + \varepsilon) < 1$ . Then there is a unique homeomorphism  $\Lambda$  such that  $\Lambda - I \in X$  and*

$$(6.2.1) \quad L \circ (I + F) \circ \Lambda = \Lambda \circ L \circ (I + G).$$

PROOF. In this proof, we omit the “ $\circ$ ” for composition. Thus, for example,  $FG$  will stand for  $F \circ G$ .

We are going to construct  $\Lambda$  in the form  $\Lambda = I + h$ , with  $h \in X$ .

Equation (6.2.1) is the same as

$$L(I + F)(I + h) = (I + h)L(I + G),$$

or taking projections,

$$(6.2.2) \quad P_s L(I + F)(I + h) = P_s(I + h)L(I + G)$$

$$(6.2.3) \quad P_u L(I + F)(I + h) = P_u(I + h)L(I + G).$$

By the previous lemma,  $I + G$  is invertible, and  $(I + G)^{-1} = I + S(G)$ , with  $S(G) \in X$ . So right-composing  $(I + G)^{-1}L^{-1}$  with both sides of (6.2.2) and left-composing  $L^{-1}$  with both sides of (6.2.3) yields

$$(6.2.4) \quad P_s(I + h) = P_s L(I + F)(I + h)(I + S(G))L^{-1}$$

$$(6.2.5) \quad P_u(I + F)(I + h) = P_u L^{-1}(I + h)L(I + G)$$

After a little bit of manipulation, we can cancel  $P_s$  from both sides of (6.2.4) and  $P_u$  from both sides of (6.2.5). This leads to

$$\begin{aligned} P_s h &= P_s L S(G) L^{-1} + P_s L h (I + S(G)) L^{-1} \\ &\quad + P_s L F(I + h) (I + S(G)) L^{-1} \\ P_u h &= P_u G + P_u L^{-1} h L (I + G) - P_u F(I + h). \end{aligned}$$

Arnold calls this the homological equation. Adding these two equations together, we see that (6.2.1) is equivalent to the equation

$$\begin{aligned} h &= P_s L S(G) L^{-1} + P_s L h (I + S(G)) L^{-1} + P_s L F(I + h) (I + S(G)) L^{-1} \\ &\quad + P_u G + P_u L^{-1} h L (I + G) - P_u F(I + h) \\ &\equiv T(F, G, h). \end{aligned}$$

So solving (6.2.1) can be accomplished by finding a fixed point of  $T$ .

$T(F, G, h)$  is defined for all  $F, G \in Y$  ( $\subset X$ ) with  $Y$ -norm smaller than  $\mu$  and all  $h \in X$ . Since the terms in the definition of  $T$  all have the form  $\beta \circ \gamma$  with  $\beta \in X$ ,  $\gamma$  continuous, and  $\beta(0) = \gamma(0) = 0$ , we see that  $T(F, G, h) \in X$ . Let  $Y_\mu = \{F \in Y : \|F\|_Y \leq \mu\}$ . Then

$$T : Y_\mu \times Y_\mu \times X \rightarrow X.$$

Next we are going to show that  $T$  is a uniform contraction in the  $X$  variable. Let  $F, G \in Y_\mu$  and let  $h_1, h_2 \in X$ . We make use of the fact that

$$\|Fh_1 - Fh_2\|_X \leq \|F\|_Y \|h_1 - h_2\|_X \leq \mu \|h_1 - h_2\|_X,$$

as well as  $\|P_s L\|, \|P_u L^{-1}\| \leq \varepsilon$ . Thus, we have

$$\begin{aligned}
& \|T(F, G, h_1) - T(F, G, h_2)\|_X \\
& \leq \|P_s L(h_1 - h_2)(I + S(G))L^{-1}\|_X \\
& \quad + \|P_s L F(I + h_1)(I + S(G))L^{-1} - P_s L F(I + h_2)(I + S(G))L^{-1}\|_X \\
& \quad + \|P_u L^{-1}(h_1 - h_2)L(I + G)\|_X + \|P_u F(I + h_1) - P_u F(I + h_2)\|_X \\
& \leq \|P_s L\| \|(h_1 - h_2)(I + S(G))L^{-1}\|_X \\
& \quad + \|P_s L\| \|F\|_Y \|(h_1 - h_2)S(G)L^{-1}\|_X \\
& \quad + \|P_u L^{-1}\| \|(h_1 - h_2)L(I + G)\|_X \\
& \quad + \|F\|_Y \|h_1 - h_2\|_X \\
& \leq (\varepsilon + \varepsilon\mu + \varepsilon + \mu) \|h_1 - h_2\|_X \\
& = \alpha \|h_1 - h_2\|_X,
\end{aligned}$$

with  $\alpha < 1$ . So by the uniform contraction principle, there exists a unique  $h = h_{FG} \in X$  such that  $h_{FG} = T(F, G, h_{FG})$ , and the map  $(F, G) \mapsto h_{FG}$  is continuous from  $Y_\mu \times Y_\mu$  to  $X$ .

Set  $\Lambda_{FG} = I + h_{FG}$ . We have that  $\Lambda_{FG}$  is continuous and

$$L(I + F)\Lambda_{FG} = \Lambda_{FG}L(I + G).$$

We still need to show that  $\Lambda_{FG}$  has a continuous inverse. If  $\Lambda_{GF} = I + h_{GF}$ , then we also have that

$$L(I + G)\Lambda_{GF} = \Lambda_{GF}L(I + F).$$

Therefore,

$$\Lambda_{FG}\Lambda_{GF}L(I + F) = \Lambda_{FG}L(I + G)\Lambda_{GF} = L(I + F)\Lambda_{FG}\Lambda_{GF}.$$

Thus by uniqueness,  $\Lambda_{FG}\Lambda_{GF} = \Lambda_{FF} = I + h_{FF}$ . However,  $T(F, F, 0) = 0$ , so again by uniqueness  $h_{FF} = 0$ . This shows that  $\Lambda_{FG}\Lambda_{GF} = I$ . By symmetry, we also have  $\Lambda_{GF}\Lambda_{FG} = I$ . Thus,  $\Lambda_{FG}$  has a continuous inverse.  $\square$

With lemma 6.2.2 in hand, we can finish the proof of the Hartman-Grobman theorem.

PROOF OF THEOREM 6.2.1. By variation of parameters, we have

$$(6.2.6) \quad \psi_t(p) = x(t, p) = \exp At p + \int_0^t \exp A(t-s)f(x(s, p))ds.$$

Choose  $T > 0$  as before so that  $L = \exp AT = \phi_T$  satisfies the bounds  $\|P_s L\|, \|P_u L^{-1}\| \leq \varepsilon < 1/2$ . With  $t = T$ , (6.2.6) can be written as

$$\psi_T(p) = L(p + F(p)),$$

with  $F(p) = \int_0^T \exp(-As)f(x(s, p))ds$ .

We claim that given any  $\mu > 0$ , there is a  $\delta > 0$  such that if  $f \in Y$  and  $\|f\|_Y < \delta$ , then  $F \in Y$  and  $\|F\|_Y < \mu$ .

This can be seen from the following estimates:

$$\|F(p)\| \leq \sup_{0 \leq s \leq T} \|f(x(s, p))\| \int_0^T \|\exp(-As)\| ds \leq C_1 \|f\|_X \leq C_1 \delta,$$

$$\begin{aligned} \|F(p) - F(q)\| &\leq \sup_{0 \leq s \leq T} \|f(x(s, p)) - f(x(s, q))\|_X \int_0^T \|\exp(-As)\| ds \\ &\leq C_1 \|f\|_Y \sup_{0 \leq s \leq T} \|x(s, p) - x(s, q)\|, \end{aligned}$$

and by Gronwall,

$$\sup_{0 \leq s \leq T} \|x(s, p) - x(s, q)\| \leq \|L\| \|p - q\| e^{C_1 \delta}.$$

We have shown that  $\|F\|_Y \leq C_2 \delta$ . Thus, we take  $\delta = \mu/C_2$ , with  $\mu$  as in the previous lemma.

Apply the lemma with  $G = 0$  to get a homeomorphism  $\Lambda$  such that

$$(6.2.7) \quad \Lambda \psi_T = \Lambda L(I + F) = L\Lambda = \phi_T \Lambda.$$

We need to verify that this holds for all  $t \in \mathbb{R}$ .

Define

$$\Lambda_1 = \frac{1}{T} \int_0^T \phi_{-s} \Lambda \psi_s ds.$$

Note that by linearity of  $\phi_t$  and (6.2.6) we have

$$\Lambda_1 - I = \frac{1}{T} \int_0^T \left[ \phi_{-s} (\Lambda - I) \psi_s + \int_0^s \phi_{-\sigma} f \psi_\sigma d\sigma \right] ds,$$

from which it follows that  $\Lambda_1 - I \in X$ . We make the following straightforward calculation which uses the formula (6.2.7).

$$\begin{aligned}
\phi_t \Lambda_1 \psi_{-t} &= \frac{1}{T} \int_0^T \phi_{t-s} \Lambda \psi_{s-t} ds \\
&= \frac{1}{T} \int_{-t}^{T-t} \phi_{-\sigma} \Lambda \psi_{\sigma} d\sigma \\
&= \frac{1}{T} \int_{-t}^0 \phi_{-\sigma} \Lambda \psi_{\sigma} d\sigma + \frac{1}{T} \int_0^{T-t} \phi_{-\sigma} \Lambda \psi_{\sigma} d\sigma \\
&= \frac{1}{T} \int_{T-t}^T \phi_{-\tau+T} \Lambda \psi_{\tau-T} d\tau + \frac{1}{T} \int_0^{T-t} \phi_{-\sigma} \Lambda \psi_{\sigma} d\sigma \\
&= \frac{1}{T} \int_{T-t}^T \phi_{-\tau} \Lambda \psi_{\tau} d\tau + \frac{1}{T} \int_0^{T-t} \phi_{-\sigma} \Lambda \psi_{\sigma} d\sigma \\
&= \Lambda_1
\end{aligned}$$

Thus,  $\phi_t \Lambda_1 = \psi_t \Lambda_1$  for all  $t \in \mathbb{R}$ , including  $t = T$ . By uniqueness,  $\Lambda_1 = \Lambda$ .  $\square$

**THEOREM 6.2.2.** *Let  $A$  and  $B$  be hyperbolic  $n \times n$  real matrices. There exists a homeomorphism  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $f \circ \exp At = \exp Bt \circ f$  for all  $t \in \mathbb{R}$  if and only if the stable subspaces of  $A$  and  $B$  have the same dimension.*

**PROOF.** Let  $E_s, E_u$  be the stable and unstable subspaces of  $A$ , and let  $P_s, P_u$  be the respective projections. Define the linear transformation  $L = -P_s + P_u$ .

The first part of the proof will be to find a homeomorphism  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$h \circ \exp At = \exp Lt \circ h = e^{-t} P_s \circ h + e^t P_u \circ h.$$

This will be done by constructing homeomorphisms

$$h_s : E_s \rightarrow E_s \quad \text{and} \quad h_u : E_u \rightarrow E_u$$

such that

$$h_s \circ P_s \circ \exp At = e^{-t} h_s \circ P_s \quad \text{and} \quad h_u \circ P_u \circ \exp At = e^t h_u \circ P_u,$$

for then, if we set  $h = h_s \circ P_s + h_u \circ P_u$ ,  $h$  is a homeomorphism and

$$\begin{aligned}
h \circ \exp At &= h_s \circ P_s \circ \exp At + h_u \circ P_u \circ \exp At \\
&= e^{-t} h_s \circ P_s + e^t h_u \circ P_u \\
&= \exp Lt \circ h.
\end{aligned}$$

Construction of  $h_s$ . Let  $\{e_i\}_{i=1}^p$  be a basis of generalized eigenvectors in  $E_s$ . Then

$$Ae_i = \lambda_i e_i + \delta_i e_{i-1}, \quad i = 1, \dots, p$$

where  $\operatorname{Re} \lambda_i < 0$  and  $\delta_i = 0$  or  $1$ . Given  $\mu > 0$ , set  $f_i = \mu^{-i} e_i$ . Then

$$Af_i = \lambda_i f_i + \frac{\delta_i}{\mu} f_{i-1}.$$

If  $T$  is the  $n \times p$  matrix whose columns are the  $f_i$ , then

$$AT = T(D + N),$$

with  $D = \operatorname{diag}(\lambda_1, \dots, \lambda_p)$  and  $N$  nilpotent. The  $p \times p$  nilpotent matrix  $N$  has zero entries except possibly  $1/\mu$  may occur above the main diagonal. Let  $\{g_i\}_{i=1}^p$  be the dual basis, i.e.  $\langle f_i, g_j \rangle = \delta_{ij}$ . Let  $S$  be the  $p \times n$  matrix whose rows are the  $g_i$ . Then  $ST = I$  ( $p \times p$ ) and

$$SAT = (D + N).$$

Thus,  $\operatorname{Re} SAT$  is negative definite for  $\mu$  large enough.

For  $x \in E_s$ , define

$$\phi(x) = \frac{1}{2} \|Sx\|^2$$

and

$$\Sigma = \{x \in E_s : \phi(x) = 1\}.$$

Then  $D_x \phi(x)y = \operatorname{Re} \langle Sx, Sy \rangle$ . So since  $TS = P_s$ , we have that

$$\frac{d}{dt} \phi(\exp Atx) = \operatorname{Re} \langle S \exp Atx, SA \exp Atx \rangle = \operatorname{Re} \langle y, SATy \rangle,$$

in which  $y = S \exp Atx$ . Thus, we can find constants  $0 < c_1 < c_2$  such that

$$-c_2 \phi(\exp Atx) \leq \frac{d}{dt} \phi(\exp Atx) \leq -c_1 \phi(\exp Atx).$$

We see that  $\phi(\exp Atx)$  is strictly decreasing as a function of  $t$ , and

$$(6.2.8) \quad \begin{aligned} \frac{1}{2} \|Sx\|^2 e^{-c_2 t} = \phi(x) e^{-c_2 t} \\ \leq \phi(\exp Atx) \leq \phi(x) e^{-c_1 t} = \frac{1}{2} \|Sx\|^2 e^{-c_1 t}. \end{aligned}$$

Given  $0 \neq x \in \mathbb{R}^n$ , it follows that there exists a unique  $t(x)$  such that  $\phi(\exp At(x)x) = 1$ . The function  $t(x)$  is continuous for  $x \neq 0$ . (Actually, it is  $C^1$  by the implicit function theorem.) Now define

$$h_s(x) = \begin{cases} e^{t(x)} \exp At(x)x, & 0 \neq x \in E_s \\ 0, & x = 0. \end{cases}$$

Since  $\|\exp At(x)x\| = 1$ , it follows from (6.2.8) that  $\lim_{x \rightarrow 0} \|h_s(x)\| = 0$ . Thus,  $h_s$  is continuous on all of  $\mathbb{R}^n$ .

Given  $0 \neq y \in E_s$ , there is a unique time  $t_0 \in \mathbb{R}$  such that  $e^{-t_0}y \in \Sigma$ , namely  $\frac{1}{2}e^{-2t_0}\|Sy\|^2 = 1$ . Setting  $x = e^{-t_0} \exp -At_0y$ , we have  $t(x) = t_0$  and  $h_s(x) = y$ . We have shown that  $h_s : E_s \rightarrow E_s$  is one-to-one and onto. Continuity of  $h_s^{-1}$  follows from its explicit formula. Therefore,  $h_s$  is a homeomorphism.

For any  $0 \neq x \in \mathbb{R}^n$ , set  $x_s = P_s x$  and  $y = \exp Atx_s$ . Note that  $t(y) = t(x_s) - t$ . From the definition of  $h_s$ , we have

$$\begin{aligned} h_s \circ P_s \circ \exp At(x) &= h_s(\exp Atx_s) \\ &= e^{t(y)} \exp At(y)y \\ &= e^{t(x_s)-t} \exp A[t(x) - t] \exp Atx_s \\ &= e^{-t} h_s \circ P_s(x), \end{aligned}$$

so  $h_s$  has all of the desired properties.

In a similar fashion, we can construct  $h_u$ , and as explained above we get a homeomorphism  $h \circ \exp At = \exp Lt \circ h$ .

Let  $\tilde{E}_s, \tilde{E}_u$  be the stable and unstable subspaces for  $B$ , with their projections  $\tilde{P}_s, \tilde{P}_u$ . Set  $\tilde{L} = -\tilde{P}_s + \tilde{P}_u$ . Let  $g$  be a homeomorphism such that  $g \circ \exp Bt = \exp \tilde{L}t \circ g$ .

Since  $E_s$  and  $\tilde{E}_s$  have the same dimension, there is an isomorphism  $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $ME_s = \tilde{E}_s$  and  $ME_u = \tilde{E}_u$ . Define  $f = g^{-1} \circ M \circ h$ . Then

$$\begin{aligned} f \circ \exp At &= g^{-1} \circ M \circ h \circ \exp At \\ &= g^{-1} \circ M \circ \exp Lt \circ h \\ &= g^{-1} \circ (M \circ \exp Lt \circ M^{-1}) \circ M \circ h \\ &= g^{-1} \circ \exp \tilde{L}t \circ M \circ h \\ &= \exp Bt \circ g^{-1} \circ M \circ h \\ &= \exp Bt \circ f. \end{aligned}$$

Conversely, suppose that  $f$  is a homeomorphism which conjugates the flows,  $f \circ \exp At = \exp Bt \circ f$ . If  $x_s \in E_s$  then  $\lim_{t \rightarrow \infty} \|\exp Bt f(x_s)\| = \lim_{t \rightarrow \infty} \|f(\exp Atx)\| = 0$ . Thus,  $f : E_s \rightarrow \tilde{E}_s$ . By symmetry,  $f^{-1} : \tilde{E}_s \rightarrow E_s$ . Thus,  $E_s$  and  $\tilde{E}_s$  are homeomorphic. By the Invariance of Domain Theorem,  $E_s$  and  $\tilde{E}_s$  have the same dimension.  $\square$

This result is somewhat surprising. For example, it says that the spiral sink and the star-shaped node are topologically conjugate.

**THEOREM 6.2.3.** *Let  $A$  and  $B$  be hyperbolic  $n \times n$  matrices whose stable subspaces have the same dimension. Let  $f, g \in Y$  with  $\|f\|_Y, \|g\|_Y \leq \mu$ . If  $\mu$  is sufficiently small, then the two flows generated by the vector fields  $Ax + f(x)$ ,  $Bx + g(x)$  are topologically conjugate.*

**THEOREM 6.2.4.** *Let  $A$  be hyperbolic, and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be  $C^1$  with  $f(0) = 0$  and  $Df(0) = 0$ . Let  $\phi_t(p)$  and  $\psi_t(p)$  be the flows generated by  $Ax$  and  $Ax + f(x)$ , respectively. There is a homeomorphism  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a neighborhood  $0 \in U \subset \mathbb{R}^n$  such that*

$$h \circ \psi_t(p) = \phi_t \circ h(p),$$

when  $\psi_t(p) \in U$ .

**PROOF.** The idea is simple. Modify the nonlinear function by smoothly cutting it off away from the origin, so that the modification  $\tilde{f}$  lies in  $Y$  with small norm. If the cut-off is done outside a sufficiently small neighborhood, then  $\|\tilde{f}\|_Y$  will be small. The Hartman-Grobman theorem says that the flow  $\tilde{\psi}_t$  of  $Ax + \tilde{f}(x)$  and  $\phi_t$  are topologically conjugate. Since  $f = \tilde{f}$  near 0,  $\psi_t = \tilde{\psi}_t$  in some neighborhood of the origin, by uniqueness. □

Remark: Note that if  $q \in E_s$ , the stable subspace of  $A$ , then

$$\lim_{t \rightarrow \infty} \phi_t(q) = 0.$$

Thus, if  $\|q\| \leq r$  for  $r$  small enough, then  $\phi_t(q) \in h^{-1}(U) \cap E_s$  for all  $t > 0$ . By 6.2.4, if  $q = h(p)$ , then  $\psi_t(p) \in U \cap h^{-1}(E_s)$  for all  $t > 0$ .

Similarly, if  $h(p) \in E_u$  with  $\|h(p)\| < r$ , then  $\psi_t(p)$  is defined for all  $t < 0$  and remains in  $U \cap h^{-1}E_u$ .

The sets  $U \cap h^{-1}(E_s)$  and  $U \cap h^{-1}(E_u)$  are called the local stable and unstable manifolds, respectively.

Example. Consider the nonlinear system

$$\begin{aligned} x_1' &= x_1, & x_1(0) &= p_1 \\ x_2' &= \alpha x_2 + x_1^2, & x_2(0) &= p_2 \end{aligned}$$

where  $\alpha > 0$ . The first equation is linear and uncoupled with the second, and so  $x_1$  can be found explicitly. Substituting the result in the second equation then yields a linear equation for  $x_2$  which can also be solved.

When  $\alpha \neq 2$ , the nonlinear flow is

$$\psi_t(p) = \left( e^t p_1, e^{\alpha t} \left( p_2 - \frac{p_1^2}{2 - \alpha} \right) + e^{2t} \frac{p_1^2}{2 - \alpha} \right).$$

Of course, the associated linear flow is

$$\phi_t(p) = (e^t p_1, e^{\alpha t} p_2).$$

Now define

$$\Lambda(p) = \left( p_1, p_2 + \frac{p_1^2}{2 - \alpha} \right).$$

Then  $\Lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a homeomorphism and

$$\psi_t(\Lambda(p)) = \Lambda(\phi_t(p)),$$

confirming Theorem 6.2.4. Notice that  $\Lambda \in C^\infty$ .

When  $\alpha = 2$ , substitution of  $x_1$  into the second equation leads to a resonant term. The nonlinear flow is now

$$\psi_t(p) = (e^t p_1, e^{2t}(p_2 + t p_1^2)).$$

According to Theorem 6.2.4, there exists a homeomorphism  $\Lambda$  on some neighborhood of the origin such that

$$\psi_t(\Lambda(p)) = \Lambda(\phi_t(p)).$$

The second component of this equation reads

$$e^{2t}[\Lambda_2(p_1, p_2) + t\Lambda_1^2(p_1, p_2)] = \Lambda_2(e^t p_1, e^{2t} p_2).$$

Setting  $\beta = e^t$ , we have that

$$\beta^2[\Lambda_2(p_1, p_2) + \ln \beta \Lambda_1^2(p_1, p_2)] = \Lambda_2(\beta p_1, \beta^2 p_2),$$

for all sufficiently small  $\beta > 0$ . If  $\Lambda_2 \in C^2$ , then we could differentiate this equation twice with respect to  $\beta$ , with the result that

$$\begin{aligned} & 2\Lambda_2(p_1, p_2) + 3\Lambda_1(p_1, p_2) + 2 \ln \beta \Lambda_1(p_1, p_2) \\ &= D_1^2 \Lambda_2(\beta p_1, \beta^2 p_2) + 2\beta D_1 D_2 \Lambda_2(\beta p_1, \beta^2 p_2) \\ & \quad + 4D_2^2 \Lambda_2(\beta p_1, \beta^2 p_2) + 2D_1^2 \Lambda_2(\beta p_1, \beta^2 p_2). \end{aligned}$$

For  $\Lambda \in C^2$ , the right-hand side has a limit as  $\beta \rightarrow 0^+$ , whereas the left-hand side does not. This contradiction shows that any homeomorphism which conjugates this flow cannot be  $C^2$  at the origin.

### 6.3. Invariant Manifolds

Last changed 02.22.10

**DEFINITION 6.3.1.** *Let  $F(x)$  be a  $C^1$  autonomous vector field defined on some open set  $\mathcal{O} \subset \mathbb{R}^n$ . A set  $A \subset \mathcal{O}$  is called an invariant set if  $A$  contains the orbit through each of its points; i.e.  $x(t, x_0) \in A$  for all  $\alpha(x_0) < t < \beta(x_0)$ . A set  $A$  is positively (negatively) invariant if  $x(t, x_0) \in A$  for all  $0 < t < \beta(x_0)$  ( $\alpha(x_0) < t < 0$ ).*

Assume that the vector field  $F : \mathcal{O} \rightarrow \mathbb{R}^n$  has a hyperbolic equilibrium at  $x_0 = 0$ . Set  $A = DF(0)$ , and write  $F(x) = Ax + f(x)$ . Then  $f(0) = 0$  and  $D_x f(0) = 0$ . Let  $x(t, x_0)$  denote the (possibly local) solution of the initial value problem

$$x' = Ax + f(x), \quad x(0) = x_0.$$

Let  $E_s$  and  $E_u$  be the stable and unstable subspaces of the hyperbolic matrix  $A$  with their projections  $P_s$  and  $P_u$ . Recall that by the remark following Theorem 1.4.1, there exists a  $\lambda > 0$  such that

$$(6.3.1) \quad \|e^{At} P_s x\| \leq C_0 e^{-\lambda t} \|x\|, \quad t \geq 0,$$

and

$$(6.3.2) \quad \|e^{At} P_u x\| \leq C_0 e^{\lambda t} \|x\|, \quad t \leq 0,$$

for all  $x \in \mathbb{R}^n$ .

The stable and unstable subspaces are invariant under the linear flow. We shall be interested in what happens to these sets under nonlinear perturbations. In fact, we already see that by the Hartman-Grobman Theorem, that there is a neighborhood of the origin in  $E_s$  which is homeomorphic to positively invariant set under the nonlinear flow. Similarly,  $E_u$  corresponds to a negatively invariant set for the nonlinear flow. The next result shows that these invariant sets are  $C^1$  manifolds which are tangent to the corresponding subspaces at the origin.

**DEFINITION 6.3.2.** *The stable manifold is the set*

$$W_s(0) = \{x_0 \in \mathbb{R}^n : x(t, x_0) \text{ exists for all } t \geq 0 \text{ and } \lim_{t \rightarrow \infty} x(t, x_0) = 0\}.$$

Remarks:

- The stable manifold is invariant under the flow.
- If  $x_0 \in W_s(0)$ , then  $\sup_{t \geq 0} \|x(t, x_0)\| < \infty$ .

**THEOREM 6.3.1 (Stable Manifold Theorem).** *There exists a function  $\eta$  defined and  $C^1$  on a neighborhood of the origin  $U \subset E_s$  with the following properties:*

- (i)  $\eta : U \rightarrow E_u$ ,  $\eta(0) = 0$ , and  $D_x \eta(0) = 0$ .
- (ii) If  $x_0 \in W_s^{loc}(0) \equiv \{x_0 \in \mathbb{R}^n : P_s x_0 \in U, P_u x_0 = \eta(P_s x_0)\}$ , then  $x(t, x_0)$  is defined for all  $t \geq 0$ , and

$$\|x(t, x_0)\| \leq C \|P_s x_0\| \exp(-\lambda t/2).$$

Hence,  $W_s^{loc}(0) \subset W_s(0)$ .

- (iii)  $W_s^{loc}(0)$  is positively invariant.

(iv) If  $x_0 \in W_s^{loc}(0)$ , then  $x(t, x_0)$  is defined for all  $t \geq 0$ , and  $y(t) = P_s x(t, x_0)$  solves the initial value problem

$$y'(t) = Ay(t) + P_s f(y(t) + \eta(y(t))), \quad y(0) = P_s x_0.$$

Remarks:

- The set  $W_s^{loc}(0)$  is called the *local stable manifold*. It is a  $C^1$  submanifold of  $\mathbb{R}^n$  of the same dimension as  $E_s$ , and tangent to  $E_s$  at the origin.
- If  $f \in C^k(\mathbb{R}^n)$  then  $\eta \in C^k(U)$ .
- There is also a notion of local unstable manifold with the analogous properties. The local unstable manifold can be constructed simply by reversing time (i.e.  $t \rightarrow -t$ ) in the equation and then applying the above result.
- The initial value problem (iv) governs the flow on the local stable manifold.

PROOF. Suppose that  $x_0 \in \mathbb{R}^n$  is such that  $x(t, x_0)$  is defined for all  $t \geq 0$  and  $\sup_{t \geq 0} \|x(t, x_0)\| < \infty$ . Using variation of parameters, we can write, for all  $t \geq 0$ ,

$$(6.3.3) \quad x(t, x_0) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} f(x(\tau, x_0)) d\tau.$$

Applying  $P_u$  to both sides and factoring out  $e^{At}$  on the right, this becomes

$$(6.3.4) \quad P_u x(t, x_0) = e^{At} P_u \left[ x_0 + \int_0^t e^{-A\tau} f(x(\tau, x_0)) d\tau \right].$$

Because of (6.3.1), we have for any  $z \in \mathbb{R}^n$ ,  $t \geq 0$

$$\|P_u z\| = \|e^{-At} e^{At} P_u z\| \leq C_0 e^{-\lambda t} \|e^{At} P_u z\|.$$

Rearranging this inequality, we have

$$\|e^{At} P_u z\| \geq C e^{\lambda t} \|P_u z\|.$$

If we combine this lower bound with (6.3.4), we obtain

$$\|P_u x(t, x_0)\| \geq C e^{\lambda t} \left\| P_u \left[ x_0 + \int_0^t e^{-A\tau} f(x(\tau, x_0)) d\tau \right] \right\|.$$

Now  $\|P_u x(t, x_0)\| \leq C \|x(t, x_0)\| \leq Const.$ , for all  $t \geq 0$ , by assumption. Therefore, sending  $t \rightarrow \infty$  implies that

$$P_u \left[ x_0 + \int_0^\infty e^{-A\tau} f(x(\tau, x_0)) d\tau \right] = 0.$$

As a consequence of this calculation, we can rewrite (6.3.3), for bounded solution  $x(t, x_0)$ , as

$$(6.3.5) \quad x(t, x_0) = e^{At} P_s x_0 + \int_0^t e^{A(t-\tau)} P_s f(x(\tau, x_0)) d\tau \\ - \int_t^\infty e^{A(t-\tau)} P_u f(x(\tau, x_0)) d\tau.$$

This is a big hint for how to find the local stable manifold.

Let  $C_b^0(\mathbb{R}^+, \mathbb{R}^n)$  denote the Banach space of bounded continuous functions from  $\mathbb{R}^+$  to  $\mathbb{R}^n$  with the sup norm

$$\|y\|_\infty = \sup_{t \geq 0} \|y(t)\|.$$

Given  $y_0 \in E_s$ ,  $y \in C_b^0$ , define the mapping

$$(6.3.6) \quad T(y_0, y)(t) = y(t) - e^{At} y_0 - \int_0^t e^{A(t-\tau)} P_s f(y(\tau)) d\tau \\ + \int_t^\infty e^{A(t-\tau)} P_u f(y(\tau)) d\tau.$$

Let's verify that

$$T : E_s \times C_b^0(\mathbb{R}^+, \mathbb{R}^n) \rightarrow C_b^0(\mathbb{R}^+, \mathbb{R}^n).$$

Since  $f$  is  $C^1$  and  $f(0) = 0$ , we have if  $\|x\| \leq r$

$$\|f(x)\| = \left\| \int_0^1 Df(\sigma x) x d\sigma \right\| \leq \max_{\|p\| \leq r} \|Df(p)\| \|x\| = C_r \|x\|.$$

Note that  $C_r \rightarrow 0$  as  $r \rightarrow 0$ , because  $Df(0) = 0$ . Given  $y \in C_b^0$ , let  $\rho = \|y\|_\infty$ . Then

$$\|f(y(\tau))\| \leq C_\rho \|y(\tau)\|.$$

Using the bounds (6.3.1), we get for  $\tau \leq t$

$$(6.3.7) \quad \|e^{A(t-\tau)} P_s f(y(\tau))\| \leq C_0 e^{-\lambda(t-\tau)} \|f(y(\tau))\| \\ \leq C_0 e^{-\lambda(t-\tau)} C_\rho \|y(\tau)\| \leq C(\rho) e^{-\lambda(t-\tau)},$$

and for  $\tau \geq t$

$$(6.3.8) \quad \|e^{A(t-\tau)} P_u f(y(\tau))\| \leq C_0 e^{\lambda(t-\tau)} \|f(y(\tau))\| \\ \leq C_0 e^{\lambda(t-\tau)} C_\rho \|y(\tau)\| \leq C(\rho) e^{\lambda(t-\tau)}.$$

Recalling that  $y_0 \in E_s$ , we can estimate the mapping  $T$  in (6.3.6)

$$\begin{aligned} \|T(y_0, y)\|_\infty &\leq \sup_{t \geq 0} \left[ \|y(t)\| + \|e^{At}y_0\| \right. \\ &\quad \left. + C(\rho) \left( \int_0^t e^{-\lambda(t-\tau)} d\tau + \int_t^\infty e^{\lambda(t-\tau)} d\tau \right) \right] \\ &\leq \sup_{t \geq 0} \left[ \|y(t)\| + C_0 e^{-\lambda t} \|y_0\| + 2C(\rho) \int_0^\infty e^{-\lambda s} ds \right] \\ &\leq C(\|y_0\|, \|y\|_\infty) < \infty. \end{aligned}$$

Thus,  $T(y_0, y) \in C_b^0$ .

Now we proceed to check that  $T$  fulfills the hypotheses of the implicit function theorem 4.4.1 at  $(0, 0)$ . From the definition (6.3.6) we see that  $T(0, 0) = 0$ .  $T$  is continuously Fréchet differentiable and

$$\begin{aligned} D_y T(y_0, y)z(t) &= z(t) - \int_0^t e^{A(t-\tau)} P_s D_x f(y(\tau)) z(\tau) d\tau \\ &\quad + \int_t^\infty e^{A(t-\tau)} P_u D_x f(y(\tau)) z(\tau) d\tau, \end{aligned}$$

so that  $D_y T(0, 0) = I$ , since  $D_x f(0) = 0$ .

By the implicit function theorem, there are neighborhoods  $0 \in U_1 \subset E_s$ ,  $0 \in V_1 \subset C_b^0$ , and a  $C^1$  mapping  $\phi : U_1 \rightarrow V_1$  such that

$$\phi(0) = 0 \quad \text{and} \quad T(y_0, \phi(y_0)) = 0, \quad y_0 \in U_1.$$

Moreover, if  $y_0 \in U_1$ ,  $y \in V_1$  and  $T(y_0, y) = 0$ , then  $y = \phi(y_0)$ .

Choose  $r > 0$  small enough such that

$$\{y_0 \in E_s : \|y_0\| < r\} \subset U_1, \quad \{y \in C_b^0 : \|y\|_\infty < r\} \subset V_1, \quad 4C_0 C_r / \lambda < 1.$$

Define

$$V = \{y \in C_b^0 : \|y\|_\infty < r, \|P_s y\|_\infty < r\} \subset V_1$$

and

$$(6.3.9) \quad U = \{y_0 \in U_1 : \|\phi(y_0)\|_\infty < r, \|P_s \phi(y_0)\|_\infty < r\} = \phi^{-1}(V).$$

All of the statements in the preceding paragraph hold on the smaller pair of neighborhoods  $U$  and  $V$ . The reason for these choices will become clear below.

For  $y_0 \in U_1$ , set  $y(t, y_0) = \phi(y_0)(t)$ . Then from (6.3.6)

$$(6.3.10) \quad y(t, y_0) = e^{At}y_0 + \int_0^t e^{A(t-\tau)}P_s f(y(\tau, y_0))d\tau \\ - \int_t^\infty e^{A(t-\tau)}P_u f(y(\tau, y_0))d\tau.$$

Letting  $t = 0$ , we have

$$(6.3.11) \quad y(0, y_0) = P_s y(0, y_0) + P_u y(0, y_0) \\ = y_0 - \int_0^\infty e^{-A\tau}P_u f(y(\tau, y_0))d\tau.$$

Define

$$(6.3.12) \quad \eta(y_0) = - \int_0^\infty e^{-A\tau}P_u f(y(\tau, y_0))d\tau.$$

From this definition, we see that  $\eta(0) = 0$  and  $\eta : U_1 \rightarrow E_u$ . From the fact that the map  $y_0 \mapsto \phi(y_0)$  is  $C^1$ , it follows that  $y(t, y_0)$  is  $C^1$  in  $y_0$ , and, in particular,  $\eta(y_0) = P_u y(0, y_0)$  is  $C^1$ . So we can compute

$$D\eta(y_0) = - \int_0^\infty e^{-A\tau}P_u D_x f(y(\tau, y_0))D_{y_0}y(\tau, y_0)d\tau.$$

Since  $y(\tau, 0) \equiv 0$ , we have  $D\eta(0) = 0$ . This verifies the assertions of (i).

Now we show that, for  $y_0 \in U_1$ ,  $y(t, y_0)$  is a solution of the differential equation. Since from (6.3.10)

$$(6.3.13) \quad y(t, y_0) = e^{At}y_0 + \int_0^t e^{A(t-\tau)}P_s f(y(\tau, y_0))d\tau \\ - e^{At} \int_0^\infty e^{-A\tau}P_u f(y(\tau, y_0))d\tau \\ + \int_0^t e^{A(t-\tau)}P_u f(y(\tau, y_0))d\tau \\ = e^{At}[y_0 + \eta(y_0)] + \int_0^t e^{A(t-\tau)}f(y(\tau, y_0))d\tau,$$

it follows by uniqueness of solution to the initial value problem that if  $y_0 \in U_1$ , then  $y(t, y_0) = x(t, y_0 + \eta(y_0))$ .

Define

$$W_s^{loc}(0) = \{x_0 \in \mathbb{R}^n : P_s x_0 \in U, P_u x_0 = \eta(P_s x_0)\},$$

and the following sets

$$\begin{aligned}\mathcal{B} &= \{x_0 \in \mathbb{R}^n : P_s x_0 \in U, x(t, x_0) = y(t, P_s x_0)\} \\ \mathcal{C} &= \{x_0 \in \mathbb{R}^n : P_s x_0 \in U, \sup_{t \geq 0} \|x(t, x_0)\| < r\}.\end{aligned}$$

We will now show that these three sets are equal.

The calculation (6.3.13) shows that  $W_s^{loc}(0) \subset \mathcal{B}$ . To see the reverse inclusion, suppose that  $P_s x_0 \in U$  and  $x(t, x_0) = y(t, P_s x_0)$ . Then by (6.3.11), (6.3.12), we have  $x_0 = x(0, x_0) = y(0, P_s x_0) = P_s x_0 + \eta(P_s x_0)$ . Therefore, we have that  $\mathcal{B} \subset W_s^{loc}(0)$ , and hence  $W_s^{loc}(0) = \mathcal{B}$ .

Now let  $x_0 \in \mathcal{B}$ . Then  $y_0 = P_s x_0 \in U$ , so since  $y(t, y_0) = \phi(y_0)(t)$  and  $\phi(y_0) \in V$ , we have that  $\sup_{t \geq 0} \|x(t, x_0)\| < r$ . Thus,  $\mathcal{B} \subset \mathcal{C}$ .

On the other hand, if  $P_s x_0 \in U$  and  $\sup_{t \geq 0} \|x(t, x_0)\| < r$ , then by (6.3.5), bounded solutions satisfy  $T(P_s x_0, x(t, x_0)) = 0$ . Since  $(P_s x_0, x(\cdot, x_0)) \in U_1 \times V_1$ , we have by uniqueness in the implicit function theorem that  $x(t, x_0) = y(t, P_s x_0)$ . Thus,  $x_0 \in \mathcal{B}$ , and we have that  $\mathcal{C} \subset \mathcal{B}$ .

This proves that  $W_s^{loc}(0) = \mathcal{B} = \mathcal{C}$ .

We now establish the inequality of (ii) and show that  $W_s^{loc}(0) \subset W_s(0)$ . If  $x_0 \in W_s^{loc}(0)$ , set  $y_0 = P_s x_0$ . Then  $y(t, y_0) = x(t, x_0)$  is defined for all  $t \geq 0$ , and  $\sup_{t \geq 0} \|x(t, x_0)\| < r$ . From (6.3.10), (6.3.8), and (6.3.7), we have the estimate

$$(6.3.14) \quad \begin{aligned}\|y(t, y_0)\| &\leq C_0 e^{-\lambda t} \|y_0\| + C_0 C_r \int_0^t e^{-\lambda(t-\tau)} \|y(\tau, y_0)\| d\tau \\ &\quad + C_0 C_r \int_t^\infty e^{\lambda(t-\tau)} \|y(\tau, y_0)\| d\tau.\end{aligned}$$

Recall that we have chosen  $r$  so small that  $4C_0 C_r / \lambda < 1$ . With  $A = C_0 \|y_0\|$  and  $B = C_0 C_r$ , we get from lemma 6.3.1 below

$$\|x(t, x_0)\| = \|y(t, y_0)\| \leq 2C_0 \|y_0\| \exp(-\lambda t/2).$$

This demonstrates the exponential decay, and hence we also have that  $W_s^{loc}(0) \subset W_s(0)$ .

The next step is to establish the invariance of the local stable manifold. Let  $x_0 \in W_s^{loc}(0)$ . Then  $x(t, x_0)$  is defined for all  $t \geq 0$ ,  $x(t, x_0) = y(t, P_s x_0)$ , and since  $P_s x_0 \in U$ , we have by (6.3.9)

$$(6.3.15) \quad \|x(t, x_0)\| = \|y(t, P_s x_0)\| < r, \quad \text{for all } t \geq 0,$$

and

$$(6.3.16) \quad \|P_s x(t, x_0)\| = \|P_s y(t, P_s x_0)\| < r, \quad \text{for all } t \geq 0.$$

Fix  $s > 0$  and set  $q = x(s, x_0)$ . We must show that  $q \in W_s^{loc}(0)$ . Now  $\|P_s q\| = \|P_s x(s, x_0)\| < r$ , so  $P_s q \in U_1$ . It follows from (6.3.13)

that  $y(t, P_s q) = x(t, q)$ . But by the properties of autonomous flow,  $x(t, q) = x(t + s, x_0)$ , and so by (6.3.15), (6.3.16), we have

$$\|y(t, P_s q)\| = \|x(t + s, x_0)\| < r, \quad \text{for all } t \geq 0,$$

and

$$\|P_s y(t, P_s q)\| = \|P_s x(t + s, x_0)\| < r, \quad \text{for all } t \geq 0.$$

Thus,  $P_s q \in U$ . But now since  $\|x(t, q)\| < r$ , we have that  $q \in \mathcal{C} = W_s^{loc}(0)$ .

Finally, if  $x_0 \in W_s^{loc}(0)$ , then  $x(t, x_0)$  is defined for all  $t \geq 0$ . If  $y(t) = P_s x_s(t, x_0)$ , then  $y(t) \in U$  and  $x(t, x_0) = y(t) + \eta(y(t))$ . So (iv) follows by applying  $P_s$  to both sides of the differential equation.  $\square$

We record the following fact which emerged during the proof of theorem 6.3.1.

**COROLLARY 6.3.1** (Uniqueness of the Stable Manifold). *The local stable manifold is unique. It has the intrinsic characterization*

$$W_s^{loc}(0) = \{x_0 \in \mathbb{R}^n : P_s x_0 \in U, \sup_{t \geq 0} \|x(t, x_0)\| < r\},$$

for a certain  $r > 0$  and neighborhood  $0 \in U \subset E_s$ .

The next lemma was used in the proof of exponential decay. It is a variant of Gronwall's inequality.

**LEMMA 6.3.1.** *Let  $A, B, \lambda$  be positive constants. Suppose that  $u(t)$  is a nonnegative, bounded continuous function such that*

$$u(t) \leq A e^{-\lambda t} + B \int_0^t e^{-\lambda(t-\tau)} u(\tau) d\tau + B \int_t^\infty e^{\lambda(t-\tau)} u(\tau) d\tau,$$

for all  $t \geq 0$ . If  $4B < \lambda$ , then  $u(t) \leq 2A \exp(-\lambda t/2)$ .

**PROOF.** Define  $\rho(t) = \sup_{\tau \geq t} u(\tau)$ .  $\rho(t)$  is finite since  $u(t)$  is bounded.  $\rho(t)$  is nonincreasing, nonnegative, and continuous (verify!). Fix  $t \geq 0$ . For any  $\varepsilon > 0$ , there is a  $t_1 \geq t$  such that  $\rho(t) < u(t_1) + \varepsilon$ . Therefore,

using this as well as the fact that  $u(t) \leq \rho(t)$ , we have

$$\begin{aligned}
\rho(t) - \varepsilon < u(t_1) &\leq Ae^{-\lambda t_1} + B \int_0^{t_1} e^{-\lambda(t_1-\tau)} u(\tau) d\tau \\
&\quad + B \int_{t_1}^{\infty} e^{\lambda(t_1-\tau)} u(\tau) d\tau \\
&\leq Ae^{-\lambda t_1} + B \int_0^{t_1} e^{-\lambda(t_1-\tau)} \rho(\tau) d\tau \\
&\quad + B \int_{t_1}^{\infty} e^{\lambda(t_1-\tau)} \rho(\tau) d\tau \\
&\leq Ae^{-\lambda t_1} + B \int_0^t e^{-\lambda(t_1-\tau)} \rho(\tau) d\tau \\
&\quad + B \int_t^{t_1} e^{-\lambda(t_1-\tau)} \rho(\tau) d\tau + B \int_{t_1}^{\infty} e^{\lambda(t_1-\tau)} \rho(\tau) d\tau \\
&\leq Ae^{-\lambda t} + B \int_0^t e^{-\lambda(t-\tau)} \rho(\tau) d\tau \\
&\quad + B\rho(t) \int_t^{t_1} e^{-\lambda(t_1-\tau)} d\tau + B\rho(t) \int_{t_1}^{\infty} e^{\lambda(t_1-\tau)} d\tau \\
&\leq Ae^{-\lambda t} + Be^{-\lambda t} \int_0^t e^{\lambda\tau} \rho(\tau) d\tau + (2B/\lambda)\rho(t).
\end{aligned}$$

This holds for every  $\varepsilon > 0$ . So if we send  $\varepsilon \rightarrow 0$  and set  $z(t) = e^{\lambda t} \rho(t)$ , we get

$$(1 - 2B/\lambda)z(t) \leq A + B \int_0^t z(\tau) d\tau.$$

Our assumption  $\lambda > 4B$  implies that  $(1 - 2B/\lambda)^{-1} < 2$  and  $B(1 - 2B/\lambda)^{-1} < \lambda/2$ . Thus,

$$z(t) \leq 2A + (\lambda/2) \int_0^t z(\tau) d\tau.$$

Ye Olde Gronwall implies that  $z(t) \leq 2A \exp(\lambda t/2)$ , so that  $u(t) \leq \rho(t) \leq 2A \exp(-\lambda t/2)$ .  $\square$

Now let's consider the problem of approximating the function  $\eta$  which defines the local stable manifold. Let  $x_0 \in W_s^{loc}(0)$ . Then  $x(t, x_0)$  is defined for all  $t \geq 0$ . Set  $x(t) = x(t, x_0)$  and  $y(t) = P_s x(t, x_0)$ . Then we have

$$P_u x(t) = \eta(y(t)).$$

If we differentiate:

$$P_u x'(t) = D\eta(y(t))y'(t),$$

use the equations:

$$P_u[Ax(t) + f(x(t))] = D\eta(y(t))[Ay(t) + f(y(t) + \eta(y(t))],$$

and set  $t = 0$ , then

$$\begin{aligned} A\eta(P_s x_0) + P_u f(P_s x_0 + \eta(P_s x_0)) = \\ D\eta(P_s x_0)[AP_s x_0 + P_s f(P_s x_0 + \eta(P_s x_0))], \end{aligned}$$

for all  $x_0 \in W_s^{loc}(0)$ . This is useful in approximating the function  $\eta$ .

**THEOREM 6.3.2 (Approximation).** *Let  $U \subset E_s$  be a neighborhood of the origin. Suppose that  $h : U \rightarrow E_u$  is a  $C^1$  mapping such that  $h(0) = 0$  and  $Dh(0) = 0$ . If*

$$Ah(x) + P_u f(x + h(x)) - Dh(x)[Ax + P_s f(x + h(x))] = O(\|x\|^k),$$

then

$$\eta(x) - h(x) = O(\|x\|^k),$$

for  $x \in U$  and  $\|x\| \rightarrow 0$ .

For a proof of this result, see the book of Carr. The way in which this is used is to plug in a finite Taylor expansion for  $h$  into the equation for  $\eta$  and grind out the coefficients. The theorem says that this procedure is correct.

**Example.** Consider the nonlinear system

$$x' = x + y^3, \quad y' = -y + x^2.$$

The origin is a hyperbolic equilibrium, and the stable and unstable subspaces for the linearized problem are

$$E_s = \{(0, y) \in \mathbb{R}^2\} \quad \text{and} \quad E_u = \{(x, 0) \in \mathbb{R}^2\}.$$

The local stable manifold has the description

$$W_s^{loc}(0) = \{(x, y) \in \mathbb{R}^2 : x = f(y)\},$$

for some function  $f$  such that  $f(0) = f'(0) = 0$ . Following the procedure described above, we find upon substitution,

$$f(y) + y^3 = f'(y)[-y + f(y)^2].$$

If we use the approximation  $f(y) \approx Ay^2 + By^3$ , we obtain

$$A = 0 \quad \text{and} \quad B = -1/4.$$

Thus, by Theorem 6.3.2, we have  $f(y) \approx (-1/4)y^3$ .

In the same fashion, we find

$$W_u^{loc}(0) = \{(x, y) \in \mathbb{R}^2 : y = g(x)\}, \quad \text{with} \quad g(x) \approx (1/3)x^2.$$

## CHAPTER 7

### Periodic Solutions

#### 7.1. Existence of Periodic Solutions in $\mathbb{R}^n$ – Noncritical Case

Suppose that we have an autonomous vector field with a critical point. By making a small  $T$ -periodic perturbation of this autonomous vector field, we will find a  $T$ -periodic solution of the perturbed equation near the critical point. The fundamental assumption will be that the linearized equation at the critical point has no  $T$ -periodic solutions. That is, the perturbation is noncritical.

Let  $f(t, x, \varepsilon)$  be a  $C^2$  map from  $\mathbb{R} \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0)$  into  $\mathbb{R}^n$  such that

- (i) There is a  $T > 0$ , such that  $f(t + T, x, \varepsilon) = f(t, x, \varepsilon)$  for all  $(t, x, \varepsilon) \in \mathbb{R} \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0)$ .
- (ii)  $f(t, x, 0) = f_0(x)$  is autonomous and  $f_0(0) = 0$ .

It follows that we can write

$$f(t, x, \varepsilon) = f_0(x) + \varepsilon \tilde{f}(t, x, \varepsilon), \quad \text{where} \quad \tilde{f}(t, x, \varepsilon) = \int_0^1 \frac{\partial f}{\partial \varepsilon}(t, x, \sigma \varepsilon) d\sigma.$$

Notice that  $\tilde{f}$  is  $C^1$  and  $\tilde{f}(t + T, x, \varepsilon) = \tilde{f}(t, x, \varepsilon)$ .

Example:  $f(t, x, \varepsilon) = f_0(x) + \varepsilon p(t)$ , with  $f_0(0) = 0$  and  $p(t + T) = p(t)$ .

**THEOREM 7.1.1.** *Set  $A = Df_0(0)$ . If  $T\lambda \notin 2\pi i\mathbb{Z}$  for each eigenvalue of  $A$ , then there exists a neighborhood of  $U$  of  $0 \in \mathbb{R}^n$  and an  $0 < \varepsilon_1 < \varepsilon_0$  such that for every  $|\varepsilon| < \varepsilon_1$  there is a unique initial point  $p = p(\varepsilon) \in U$  for which the solution,  $x(t, p, \varepsilon)$ , of the initial value problem*

$$x' = f(t, x, \varepsilon), \quad x(0, p, \varepsilon) = p$$

*is  $T$ -periodic.*

Remarks:

- The assumption means that the linear equation  $x' = Ax$  has no  $T$ -periodic solutions.

- Since the vector field is nonautonomous for  $\varepsilon \neq 0$ , the notion of orbit is not available. Thus, the uniqueness of the initial point of the periodic solution makes sense.

PROOF. Let  $\alpha(p, \varepsilon) < t < \beta(p, \varepsilon)$  be the existence interval for the solution  $x(t, p, \varepsilon)$ . Since  $f$  is  $C^1$ ,  $x$  is  $C^1$  in  $(t, p, \varepsilon)$ . Moreover, since  $x(t, 0, 0) \equiv 0$ , we have  $\alpha(0, 0) = -\infty$  and  $\beta(0, 0) = \infty$ . Therefore, by continuous dependence, we know that  $\beta(p, \varepsilon) > T$  for all  $(p, \varepsilon)$  sufficiently small, say in a neighborhood  $U \times (-\varepsilon_1, \varepsilon_1)$  of  $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$ .

Let  $Y(t) = D_p x(t, 0, 0)$ . Then since  $D_p x(t, p, \varepsilon)$  solves the linear variational equation

$$\frac{d}{dt} D_p x(t, p, \varepsilon) = D_x f(t, x(t, p, \varepsilon), \varepsilon) D_p x(t, p, \varepsilon), \quad D_p x(0, p, \varepsilon) = I,$$

we have that  $Y(t)$  solves

$$Y'(t) = D_x f(t, 0, 0) Y(t) = D_x f_0(0) Y(t) = A Y(t), \quad Y(0) = I.$$

Thus, we see that  $Y(t) = e^{At}$  is the fundamental matrix of  $A$ .

For  $(p, \varepsilon) \in U \times (-\varepsilon_1, \varepsilon_1)$ , the “time  $T$ ” map

$$\Pi(p, \varepsilon) = x(T, p, \varepsilon)$$

is well-defined. Note that  $\Pi(0, 0) = 0$ , and by smooth dependence,  $\Pi : U \times (-\varepsilon_1, \varepsilon_1) \rightarrow \mathbb{R}^n$  is  $C^1$ . By our definitions, we have

$$D_p \Pi(0, 0) = D_p x(T, p, \varepsilon) \Big|_{(p, \varepsilon) = (0, 0)} = Y(T) = e^{AT}.$$

Now set  $Q(p, \varepsilon) = \Pi(p, \varepsilon) - p$ . If we transfer the properties of  $\Pi$  to  $Q$ , we see that  $Q : U \times (-\varepsilon_1, \varepsilon_1) \rightarrow \mathbb{R}^n$  is  $C^1$ ,  $Q(0, 0) = 0$ , and  $D_p Q(0, 0) = e^{AT} - I$ . Our assumptions on  $A$  ensure that this matrix is invertible. By the implicit function theorem 4.4.1, there is a smooth curve of initial points  $p(\varepsilon)$  defined for  $|\varepsilon| < \varepsilon_1$  (with a smaller value for  $\varepsilon_1$ ) such that  $Q(p(\varepsilon), \varepsilon) = 0$ . This is the same as saying that  $\Pi(p(\varepsilon), \varepsilon) = p(\varepsilon)$  which, in turn, is equivalent to  $x(T, p(\varepsilon), \varepsilon) = p(\varepsilon)$ . However, this implies that  $x(t + T, p(\varepsilon), \varepsilon) = x(t, p(\varepsilon), \varepsilon)$  for all  $t \in \mathbb{R}$ , by the periodicity of the vector field  $f(t, x, \varepsilon)$  in  $t$  and uniqueness. (Notice that this is the only place in the argument that periodicity is used.)

□

Example: If the matrix  $A$  is hyperbolic, then there are no eigenvalues on the imaginary axis and the condition of theorem 7.1.1 is met.

Example: Consider Newton’s equation  $u'' + g(u) = \varepsilon p(t)$ , in which  $g(0) = 0$ ,  $g'(0) \neq 0$ , and  $p(t + T) = p(t)$  for all  $t \in \mathbb{R}$ .

If  $g'(0) = -\gamma^2$ , then the matrix  $A$  is hyperbolic. There exists a small  $T$ -periodic solutions for small enough  $\varepsilon$ .

If  $g'(0) = \gamma^2$ , then there is a  $T$ -periodic solution for small  $\varepsilon$  provided that  $\gamma T/2\pi$  is not an integer.

Consider the linear equation,

$$u'' + u = \varepsilon \cos(\omega t).$$

Here we have  $g'(0) = 1$  and  $T = 2\pi/\omega$ . According to theorem 7.1.1, for small  $\varepsilon$  there is a  $T$ -periodic solution if  $1/\omega$  is not an integer. This is consistent with (but weaker than) the familiar nonresonance condition which says that if  $\omega^2 \neq 1$ , then

$$u_\varepsilon(t) = \frac{\varepsilon}{1 - \omega^2} \cos(\omega t)$$

is the unique  $T$ -periodic solution for every  $\varepsilon$ .

Example: Consider the Duffing equation

$$u'' + \alpha u' + \gamma^2 u - \varepsilon u^3 = B \cos \omega t.$$

Notice that the periodic forcing term does not have small amplitude. The nonlinear term is “small”, however, which allows us to rewrite the equation in a form to which the theorem applies. Let  $v = \varepsilon^{1/2}u$ . If we multiply the equation by  $\delta = \varepsilon^{1/2}$ , then there results

$$v'' + \alpha v' + \gamma^2 v - v^3 = B\delta \cos \omega t.$$

When this equation is written as a first order system, it has the form

$$x' = f(x) + \delta p(t), \quad p(t + 2\pi/\omega) = p(t)$$

with  $Df(0)$  hyperbolic, so that theorem 7.1.1 ensures the existence of a  $2\pi/\omega$ -periodic solution.

## 7.2. Stability of Periodic Solutions to Nonautonomous Periodic Systems

Having established the existence of periodic solutions to certain periodic nonautonomous systems, we now examine their stability.

Let  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  nonautonomous vector field that is  $T$ -periodic:  $f(t + T, x) = f(t, x)$  for all  $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ . Assume that  $\varphi(t)$  is a  $T$ -periodic solution of  $x' = f(t, x)$ . Let  $x(t, \tau, x_0)$  denote the solution of the initial value problem

$$x' = f(t, x), \quad x(\tau) = x_0.$$

DEFINITION 7.2.1. A periodic orbit  $\varphi(t)$  is stable if for every  $\varepsilon > 0$  and every  $\tau \in \mathbb{R}$  there is a  $\delta > 0$  such that  $\|x_0 - \varphi(\tau)\| < \delta$  implies that

$$\|x(t, \tau, x_0) - \varphi(t)\| < \varepsilon,$$

for all  $t \geq \tau$ .

If in addition,  $\lim_{t \rightarrow \infty} \|x(t, \tau, x_0) - \varphi(t)\| = 0$ , then  $\varphi(t)$  is asymptotically stable.

The matrix  $A(t) = D_x f(t, \varphi(t))$  is  $T$ -periodic. Let  $Y(t)$  be a fundamental matrix for  $A(t)$ . That is,

$$Y'(t) = A(t)Y(t), \quad Y(0) = I.$$

This is the situation to which Floquet theory applies. There exist real matrices  $P(t)$ ,  $B$ , and  $R$  such that

$$\begin{aligned} Y(t) &= P(t)e^{Bt} \\ P(t+T) &= P(t)R \\ (7.2.1) \quad R^2 &= I \quad BR = RB. \end{aligned}$$

The eigenvalues of  $e^{BT}$  are called the Floquet multipliers of  $\varphi$ . They are unique. The eigenvalues of  $B$  are called the Floquet exponents of  $\varphi$ . They are unique up to integer multiples of  $2\pi i/T$ . If there are no Floquet multipliers in the interval  $(-\infty, 0)$ , then  $R = I$  and  $P(t)$  is  $T$ -periodic. Otherwise,  $P(t)$  is  $2T$ -periodic. The Floquet multipliers of the periodic solution  $\varphi$  are independent of the initial time used in computing the fundamental matrix (exercise).

THEOREM 7.2.1. If the Floquet exponents of  $\varphi$  satisfy  $\operatorname{Re} \lambda < -\lambda_0 < 0$  (equivalently, if the Floquet multipliers satisfy  $|\mu| < e^{-\lambda_0} < 1$ ), then  $\varphi$  is asymptotically stable. There exist  $C > 0$ ,  $\delta > 0$  such that if  $\|x_0 - \varphi(\tau)\| < \delta$  for some  $\tau \in \mathbb{R}$ , then

$$\|x(t, \tau, x_0) - \varphi(t)\| \leq C e^{-\lambda_0(t-\tau)} \|x_0 - \varphi(\tau)\|,$$

for all  $t \geq \tau$ .

PROOF. Consider a solution  $x(t)$  that is “close to”  $\varphi(t)$ . If  $y = x - \varphi$ , then  $y$  should be small. What equation does the small perturbation  $y(t)$  satisfy?

$$\begin{aligned} y' &= x' - \varphi' \\ &= f(t, x) - f(t, \varphi) \\ &= f(t, \varphi + y) - f(t, \varphi) \\ &\equiv g(t, y). \end{aligned}$$

Note that  $g(t+T, y) = g(t, y)$ , for all  $(t, y) \in \mathbb{R} \times \mathbb{R}^n$ , and that  $g(t, 0) = 0$ , for all  $t \in \mathbb{R}$ . We have that  $A(t) = D_x f(t, \varphi(t)) = D_y g(t, 0)$ , so this becomes

$$\begin{aligned} y' &= A(t)y + [g(t, y) - A(t)y] \\ &\equiv A(t)y + h(t, y), \end{aligned}$$

in which  $h(t, 0) = 0$ ,  $D_y h(t, 0) = 0$ , and  $h(t+T, y) = h(t, y)$ .

As a final reduction, set  $y(t) = P(t)z(t)$ . This makes sense since  $P(t)$  is invertible for all  $t \in \mathbb{R}$ . Then

$$\begin{aligned} P'(t)z(t) + P(t)z'(t) &= y'(t) \\ &= A(t)y(t) + h(t, y(t)) \\ &= A(t)P(t)z(t) + h(t, P(t)z(t)), \end{aligned}$$

so that

$$z'(t) = P^{-1}(t)[A(t)P(t) - P'(t)]z(t) + P^{-1}(t)h(t, P(t)z(t)).$$

Now  $P^{-1}(t)[A(t)P(t) - P'(t)] = B$ . Plug this in above:

$$\begin{aligned} z'(t) &= Bz(t) + P^{-1}(t)h(t, P(t)z(t)) \\ &\equiv Bz(t) + H(t, z(t)). \end{aligned}$$

The nonlinear function  $H(t, z) = h(t, P(t)z)$  has the following properties:

$$(7.2.2) \quad \begin{aligned} H(t, 0) &= 0, \quad D_z H(t, 0) = 0, \\ \text{and } H(t+T, z) &= R^{-1}H(t, Rz) = RH(t, Rz). \end{aligned}$$

Recall the Theorem 2.9.2 which said that if  $F(x)$  is a  $C^1$  autonomous vector field on  $\mathbb{R}^n$  with  $F(0) = 0$ ,  $DF(0) = 0$ , and  $A$  is an  $n \times n$  matrix whose eigenvalues all have negative real part, then the origin is asymptotically stable for  $x' = Ax + F(x)$ . The present situation differs in that the vector field  $H(t, z)$  is nonautonomous. However, the fact that  $H(t+2T, z) = H(t, z)$  gives uniformity in the  $t$  variable which permits us to use the same argument as in the autonomous case. The outcome is the following statement:

There exist  $C_0 > 0$ ,  $\delta > 0$  such that if  $\|z_0\| < \delta$ , then

$$\|z(t, \tau, z_0)\| < C_0 e^{-\lambda_0(t-\tau)} \|z_0\|,$$

for every  $t \geq \tau$ .

The same estimate holds for  $P(t)z(t, \tau, z_0)$  (with different constants  $C_0$  and  $\delta$ ), since  $P$  is  $2T$ -periodic and invertible. Theorem 7.2.1 follows now because

$$x(t, \tau, x_0) = \varphi(t) + P(t)z(t, \tau, z_0),$$

with  $z_0 = P(\tau)^{-1}[x_0 - \varphi(\tau)]$ .  $\square$

Remark: The defect of this theorem is that it is necessary to at least estimate the Floquet multipliers. We have already seen that this can be difficult. However, suppose that we are in the situation of section 7.1, namely the vector field has the form

$$f(t, x, \varepsilon) = f_0(x) + \varepsilon \tilde{f}(t, x, \varepsilon),$$

with  $f_0(0) = 0$  and  $\tilde{f}(t + T, x, \varepsilon) = \tilde{f}(t, x, \varepsilon)$ . Let  $\varphi_\varepsilon$  be the unique  $T$ -periodic orbit of  $f(t, x, \varepsilon)$  near the origin. Then by continuous dependence

$$A_\varepsilon(t) = D_x f(t, \varphi_\varepsilon(t), \varepsilon) \approx D_x f_0(0) = A.$$

Again by continuous dependence, the fundamental matrix  $Y_\varepsilon(t)$  of  $A_\varepsilon(t)$  is close to  $e^{At}$ . So for  $\varepsilon$  sufficiently small the Floquet multipliers of  $\varphi_\varepsilon$  are close to the eigenvalues of  $e^{AT}$ . If the eigenvalues of  $A$  all have  $\operatorname{Re} \lambda < 0$ , then  $\varphi_\varepsilon$  is asymptotically stable, for  $\varepsilon$  small.

### 7.3. Stable Manifold Theorem for Nonautonomous Periodic Systems

We continue the notation and definitions of the previous section.

In particular, let  $B$  and  $R$  be  $n \times n$  matrices which satisfy (7.2.1). Suppose that  $B$  is hyperbolic. Let  $P_s, P_u$  be the usual projections onto the stable and unstable subspaces  $E_s, E_u$  of  $B$ . Then there exist constants  $C_0, \lambda > 0$  such that

$$\|P_s e^{Bt}\| \leq C_0 e^{-\lambda t}, \quad t \geq 0 \quad \text{and} \quad \|P_u e^{Bt}\| \leq C_0 e^{\lambda t}, \quad t \leq 0.$$

Let  $H : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  function which satisfies the conditions in (7.2.2). Let  $z(t, \tau, z_0)$  be the flow of the IVP

$$z' = Bz + H(t, z), \quad z(0, \tau, z_0) = z_0.$$

**THEOREM 7.3.1.** *There is a neighborhood of the origin  $U \subset E_s$  and a  $C^1$  function  $\eta : \mathbb{R}^+ \times U \rightarrow E_u$  such that*

- (i)  $\eta(\tau, 0) = 0, D_y \eta(\tau, 0) = 0$  for all  $\tau \in \mathbb{R}^+$ , and  $\eta(\tau + T, y) = R\eta(\tau, Ry)$  for all  $(\tau, y) \in \mathbb{R}^+ \times U$ .
- (ii) Define

$$W_s^{loc}(0) \equiv \{(\tau, z_0) \in \mathbb{R}^+ \times \mathbb{R}^n : P_s z_0 \in U, P_u z_0 = \eta(\tau, P_s z_0)\}.$$

If  $(\tau, z_0) \in W_s^{loc}(0)$ , then  $z(t, \tau, z_0)$  is defined for all  $t \geq \tau$  and

$$\|z(t, \tau, z_0)\| \leq C e^{-[\lambda(t-\tau)/2]}.$$

- (iii)  $W_s^{loc}(0)$  is positively invariant in the sense that if  $(\tau, z_0) \in W_s^{loc}(0)$ , then  $(t, P_s z(t, \tau, z_0)) \in W_s^{loc}(0)$ , for  $t > \tau$ .

SKETCH OF PROOF. The steps of the proof are similar to the autonomous case, see Theorem 6.3.1.

Suppose that  $z(t, \tau, z_0)$  is defined and uniformly bounded for  $t \geq \tau$ . Then it follows from the variation of parameters formula that

$$P_u \left[ z_0 - \int_{\tau}^{\infty} e^{B(\tau-s)} H(s, z(s, \tau, z_0)) ds \right] = 0.$$

This implies that

$$\begin{aligned} z(t, \tau, z_0) &= e^{B(t-\tau)} P_s z_0 + \int_{\tau}^t e^{B(t-s)} P_s H(s, z(s, \tau, z_0)) ds \\ &\quad - \int_t^{\infty} e^{B(t-s)} P_u H(s, z(s, \tau, z_0)) ds, \end{aligned}$$

for all  $t \geq \tau$ .

Let  $Y$  denote the Banach space of bounded continuous functions  $y(t, \tau)$  from the domain  $\{(t, \tau) \in \mathbb{R}^2 : t \geq \tau \geq 0\}$  into  $\mathbb{R}^n$  with the *sup* norm. Define the mapping  $F : E_s \times Y \rightarrow Y$  by

$$\begin{aligned} F(y_0, y)(t, \tau) &= y(t, \tau) - e^{B(t-\tau)} y_0 \\ &\quad - \int_{\tau}^t e^{B(t-s)} P_s H(s, y(s, \tau)) ds \\ &\quad + \int_t^{\infty} e^{B(t-s)} P_u H(s, y(s, \tau)) ds. \end{aligned}$$

$F$  is a  $C^1$  mapping,  $F(0, 0) = 0$ , and  $D_y(0, 0) = I$  is invertible. By the implicit function theorem 4.4.1, there exist neighborhoods of the origin  $U \subset E_s$  and  $V \subset Y$  and a  $C^1$  map  $\phi : U \rightarrow V$  such that  $\phi(0) = 0$  and  $F(y_0, \phi(y_0)) = 0$ , for all  $y_0 \in U$ . In addition, if  $(y_0, y) \in U \times V$  and  $F(y_0, y) = 0$ , then  $y = \phi(y_0)$ . Write  $y(t, \tau, y_0) = \phi(y_0)(t, \tau)$ .

Define  $\eta : \mathbb{R}^+ \times U \rightarrow E_u$  by

$$\eta(y, \tau, y_0) = - \int_{\tau}^{\infty} e^{A(s-\tau)} P_u H(s, y(s, \tau, y_0)) ds.$$

Then  $\eta$  is continuous in  $(\tau, y_0)$  and  $C^1$  in  $y_0$ .  $\eta(\tau, 0) = 0$  and  $D_{y_0}(\tau, 0) = 0$  for all  $\tau \in \mathbb{R}^+$ .

Suppose that  $z_0 \in \mathbb{R}^n$  and  $y_0 = P_s z_0 \in U$ . Then  $z(t, \tau, z_0)$  is defined for all  $t \geq \tau$  and remains uniformly bounded if and only if

$$z(t, \tau, z_0) = y(t, \tau, y_0).$$

This, in turn, is equivalent to

$$P_u z_0 = \eta(\tau, y_0).$$

(By smooth dependence of  $z(t, \tau, z_0)$  on  $\tau$ , it now follows that  $\eta$  is  $C^1$  in  $\tau$ , also.)

By the uniqueness of solutions to the initial value problem, we have that

$$z(t + T, \tau, z_0) = Rz(t, \tau - T, Rz_0),$$

for all  $t \geq \tau - T$ . Shrink the neighborhood  $U$  to  $U' = U \cap RU \cap \phi^{-1}(RV)$ . It follows by the uniqueness part of the implicit function theorem that if  $y_0 \in U'$ , then

$$y(t + T, \tau, y_0) = Ry(t, \tau - T, Ry_0),$$

for all  $t \geq \tau - T$ . Now let  $z_0 \in \mathbb{R}^n$  with  $y_0 = P_s z_0 \in U'$ . Then

$$\begin{aligned} z(t, \tau - T, Rz_0) &= Rz(t + T, \tau, z_0) \\ &= Ry(t + T, \tau, y_0) \\ &= y(t, \tau - T, Ry_0) \\ &= z(t, \tau - T, z_1), \end{aligned}$$

with  $z_1 = Ry_0 + \eta(\tau - T, Ry_0)$ . By uniqueness to the initial value problem, we must have  $\eta(\tau - T, Ry_0) = P_u z_1 = RP_u z_0 = R\eta(\tau, y_0)$ . This shows all of the properties in (i).

The proof of exponential decay and invariance is similar to the autonomous case. As in the previous case, further shrinkage of the neighborhood  $U'$  will be necessary.  $\square$

Remarks:

- By reversing the sense of time in the system, this theorem immediately ensures the existence of a local unstable manifold.
- If we return to the situation where  $z$  represents local coordinates near a hyperbolic periodic orbit, the theorem gives us an invariant manifold  $\varphi(\tau) + P(\tau)z_0$  parametrized by  $(\tau, z_0) \in W_s^{loc}(0)$  near the periodic orbit  $\varphi$  of the original system  $x' = f(t, x)$ .
- The periodic orbit is the intersection of the local stable and unstable manifolds.
- If  $R = I$ , i.e. there are no Floquet multipliers on the negative real axis, then the local stable manifold is  $T$ -periodic. It follows that the stable manifold makes an even number of twists as it revolves around the periodic orbit. If  $R \neq I$ , i.e. there are Floquet multipliers on the negative real axis, then the local stable manifold makes an odd number of twists as it circles the periodic orbit. It is a generalized Möbius band.

**Example: Newton's Equation with Small Periodic Forcing**

Recall that Newton's equation is the second order equation,

$$u'' + g(u) = 0,$$

which is equivalent to the first order system

$$x' = f(x), \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \\ u' \end{bmatrix}, \quad f(x) = \begin{bmatrix} x_2 \\ -g(x_1) \end{bmatrix}.$$

Here we assume that  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^3$  function with  $g(0) = 0$  and  $g'(0) = \gamma^2 > 0$ , so that the origin is a non-hyperbolic critical point for this system. In fact, if  $G'(u) = g(u)$ , then solutions satisfy  $x_2^2(t)/2 + G(x_1(t)) = \text{Const.}$ , which shows that all orbits near the origin are closed, and hence periodic.

Let  $p(t)$  be a continuous  $T$ -periodic function. Consider the forced equation

$$u''(t) + g(u(t)) = \varepsilon p(t),$$

or equivalently,

$$x' = f(t, x, \varepsilon), \quad f(t, x, \varepsilon) = \begin{bmatrix} x_2 \\ -g(x_1) + \varepsilon p(t) \end{bmatrix}.$$

If  $\gamma T/2\pi$  is not an integer, then we have seen that for all small  $\varepsilon$ , there is a unique periodic solution  $x_\varepsilon(t)$  of the forced equation near the origin.

We will show that in spite of the fact that the origin is not a hyperbolic critical point, under certain restrictions on  $g$  and  $p$  the periodic solution  $x_\varepsilon(t)$  of the forced equation is hyperbolic.

The hyperbolicity of  $x_\varepsilon$  depends on showing that the Floquet multipliers lie off of the unit circle. Let  $u_\varepsilon(t)$  denote the first component of  $x_\varepsilon(t)$ . Then  $u_\varepsilon(t)$  solves

$$(7.3.1) \quad u_\varepsilon''(t) + g(u_\varepsilon(t)) = \varepsilon p(t).$$

Set

$$A(t, \varepsilon) = D_x f(t, x_\varepsilon(t), 0) = \begin{bmatrix} 0 & 1 \\ -g'(u_\varepsilon(t)) & 0 \end{bmatrix}.$$

The Floquet multipliers are the eigenvalues of  $Y(T, \varepsilon)$  where  $Y(t, \varepsilon)$  is a fundamental matrix for  $A(t, \varepsilon)$ . That is,

$$(7.3.2) \quad Y'(t, \varepsilon) = A(t, \varepsilon)Y(t, \varepsilon) \quad Y(0, \varepsilon) = I.$$

The first observation is that since

$$\det Y(T, \varepsilon) = \exp \left( \text{tr} \int_0^T A(t, \varepsilon) dt \right) = 1,$$

the Floquet multipliers  $\mu_1(\varepsilon)$  and  $\mu_2(\varepsilon)$  of  $x_\varepsilon$  satisfy

$$\mu_1(\varepsilon)\mu_2(\varepsilon) = \det Y(T, \varepsilon) = 1.$$

This means that one of the following hold:

- I.  $\mu_1(\varepsilon) = \overline{\mu_2(\varepsilon)}$ ,  $|\mu_1(\varepsilon)| = |\mu_2(\varepsilon)| = 1$
- II.  $\mu_1(\varepsilon) = \mu_2(\varepsilon)^{-1}$ 
  - A.  $\mu_1(\varepsilon) > 1 > \mu_2(\varepsilon) > 0$
  - B.  $\mu_1(\varepsilon) < -1 < \mu_2(\varepsilon) < 0$

Since the Floquet multipliers are root of the characteristic polynomial,

$$\mu^2 - \tau(\varepsilon)\mu + 1 = 0,$$

with  $\tau(\varepsilon) = \mu_1(\varepsilon) + \mu_2(\varepsilon)$ , the trace of  $Y(T, \varepsilon)$ , these cases can be distinguished as: I.  $|\tau(\varepsilon)| \leq 2$ , II A.  $\tau(\varepsilon) > 2$ , and II B.  $\tau(\varepsilon) < -2$ . The stable manifold theorem 7.3.1 applies only in case II. We shall show that case II B holds under appropriate restrictions on  $g$  and  $p$ . In case II B, the Floquet multipliers lie on the negative real axis. This means that  $R = -I$ , so the stable manifold is a Möbius band when it exists.

Our strategy will be to obtain an expansion for the trace

$$\tau(\varepsilon) = \tau(0) + \varepsilon\tau'(0) + \frac{\varepsilon^2}{2}\tau''(0) + O(\varepsilon^3).$$

Since  $\tau^{(k)}(0) = \text{tr} \left[ \frac{d^k Y}{d\varepsilon^k}(T, 0) \right]$ , this can be accomplished by first finding an expansion for the fundamental matrix  $Y(t, \varepsilon)$ :

$$\begin{aligned} Y(t, \varepsilon) &= Y(t, 0) + \varepsilon \frac{dY}{d\varepsilon}(t, 0) + \frac{\varepsilon^2}{2} \frac{d^2 Y}{d\varepsilon^2}(t, 0) + O(\varepsilon^3) \\ &= Y_0(t) + \varepsilon Y_1(t) + \frac{\varepsilon^2}{2} Y_2(t) + O(\varepsilon^3) \end{aligned}$$

The existence of this expansion is implied by the smooth dependence of  $Y(t, \varepsilon)$  on  $\varepsilon$ .

The terms  $Y_k(t)$  will be found by successive differentiation of the variational equation (7.3.2) with respect to  $\varepsilon$ . Thus, for  $k = 0$  we have

$$(7.3.3) \quad Y_0'(t) = A_0 Y_0(t), \quad Y_0(0) = I,$$

in which

$$A_0 \equiv A(t, 0) = D_x f(t, 0, 0) = \begin{bmatrix} 0 & 1 \\ -\gamma^2 & 0 \end{bmatrix}.$$

This has the solution

$$Y_0(t) = e^{A_0 t} = \begin{bmatrix} \cos \gamma t & \gamma^{-1} \sin \gamma t \\ -\gamma \sin \gamma t & \cos \gamma t \end{bmatrix}.$$

Thus,  $\tau(0) = \text{tr} Y_0(T) = 2 \cos \gamma T$ . Recall that we have  $\gamma T \neq 2j\pi$ ,  $j \in \mathbb{Z}$ . So  $\tau(0) < 2$ , and by continuous dependence  $\tau(\varepsilon) < 2$ , for small  $\varepsilon$ . This says that case II A can not occur. If  $\tau(0) > -2$ , then by

continuous dependence,  $\tau(\varepsilon) > -2$  for small  $\varepsilon$ . This is case I in which the stable manifold theorem does not apply. So in order for case II B to occur, it is necessary that  $\tau(0) = -2$ , i.e.  $\gamma T = (2j + 1)\pi$ ,  $j \in \mathbb{Z}$ .

Next let's look at  $\tau'(0)$ . For  $k = 1$ , we get

$$(7.3.4) \quad Y_1'(t) = A_0 Y_1(t) + A_1(t) Y_0(t), \quad Y_1(0) = 0,$$

with

$$(7.3.5) \quad A_1(t) = \frac{dA}{d\varepsilon}(t, 0) \\ = \frac{d}{d\varepsilon} \begin{bmatrix} 0 & 0 \\ -g'(u_\varepsilon(t)) & 0 \end{bmatrix} \Big|_{\varepsilon=0} = \begin{bmatrix} 0 & 0 \\ -g''(0)q(t) & 0 \end{bmatrix},$$

and  $q(t) = \frac{d}{d\varepsilon} u_\varepsilon(t) \Big|_{\varepsilon=0}$ .

Using the variation of parameters formula together with the previous computation, we obtain the representation

$$(7.3.6) \quad Y_1(t) = \int_0^t e^{A_0(t-s)} A_1(s) Y_0(s) ds \\ = e^{A_0 t} \int_0^t e^{-A_0 s} A_1(s) Y_0(s) ds \\ = e^{A_0 t} \int_0^t B(s) ds$$

with

$$(7.3.7) \quad B(s) = e^{-A_0 s} A_1(s) Y_0(s) = e^{-A_0 s} A_1(s) e^{A_0 s}.$$

If we set  $t = T$  and use the fact that  $e^{A_0 T} = -I$ , then

$$Y_1(T) = - \int_0^T B(s) ds.$$

Hence,  $\tau'(0) = \text{tr } Y_1(T) = - \int_0^T \text{tr } B(s) ds$ . Now  $B(s)$  is similar to  $A_1(s)$ . Similar matrices have identical traces, and  $A_1(s)$  has zero trace. Therefore,  $\tau'(0) = 0$ . We get no information from this term!

So it's on to the second order term  $\tau''(0)$ . If  $k = 2$ , then

$$(7.3.8) \quad Y_2'(t) = A_0 Y_2(t) + 2A_1(t) Y_1(t) + A_2(t) Y_0(t), \quad Y_2(0) = 0,$$

in which

$$A_2(t) = \frac{d^2 A}{d\varepsilon^2}(t, 0).$$

As before, we have

$$\begin{aligned} Y_2(T) &= \int_0^T e^{A_0(T-s)} [2A_1(s)Y_1(s) + A_2(s)Y_0(s)] ds \\ &= -2 \int_0^T e^{-A_0s} A_1(s)Y_1(s) ds - \int_0^T e^{-A_0s} A_2(s)e^{A_0s} ds. \end{aligned}$$

The second term has zero trace, by the similarity argument above. Therefore, we have by (7.3.6)

$$\begin{aligned} \tau''(0) &= \text{tr } Y_2(T, 0) \\ &= -2 \text{tr} \int_0^T e^{-A_0s} A_1(s)Y_1(s) ds \\ &= -2 \text{tr} \int_0^T e^{-A_0s} A_1(s)e^{A_0s} \left( \int_0^s B(\sigma) d\sigma \right) ds \\ &= -2 \text{tr} \int_0^T B(s) \left( \int_0^s B(\sigma) d\sigma \right) ds \\ &= -2 \int_0^T \int_0^s \text{tr } B(s)B(\sigma) d\sigma ds \\ &= -2 \int_0^T \int_\sigma^T \text{tr } B(s)B(\sigma) ds d\sigma \\ &= -2 \int_0^T \int_\sigma^T \text{tr } B(\sigma)B(s) d\sigma ds, \end{aligned}$$

where we have interchanged the order of integration. Using the definitions (7.3.5), (7.3.7), we find

$$(7.3.9) \quad B(s) = -g''(0)q(s) \begin{bmatrix} -\gamma^{-1} \cos \gamma s \sin \gamma s & -\gamma^{-2} \sin^2 \gamma s \\ \cos^2 \gamma s & \gamma^{-1} \cos \gamma s \sin \gamma s \end{bmatrix}.$$

It is straightforward to check that

$$\text{tr } B(s)B(\sigma) = \text{tr } B(\sigma)B(s).$$

Therefore, we conclude that

$$\begin{aligned} \tau_2''(0) &= -\text{tr} \int_0^T \int_0^s B(s)B(\sigma) d\sigma ds - \text{tr} \int_0^T \int_s^T B(s)B(\sigma) d\sigma ds \\ &= -\text{tr} \int_0^T \int_0^T B(s)B(\sigma) d\sigma ds \\ &= -\text{tr} \left( \int_0^T B(s) ds \right)^2. \end{aligned}$$

Let's see what this condition means. We obtain from (7.3.9)

$$\tau''(0) = - \left( \frac{g''(0)}{\gamma^2} \right)^2 \left[ \left( \int_0^T q(s) \cos \gamma s \sin \gamma s ds \right)^2 - \left( \int_0^T q(s) \cos^2 \gamma s ds \right) \left( \int_0^T q(s) \sin^2 \gamma s ds \right) \right].$$

Taking the derivative of (7.3.1), we see that the function  $q(t) = \frac{d}{d\varepsilon} u_\varepsilon|_{\varepsilon=0}$  is a  $T$ -periodic solution of the linear equation

$$q''(t) + \gamma^2 q(t) = p(t).$$

Since  $\gamma T \neq 2\pi$ , this solution is unique. If, for example, we take the period forcing term in (7.3.1) to be

$$p(t) = \sin 2\gamma t,$$

so that  $\gamma T = \pi$ , then

$$q(t) = -(1/3\gamma^2) \sin 2\gamma t.$$

The last two integrals vanish, while the first is nonzero. We therefore see that  $\tau''(0) < 0$ , and so the period orbit  $u_\varepsilon$  is hyperbolic, for small  $\varepsilon \neq 0$ .

#### 7.4. Stability of Periodic Solutions to Autonomous Systems

Our aim in this section is to investigate the orbital stability of periodic solutions to autonomous systems. Recall that in Section 7.1, Theorem 7.2.1, we established the stability of periodic solutions to periodic *nonautonomous* systems. In that case, we saw that if the Floquet multipliers of a periodic solution are all inside the unit disk, then the periodic solution is asymptotically stable. For autonomous flow, this hypothesis can never hold since, as we will see below, at least one Floquet multiplier is always equal to 1.

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  autonomous vector field. Suppose that  $\varphi(t)$  is a non-trivial  $T$ -periodic solution of the autonomous differential equation

$$(7.4.1) \quad x' = f(x).$$

Then the matrix  $A(t) = Df(\varphi(t))$  is  $T$ -periodic. Let  $Y(t)$  be its fundamental matrix:

$$Y'(t) = A(t)Y(t), \quad Y(0) = I.$$

According to the the results of Floquet theory, there are real matrices  $P(t)$ ,  $B$ , and  $R$  such that

$$Y(t) = P(t)e^{Bt},$$

with  $P(t+T) = P(t)R$ ,  $R^2 = I$ , and  $BR = RB$ . The Floquet multipliers are the eigenvalues  $\{\mu_i\}_{i=1}^n$  of the matrix  $Y(T) = Re^{BT}$ .

If we differentiate the equation  $\varphi'(t) = f(\varphi(t))$  with respect to  $t$ , we get

$$\varphi''(t) = DF(\varphi(t))\varphi'(t) = A(t)\varphi'(t).$$

This says that  $\varphi'(t)$  is a solution of the linear variational equation. The solution is given by

$$\varphi'(t) = Y(t)\varphi'(0).$$

The  $T$ -periodicity of  $\varphi$  is inherited by  $\varphi'$ , so

$$\varphi'(0) = \varphi'(T) = Y(T)\varphi'(0).$$

Note that  $\varphi'(0) = f(\varphi(0)) \neq 0$ , for otherwise,  $\varphi$  would be an equilibrium point, contrary to assumption. So we have just shown that  $\varphi'(0)$  is a eigenvector of  $Y(T)$  with eigenvalue 1. Thus, one of the Floquet multipliers,  $\mu_n$  say, is 1. It is called the trivial Floquet multiplier.

We now need some definitions.

**DEFINITION 7.4.1.** *A  $T$ -periodic orbit  $\gamma = \{\varphi(t) : 0 \leq t \leq T\}$  is said to be orbitally stable if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $\text{dist}(x_0, \gamma) < \delta$  implies that  $x(t, x_0)$  is defined for all  $t \geq 0$  and  $\text{dist}(x(t, x_0), \gamma) < \varepsilon$ , for all  $t \geq 0$ . If in addition,  $\lim_{t \rightarrow \infty} \text{dist}(x(t, x_0), \gamma) = 0$ , for all  $x_0$  sufficiently close to  $\gamma$ , then the orbit is said to be asymptotically orbitally stable.*

This is weaker than the definition of stability given in Definition 7.2.1, where solutions are compared at the same time values.

**DEFINITION 7.4.2.** *A periodic orbit  $\gamma = \{\varphi(t) : \leq t \leq T\}$  is asymptotically orbitally stable with asymptotic phase if it is asymptotically orbitally stable and there exists a  $\tau \in \mathbb{R}$  such that*

$$\lim_{t \rightarrow \infty} \|x(t, x_0) - \varphi(t + \tau)\| = 0.$$

**THEOREM 7.4.1.** *Suppose that  $\varphi(t)$  is a  $T$ -periodic solution of the autonomous equation (7.4.1) whose nontrivial Floquet multipliers satisfy  $|\mu_i| \leq \bar{\mu} < 1$ ,  $i = 1, \dots, n-1$ . Then  $\varphi(t)$  is asymptotically orbitally stable with asymptotic phase.*

PROOF. To start out with, we recall some estimates for the fundamental matrix  $e^{Bt}$ . Our assumption is equivalent to saying that the Floquet exponents, i.e. the eigenvalues of  $BT$ , have strictly negative real parts except for a simple zero eigenvalue. Thus, if  $\{\lambda_i\}_{i=1}^n$  denote the eigenvalues of  $B$ , then there is a  $\bar{\lambda} > 0$  such that

$$\operatorname{Re} \lambda_i \leq -\bar{\lambda} < 0, \quad i = 1, \dots, n-1; \quad \lambda_n = 0.$$

So  $B$  has an  $(n-1)$ -dimensional stable subspace  $E_s$ , and a 1-dimensional center subspace  $E_c$ . The vector  $v_n = \varphi'(0)$  spans  $E_c$ . Let  $P_s$  and  $P_c$  be the corresponding projections. We have the estimates

$$(7.4.2) \quad \begin{aligned} \|e^{Bt}P_sx\| &\leq C_0e^{-\bar{\lambda}t}\|P_sx\|, \quad t \geq 0 \\ \|e^{Bt}P_cx\| &\leq C_0\|P_cx\|, \quad t \leq 0, \end{aligned}$$

for all  $x \in \mathbb{R}^n$ . Note that  $e^{Bt}$  is bounded on  $E_c$ , since  $v_n$  is a simple eigenvalue.

With the preliminaries finished, we now come to the motivational sermon. Let  $x_0 \in \mathbb{R}^n$  be a given point close to  $\varphi(0)$ . We want to show that there is a small phase shift  $\tau \in \mathbb{R}$  such that

$$x(t, x_0) - \varphi(t + \tau) \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Similar to what we did in the nonautonomous case in Section 7.2, define a function  $z(t, \tau)$  by

$$P(t + \tau)z(t, \tau) = x(t, x_0) - \varphi(t + \tau).$$

We note that  $P'(t + \tau) = A(t + \tau)P(t + \tau) - P(t + \tau)B$ . Since  $f$  is autonomous,  $\varphi(t + \tau)$  is a solution of (7.4.1) for all  $\tau \in \mathbb{R}$ . After a bit of computation, we see that  $z(t, \tau)$  solves

$$(7.4.3) \quad \begin{aligned} z'(t, \tau) &= Bz(t, \tau) + P(t + \tau)^{-1}[f(\varphi(t + \tau) + P(t + \tau)z(t, \tau)) \\ &\quad - f(\varphi(t + \tau)) - A(t + \tau)P(t + \tau)z(t, \tau)] \\ &\equiv Bz(t, \tau) + H(t + \tau, z(t, \tau)), \end{aligned}$$

with  $H(\sigma, 0) = 0$ ,  $D_z H(\sigma, 0) = 0$ , and  $H(\sigma + T, z) = RH(\sigma, Rz)$ , for all  $\sigma \in \mathbb{R}$  and  $z \in \mathbb{R}^n$ .

Conversely, if  $z(t, \tau)$  is a solution of (7.4.3), then  $x(t) = \varphi(t + \tau) + P(t + \tau)z(t, \tau)$  solves (7.4.1) with initial data  $z(0, \tau) + P(\tau)\varphi(\tau)$ . Our strategy will be to construct exponentially decaying solutions of (7.4.3) and then to adjust the initial data appropriately.

By variation of parameters, a solution of (7.4.3) satisfies

$$z(t, \tau) = e^{Bt}z(0, \tau) + \int_0^t e^{B(t-\sigma)}H(\sigma + \tau, z(\sigma, \tau))d\sigma.$$

Moreover, if  $z(t, \tau)$  is exponentially decaying, then using (7.4.2), this is equivalent to

$$\begin{aligned} z(t, \tau) &= e^{Bt} P_s z(0, \tau) + \int_0^t e^{B(t-\sigma)} P_s H(\sigma + \tau, z(\sigma, \tau)) d\sigma \\ &\quad - \int_t^\infty e^{B(t-\sigma)} P_c H(\sigma + \tau, z(\sigma, \tau)) d\sigma. \end{aligned}$$

Now we come to the main set-up. Take a  $0 < \beta < \bar{\lambda}$ . Define the set of functions

$$Z = \{z \in C(\mathbb{R}^+, \mathbb{R}^n) : \|z\|_\beta \equiv \sup_{t \geq 0} e^{\beta t} \|z(t)\| < \infty\}.$$

$Z$  is a Banach space with the indicated norm. For  $\tau \in \mathbb{R}$ ,  $z_0 \in E_s$ , and  $z \in Z$ , define the mapping

$$\begin{aligned} T(\tau, z_0, z)(t) &= z(t) - e^{Bt} z_0 - \int_0^t e^{B(t-\sigma)} P_s H(\sigma + \tau, z(\sigma)) d\sigma \\ &\quad + \int_t^\infty e^{B(t-\sigma)} P_c H(\sigma + \tau, z(\sigma)) d\sigma. \end{aligned}$$

Now  $T : \mathbb{R} \times E_s \times Z \rightarrow Z$  is a well-defined  $C^1$  mapping such that  $T(0, 0, 0) = 0$  and  $D_z T(0, 0, 0) = I$ . By the implicit function theorem 4.4.1, there is a neighborhood  $U$  of the origin in  $\mathbb{R} \times E_s$  and a  $C^1$  mapping  $\psi : U \rightarrow Z$  such that

$$\psi(0, 0) = 0 \quad \text{and} \quad T(\tau, z_0, \psi(\tau, z_0)) = 0, \quad \text{for all } (\tau, z_0) \in U.$$

It follows that  $z(t, \tau, y_0) = \psi(\tau, z_0)(t)$  is a solution of (7.4.3), and since  $z(t, \tau, z_0) \in Z$ , we have  $\|z(t, \tau, z_0)\| \leq C e^{-\beta t}$ . Thus,  $x(t) = \varphi(t + \tau) + P(t + \tau)z(t, \tau, z_0)$  is a solution of (7.4.1) for all  $(\tau, z_0) \in U$ , with data  $x(0) = \varphi(\tau) + P(\tau)z(0, \tau, z_0)$ .

Define a map  $F : E_s \times \mathbb{R} \rightarrow \mathbb{R}^n$  by

$$F(\tau, z_0) = P(\tau)z(0, \tau, z_0) + \varphi(\tau).$$

Then  $F$  is  $C^1$  and  $F(0, 0) = \varphi(0)$ . We are going to show, using the inverse function theorem, that  $F$  is an invertible map from a neighborhood  $U$  of  $(0, 0) \in E_s \times \mathbb{R}$  to a neighborhood  $V$  of  $\varphi(0)$  in  $\mathbb{R}^n$ .

Since  $H(\tau, 0) = 0$  for all  $\tau$ , we have that  $D_\tau H(\tau, 0) = 0$ . We also know that  $D_z H(\tau, 0) = 0$ . It then follows from the definition of  $z(t, \tau, z_0)$  and the uniqueness portion of the implicit function theorem that

$$DF(0, 0)(\bar{\tau}, \bar{z}_0) = \varphi'(0)\bar{\tau} + \bar{z}_0.$$

We claim that this map is invertible. Given  $x \in \mathbb{R}^n$ , there is a unique decomposition  $x = P_s x + P_c x$ . Since  $E_c$  is spanned by  $\varphi'(0)$ , we have  $P_c x = \varphi'(0)\nu$ , for a unique  $\nu \in \mathbb{R}$ , and  $DF(0, 0)^{-1}(x) = (\nu, P_s x)$ .

So by the inverse function theorem, there is a neighborhood  $V \subset \mathbb{R}^n$  containing  $\varphi(0)$  and a neighborhood  $U \subset E_s \times \mathbb{R}$  containing  $(0, 0)$  such that  $F : U \rightarrow V$  is a diffeomorphism.

In other words, for every  $x_0 \in V$ , there is a  $(\tau, y_0) \in U$  such that

$$x(t, x_0) = \varphi(t + \tau) + P(t + \tau)z(t, \tau, z_0).$$

Thus, since  $\|P(t + \tau)\|$  is uniformly bounded and  $\|z(t, \tau, z_0)\| \leq Ce^{-\beta t}$ , given  $x_0 \in V$ , we have found a phase  $\tau \in \mathbb{R}$  with

$$\|x(t, x_0) - \varphi(t + \tau)\| \leq Ce^{-\beta t}, \quad t \geq 0.$$

Since the argument works for any point along the periodic orbit (not only  $\varphi(0)$ ), we have established asymptotic orbital stability with asymptotic phase. □

Remarks:

- The proof shows that the convergence to the periodic orbit occurs at an exponential rate.
- The theorem implies that if the nontrivial Floquet multipliers of a periodic orbit lie within the unit disk, then the orbit is a local attractor.
- I know of two other proofs of theorem 7.4.1. There is one using the Poincaré map to be found in Hartman (Theorem 11.1), and another based on change of coordinates near the periodic orbit in the book of Hale (Chapter 6, Theorem 2.1). I chose the one presented above because it emphasizes the similarities with the corresponding result for the center manifold.
- If the nontrivial Floquet multipliers all lie off the unit circle (but not necessarily inside), then a version of the stable/unstable manifold theorem can be formulated. More or less routine (by now) modifications of the preceding argument could be used to prove it.

### 7.5. Existence of Periodic Solutions in $\mathbb{R}^n$ – Critical Case

We now return to the question of existence of periodic solutions to systems which are periodic perturbations of an autonomous system with a critical point. Although this topic has been raised in section 13 when the perturbation is noncritical, here we will be concerned with the case of a critical perturbation.

Once again we consider a one parameter family of nonautonomous vector fields  $f(t, x, \varepsilon)$ . Assume that

- (i)  $f : \mathbb{R} \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$  is  $C^\infty$ .
- (ii) There is a  $T > 0$ , such that  $f(t + T, x, \varepsilon) = f(t, x, \varepsilon)$  for all  $(t, x, \varepsilon) \in \mathbb{R} \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0)$ .
- (iii)  $f(t, x, 0) = f_0(x)$  is autonomous and  $f_0(0) = 0$ .
- (iv) All solutions of  $x' = Ax$  are  $T$ -periodic, where  $A = Df(0)$ .

In order to avoid counting derivatives, we lazily assume that the vector field is infinitely differentiable in (i).

Of interest here is the last condition which is what is meant by a critical perturbation. Recall that before we assumed that the eigenvalues,  $\lambda$ , of  $A$  satisfy  $\lambda T \notin 2\pi\mathbb{Z}$ , and so no solution of  $x' = Ax$  is  $T$ -periodic, see Theorem 7.1.1. Admittedly, some ground is left uncovered between these two extremes. Although we will not discuss what happens when some, but not all, solutions of  $x' = Ax$  are  $T$ -periodic, a combination of the two approaches would yield a result here, as well.

There are various statements that are equivalent to assumption (iv). One is that  $\exp A(t + T) = \exp At$  for all  $t \in \mathbb{R}$ . It is also equivalent to saying that the set of eigenvalues of  $A$  are contained in  $(2\pi i/T)\mathbb{Z}$ , the set of integer multiples of  $2\pi i/T$ , and that  $A$  has a basis of eigenvectors in  $\mathbb{C}^n$ .

In the following paragraphs we transform the equation

$$(7.5.1) \quad x' = f(t, x, \varepsilon)$$

to the so-called Lagrange standard form, with which most books begin.

As in section 7.1, we can write

$$f(t, x, \varepsilon) = f_0(x) + \varepsilon \tilde{f}(t, x, \varepsilon), \quad \text{where} \quad \tilde{f}(t, x, \varepsilon) = \int_0^1 \frac{\partial f}{\partial \varepsilon}(t, x, \sigma\varepsilon) d\sigma.$$

Notice that  $\tilde{f}$  is  $C^\infty$  and  $\tilde{f}(t+T, x, \varepsilon) = \tilde{f}(t, x, \varepsilon)$ . Moreover, let's write

$$f_0(x) = Ax + [f_0(x) - Ax] \equiv Ax + f_1(x).$$

We see that  $f_1 \in C^\infty$ ,  $f_1(0) = 0$ , and  $Df_1(0) = 0$ . Using integration by parts, we obtain another expression for the function  $f_1$ :

$$\begin{aligned} f_1(x) &= \int_0^1 \frac{d}{d\sigma} [f_1(\sigma x)] d\sigma \\ &= \sum_{i=1}^n \int_0^1 \frac{\partial f_1}{\partial x_i}(\sigma x) x_i d\sigma \\ &= \sum_{i=1}^n \left( \int_0^1 \frac{d}{d\sigma} \left[ \frac{\partial f_1}{\partial x_i}(\sigma x) \right] (1 - \sigma) d\sigma \right) x_i \\ &= \sum_{i,j=1}^n \left( \int_0^1 \frac{\partial^2 f_1}{\partial x_i \partial x_j}(\sigma x) (1 - \sigma) d\sigma \right) x_i x_j. \end{aligned}$$

Notice that the expressions enclosed in parentheses above are in  $C^\infty$ , since  $f_1 \in C^\infty$ .

Now suppose that  $x(t)$  is a solution of (7.5.1) and perform the rescaling

$$x(t) = \sqrt{\varepsilon} y(t).$$

Then

$$\begin{aligned} \sqrt{\varepsilon} y' &= x' \\ &= f(t, x, \varepsilon) \\ &= Ax + f_1(x) + \varepsilon \tilde{f}(t, x, \varepsilon) \\ &= \sqrt{\varepsilon} Ay + f_1(\sqrt{\varepsilon} y) + \varepsilon \tilde{f}(t, \sqrt{\varepsilon} y, \varepsilon) \\ &\equiv \sqrt{\varepsilon} Ay + \varepsilon g_1(y, \sqrt{\varepsilon}) + \varepsilon \tilde{g}(t, y, \sqrt{\varepsilon}), \end{aligned}$$

in which

$$g_1(y, \mu) = \sum_{i,j=1}^n \left( \int_0^1 \frac{\partial^2 f_1}{\partial x_i \partial x_j}(\sigma \mu y) (1 - \sigma) d\sigma \right) y_i y_j,$$

and

$$\tilde{g}(t, y, \mu) = \tilde{f}(t, \mu y, \mu^2).$$

Both  $g_1$  and  $\tilde{g}$  are  $C^\infty$  functions of their arguments. To summarize, we have transformed the original problem to

$$y' = Ay + \sqrt{\varepsilon} g(t, y, \sqrt{\varepsilon}),$$

with  $g$  in  $C^\infty$  and  $T$ -periodic in the “ $t$ ” variable.

As a final reduction, set  $\mu = \sqrt{\varepsilon}$  and  $y(t) = e^{At} z(t)$ . Then

$$z' = \mu h(t, z, \mu),$$

with  $h$  in  $C^\infty$ , and since  $e^{A(t+T)} = e^{At}$ , we have  $h(t+T, z, \mu) = h(t, z, \mu)$  for all  $(t, z, \mu)$ , with  $|\mu| < \sqrt{\varepsilon_0}$ . This is the standard form we were after.

Having reduced (7.5.1) to the standard form, let's go back to the original variables. We have

$$(7.5.2) \quad x' = \varepsilon f(t, x, \varepsilon)$$

with  $f$  in  $C^\infty$  and  $f(t+T, x, \varepsilon) = f(t, x, \varepsilon)$  for all  $(t, x, \varepsilon)$ , with  $|\varepsilon| < \sqrt{\varepsilon_0}$ .

We will use the so-called *method of averaging*, one of the central techniques used in the study of periodic systems, to compare solutions of (7.5.2) with those of

$$(7.5.3) \quad y' = \varepsilon \bar{f}(y),$$

where  $\bar{f}$  denotes the *averaged vector field*

$$(7.5.4) \quad \bar{f}(x) = \frac{1}{T} \int_0^T f(t, x, 0) dt.$$

(Recall that having changed coordinates,  $f(t, x, 0)$  is no longer autonomous.) We will show

**THEOREM 7.5.1.** *Suppose that the averaged vector field  $\bar{f}$  has a critical point at  $p_0 \in \mathbb{R}^n$ . Let  $B = D\bar{f}(p_0)$ , and assume that  $B$  is invertible. Then there is an  $\varepsilon_0 > 0$ , such that for all  $|\varepsilon| < \varepsilon_0$  the equation (7.5.2) has a unique  $T$ -periodic solution  $u_\varepsilon(t)$  near  $p_0$ .*

*If the eigenvalues of  $B$  lie in the negative half plane (so that  $p_0$  is asymptotically stable for (7.5.3)), then the Floquet multipliers of  $u_\varepsilon(t)$  lie inside the unit disk (i.e. the periodic solution  $u_\varepsilon(t)$  is asymptotically stable for (7.5.2)).*

*If the eigenvalues of  $B$  lie off the imaginary axis (so that  $p_0$  is hyperbolic for (7.5.3)), then the Floquet multipliers of  $u_\varepsilon(t)$  lie off the unit circle (i.e. the periodic solution  $u_\varepsilon(t)$  is hyperbolic for (7.5.2)).*

Before starting the proof of the theorem, we prepare the following lemma which lies behind a crucial change of coordinates.

**LEMMA 7.5.1.** *Suppose that  $w : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $C^k$  and that for some  $T > 0$ ,  $w(t+T, y) = w(t, y)$  for all  $(t, y) \in \mathbb{R} \times \mathbb{R}^n$ . Let  $D \subset \mathbb{R}^n$  be a bounded open set.*

*Then there exists an  $\varepsilon_0 > 0$  such that for all  $|\varepsilon| < \varepsilon_0$  and  $t \in \mathbb{R}$ , the mapping*

$$y \mapsto y + \varepsilon w(t, y)$$

*is a  $T$ -periodic family of  $C^k$  diffeomorphisms on  $D$ .*

*Moreover, the Jacobian matrix*

$$I + \varepsilon D_y w(t, y)$$

is nonsingular for all  $|\varepsilon| < \varepsilon_0$ ,  $t \in \mathbb{R}$ , and  $y \in D$ . Its inverse is  $C^{k-1}$  in  $(y, t, \varepsilon)$ , is  $T$ -periodic in  $t$ , and satisfies the estimates

$$\|(I + \varepsilon D_y w(t, y))^{-1}\| \leq \frac{1}{1 - \varepsilon/\varepsilon_0}$$

and

$$\|(I + \varepsilon D_y w(t, y))^{-1} - I\| \leq \frac{\varepsilon/\varepsilon_0}{1 - \varepsilon/\varepsilon_0}.$$

PROOF. Let  $B$  be a ball containing  $D$ . Set

$$M = \max\{\|D_y w(t, y)\| : (t, y) \in \mathbb{R} \times \bar{B}\},$$

and define  $\varepsilon_0 = M^{-1}$ . For all  $y_1, y_2 \in B$ , we have

$$\begin{aligned} w(t, y_1) - w(t, y_2) &= \int_0^1 \frac{d}{d\sigma} w(t, \sigma y_1 + (1 - \sigma)y_2) d\sigma \\ &= \int_0^1 D_y w(t, \sigma y_1 + (1 - \sigma)y_2)(y_1 - y_2) d\sigma. \end{aligned}$$

Note that  $\sigma y_1 + (1 - \sigma)y_2 \in B$  for all  $0 \leq \sigma \leq 1$ . Hence,

$$\|w(t, y_1) - w(t, y_2)\| \leq M\|y_1 - y_2\|.$$

So if  $y_1, y_2 \in B$  and  $|\varepsilon| < \varepsilon_0$ , we have

$$\begin{aligned} \|(y_1 + w(t, y_1)) - (y_2 + w(t, y_2))\| &\geq \|y_1 - y_2\| - \varepsilon\|w(t, y_1) - w(t, y_2)\| \\ &\geq (1 - \varepsilon M)\|y_1 - y_2\| \\ &= (1 - \varepsilon/\varepsilon_0)\|y_1 - y_2\|. \end{aligned}$$

This implies that the mapping

$$y \mapsto y + \varepsilon w(t, y)$$

is one-to-one on  $B$  for any  $|\varepsilon| < \varepsilon_0$  and  $t \in \mathbb{R}$ , and is therefore invertible.

Next, for any  $|\varepsilon| < \varepsilon_0$ ,  $t \in \mathbb{R}$ , and  $y \in D$ , consider the Jacobian

$$I + \varepsilon D_y w(t, y).$$

Since  $\|D_y w(t, y)\| \leq M = \varepsilon_0^{-1}$  and  $\varepsilon/\varepsilon_0 < 1$ , we have that the Jacobian is nonsingular.

By the inverse function theorem, the mapping  $y + \varepsilon w(t, y)$  is a local diffeomorphism in a neighborhood of any point  $y \in B$ . But since we have shown that this map is one-to-one on all of  $B$ , it is a diffeomorphism on all of  $B$ .

In general, when  $A$  in an  $n \times n$  matrix with  $\|A\| < 1$ , we know that  $(I + A)^{-1} = \sum_{k=0}^{\infty} (-A)^k$ . Thus,

$$\|(I + A)^{-1}\| \leq \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1 - \|A\|},$$

and

$$\|(I + A)^{-1} - 1\| \leq \sum_{k=1}^{\infty} \|A\|^k = \frac{\|A\|}{1 - \|A\|}.$$

Apply this with  $A = \varepsilon D_y w(t, y)$ . The estimates follow from the fact that  $\|A\| \leq \varepsilon/\varepsilon_0 < 1$ .

Finally, the smoothness of  $Z(t, y, \varepsilon) = (I + \varepsilon D_y w(t, y))^{-1}$  in  $(y, t, \varepsilon)$  follows from the implicit function theorem, since it solves the equation

$$(I + \varepsilon D_y w(t, y))Z - I = 0.$$

□

**PROOF OF THEOREM 7.5.1.** Having defined the averaged vector field  $\bar{f}$  in (7.5.4), set

$$\tilde{f}(t, x, \varepsilon) = f(t, x, \varepsilon) - \bar{f}(x).$$

Then  $\tilde{f}$  is  $C^\infty$ ,  $T$ -periodic, and

$$(7.5.5) \quad \int_0^T \tilde{f}(t, x, 0) dt = 0.$$

Define

$$(7.5.6) \quad w(t, y) = \int_0^t \tilde{f}(s, y, 0) ds.$$

Then  $w \in C^\infty$ , and thanks to (7.5.5),  $w(t + T, y) = w(t, y)$ . This function will be used to change coordinates.

Suppose that  $p_0$  is a critical point for  $\bar{f}$ . Let  $B_\delta(p_0)$  denote the ball of radius  $\delta$  centered at  $p_0$ . Set

$$\varepsilon_0^{-1} = M = \max\{\|D_y w(t, y)\| : (t, y) \in \mathbb{R} \times \bar{B}_\delta\}.$$

By the preceding lemma, we know that for all  $t \in \mathbb{R}$  and  $|\varepsilon| < \varepsilon_0$ , the mapping

$$F(y, t, \varepsilon) = y + \varepsilon w(t, y)$$

is a diffeomorphism on  $B_\delta(p_0)$ . If we further restrict  $\varepsilon_0$  so that

$$\varepsilon_0 \max\{\|w(t, y)\| : (t, y) \in \mathbb{R} \times \bar{B}_\delta\} < \delta/2,$$

then  $F(y, t, \varepsilon)$  maps  $B_\delta(p_0)$  onto an open set which contains  $B_{\delta/2}(p_0)$ , for all  $|\varepsilon| < \varepsilon_0$  and  $t \in \mathbb{R}$ , by the uniform contraction principle.

Let  $x(t)$  be any solution of (7.5.2) in  $B_{\delta/2}(p_0)$ . Then the formula

$$x(t) = y(t) + \varepsilon w(t, y(t))$$

defines a smooth curve  $y(t)$  in  $B_\delta(p_0)$ . We need to calculate the differential equation satisfied by  $y(t)$ . Using (7.5.2), we have

$$\begin{aligned} y'(t) + \varepsilon D_y w(t, y(t)) y'(t) + \varepsilon \frac{\partial w}{\partial t}(t, y(t)) \\ = x'(t) \\ = \varepsilon f(t, x(t), \varepsilon) \\ = \varepsilon \bar{f}(x(t)) + \varepsilon \tilde{f}(t, x(t), \varepsilon) \\ = \varepsilon \bar{f}(y(t) + \varepsilon w(t, y(t))) + \varepsilon \tilde{f}(t, y(t) + \varepsilon w(t, y(t)), \varepsilon). \end{aligned}$$

Notice that by (7.5.6),  $\frac{\partial w}{\partial t}(t, y(t)) = \tilde{f}(t, y(t), 0)$ , so that we have

$$\begin{aligned} [I + \varepsilon D_y w(t, y(t))] y'(t) \\ = \varepsilon \bar{f}(y(t) + \varepsilon w(t, y(t))) \\ + \varepsilon [\tilde{f}(t, y(t) + \varepsilon w(t, y(t)), \varepsilon) - \tilde{f}(t, y(t), 0)]. \end{aligned}$$

By Lemma 7.5.1, the matrix  $I + \varepsilon D_y w(t, y(t))$  is invertible, so this may be rewritten as

$$\begin{aligned} y' = \varepsilon \bar{f}(y) + \varepsilon ([I + \varepsilon D_y w(t, y)]^{-1} - I) \bar{f}(y) \\ + \varepsilon [I + \varepsilon D_y w(t, y)]^{-1} [\bar{f}(y + \varepsilon w(t, y)) - \bar{f}(y)] \\ + \varepsilon [I + \varepsilon D_y w(t, y)]^{-1} [\tilde{f}(t, y + \varepsilon w(t, y), \varepsilon) - \tilde{f}(t, y, 0)]. \end{aligned}$$

Now using the estimates given in the lemma and Taylor expansion, the last three terms on the right can be grouped and written in the form

$$\varepsilon^2 \hat{f}(t, y, \varepsilon),$$

with  $\hat{f}$  in  $C^\infty$  and  $T$ -periodic.

So now let's consider the initial value problem

$$(7.5.7) \quad y' = \varepsilon \bar{f}(y) + \varepsilon^2 \hat{f}(t, y, \varepsilon), \quad y(0) = y_0,$$

and let  $y(t, y_0, \varepsilon)$  denote its local solution. (By smooth dependence, it is  $C^\infty$  in its arguments.) As in section 7.1, we are going to look for periodic solutions as fixed points of the period  $T$  map, however the argument is a bit trickier now because the perturbation is critical.

Note that  $y(t, p, 0) = p$  is a global solution (the right hand side vanishes), so for  $|\varepsilon|$  sufficiently small,  $y(t, p, \varepsilon)$  is defined for  $0 \leq t \leq T$

for all  $p \in B_\delta(p_0)$ , by continuous dependence. For such  $p$  and  $\varepsilon$ , define

$$Q(p, \varepsilon) = \varepsilon^{-1}[y(T, p, \varepsilon) - p].$$

We are going to construct a curve of zeros  $p(\varepsilon)$  of  $Q(p, \varepsilon)$  using the implicit function theorem. Each zero corresponds to a  $T$ -periodic solution of (7.5.7), by periodicity of the vector field and uniqueness of solutions to the IVP.

As noted above,  $y(t, p, 0) = p$ . Thus, we have

$$Q(p, \varepsilon) = \varepsilon^{-1} \int_0^1 \frac{d}{d\sigma} y(T, p, \sigma\varepsilon) d\sigma = \int_0^1 \frac{\partial y}{\partial \varepsilon}(T, p, \sigma\varepsilon) d\sigma,$$

which shows that  $Q(p, \varepsilon)$  is  $C^\infty$  and also that

$$(7.5.8) \quad Q(p, 0) = \frac{\partial y}{\partial \varepsilon}(T, p, 0).$$

To further evaluate this expression, differentiate equation (7.5.7) with respect to  $\varepsilon$ :

$$\begin{aligned} \frac{d}{dt} \frac{\partial y}{\partial \varepsilon}(t, p, 0) &= \frac{\partial}{\partial \varepsilon} [\varepsilon \bar{f}(y(t, p, \varepsilon)) + \varepsilon^2 \hat{f}(y(t, p, \varepsilon))] \Big|_{\varepsilon=0} \\ &= \bar{f}(y(t, p, 0)) = \bar{f}(p), \end{aligned}$$

and

$$\frac{\partial y}{\partial \varepsilon}(0, p, 0) = \frac{\partial}{\partial \varepsilon} p \Big|_{\varepsilon=0} = 0.$$

This, of course, is easily solved to produce

$$\frac{\partial y}{\partial \varepsilon}(T, p, 0) = T \bar{f}(p),$$

which when combined with (7.5.8) gives us

$$Q(p, 0) = T \bar{f}(p).$$

and hence, in particular,

$$Q(p_0, 0) = T \bar{f}(p_0) = 0.$$

On the other hand, we have

$$D_p Q(p_0, 0) = D_p Q(p, 0)|_{p=p_0} = T D_p \bar{f}(p_0) = TB,$$

which is nonsingular, by assumption.

So by the implicit function theorem, there is a  $C^1$  curve  $p(\varepsilon)$  defined near  $\varepsilon = 0$  such that

$$p(0) = 0 \quad \text{and} \quad Q(p(\varepsilon), \varepsilon) = 0.$$

Moreover, if  $q \in B_\delta(p_0)$  and  $Q(q, \varepsilon) = 0$ , then  $q = p(\varepsilon)$ .

Thus, we have constructed a unique family  $y_\varepsilon(t) = y(t, p(\varepsilon), \varepsilon)$  of  $T$ -periodic solutions of (7.5.7) near  $p_0$ . By continuous dependence, the Floquet exponents of  $y_\varepsilon$  are close to the eigenvalues of  $\varepsilon TB$ , for  $\varepsilon$  small. This gives the statements on stability. Finally, the analysis is carried over to the original equation (7.5.2) by our change of variables.  $\square$

**Example: Duffing's Equation.** We can illustrate the ideas with the example of Duffing's equation

$$u'' + u + \varepsilon\beta u + \varepsilon\gamma u^3 = \varepsilon F \cos t,$$

which models the nonlinear oscillations of a spring. Here  $\beta$ ,  $\gamma$ , and  $F$  are fixed constants, and  $\varepsilon$  is a small parameter. Notice that when  $\varepsilon = 0$ , the unperturbed equation is

$$u'' + u = 0$$

solutions of which all have period  $2\pi$ , the same period as the forcing term.

In first order form, the system looks like

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - \varepsilon\beta x_1 - \varepsilon\gamma x_1^3 + \varepsilon F \cos t \end{bmatrix}.$$

The vector field is smooth and  $2\pi$ -periodic in  $t$ . When  $\varepsilon = 0$ , the system is linear with a critical point at  $x = 0$ :

$$x' = Ax, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

All solutions are  $2\pi$ -periodic.

We can express the system as

$$x' = Ax + \varepsilon f(t, x),$$

with

$$f(t, x) = \begin{bmatrix} 0 \\ -\beta x_1 - \gamma x_1^3 + F \cos t \end{bmatrix}.$$

Since the unperturbed system is linear, reduction to standard form is easily achieved without the rescaling step. Set

$$x = e^{At}z.$$

Then

$$z' = \varepsilon e^{-At} f(t, e^{At}z).$$

Of interest are the critical points of the averaged vector field

$$\bar{f}(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-At} f(t, e^{At}z) dt.$$

Evaluation of this formula is a rather messy, but straightforward calculation. Mathematica makes it pretty easy, however. We summarize the main steps, skipping over all of the algebra and integration. First, since

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

we have (recalling that  $f(t, x)$  depends only on  $x_1$ )

$$x_1 = z_1 \cos t + z_2 \sin t.$$

So we get

$$e^{-At} f(t, e^{At} z) = (-\beta(z_1 \cos t + z_2 \sin t) - \gamma(z_1 \cos t + z_2 \sin t)^3 + F \cos t) \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}.$$

Expanding this and averaging in  $t$ , we get

$$\bar{f}(z) = \frac{1}{2} \begin{bmatrix} (\beta + \frac{3}{4}\gamma r^2)z_2 \\ -(\beta + \frac{3}{4}\gamma r^2)z_1 + F \end{bmatrix}, \quad r^2 = z_1^2 + z_2^2.$$

From this, we see that there are nontrivial critical points when

$$\frac{3}{4}\gamma z_1^3 + \beta z_1 - F = 0, \quad \text{and} \quad z_2 = 0.$$

The first equation is called the frequency response curve. Suppose that  $\gamma > 0$ , this is the case of the so-called *hard* spring. Also let  $F > 0$ . Then there is always one positive root of the frequency response function for all  $\beta$ .

If we set  $\beta_0 = -(3/2)^{4/3} F^{2/3} \gamma^{1/3}$ , then for  $\beta < \beta_0$  there are an additional pair of negative roots of the frequency response function. This picture is the same for all  $F, \gamma > 0$ .

In order to apply Theorem 7.5.1, we need to verify that  $D\bar{f}$  is nonsingular at the equilibria determined by the frequency response function. Since  $z_2 = 0$ , it follows that at the critical points of  $\bar{f}$  we have

$$D\bar{f} = \begin{bmatrix} 0 & (\beta + \frac{3}{4}\gamma z_1^2) \\ -(\beta + \frac{3}{4}\gamma z_1^2) - \frac{3}{2}\gamma z_1^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & F/z_1 \\ (2\beta z_1 - 3F)/z_1 & 0 \end{bmatrix}.$$

This matrix is nonsingular provided that  $2\beta z_1 - 3F \neq 0$  for the zeros of the frequency response equation. This is indeed the case when  $\beta \neq \beta_0$  because then  $z_1 = 3F/2\beta$  is not a zero of the frequency response equation.

It follows that there are either one or three periodic orbits of the original equation near the origin depending on whether  $\beta > \beta_0$  or  $\beta < \beta_0$ . If  $2\beta z_1 - 3F > 0$ , then the periodic orbit will be hyperbolic

with one dimensional stable and unstable manifolds. If  $2\beta z_1 - 3F < 0$ , then stability can not be determined by the Theorem.



## CHAPTER 8

### Center Manifolds and Bifurcation Theory

#### 8.1. The Center Manifold Theorem

**DEFINITION 8.1.1.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  vector field with  $F(0) = 0$ . A center manifold for  $F$  at 0 is an invariant manifold containing 0 which is tangent to the center subspace of  $DF(0)$  at 0.*

Assume  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^1$  vector field with  $F(0) = 0$ . Set  $A = DF(0)$ , and let  $E_s$ ,  $E_u$ , and  $E_c$  be its stable, unstable, and center subspaces with their corresponding projections  $P_s$ ,  $P_u$ , and  $P_c$ . Assume that  $E_c \neq 0$ . By Theorem 1.4.1 exist constants  $C_0$ ,  $\lambda > 0$ ,  $d \geq 0$  such that

$$\begin{aligned} \|e^{At}P_s\| &\leq C_0e^{-\lambda t}, \quad t \geq 0 \\ \|e^{At}P_u x\| &\leq C_0e^{\lambda t}, \quad t \leq 0 \\ \|e^{At}P_c x\| &\leq C_0(1 + |t|^d), \quad t \in \mathbb{R}. \end{aligned}$$

Write  $F(x) = Ax + f(x)$ . Then  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $C^1$ ,  $f(0) = 0$ , and  $Df(0) = 0$ . Moreover, we temporarily assume that

$$(8.1.1) \quad \sup_{x \in \mathbb{R}^n} (\|f(x)\| + \|Df(x)\|) \leq M.$$

This restriction will be removed later, at the expense of somewhat weakening the conclusions of the next result.

As usual, we denote by  $x(t, x_0)$  the solution of the initial value problem

$$x' = Ax + f(x), \quad x(0) = x_0.$$

Thanks to the strong bound assumed for the nonlinear portion of the vector field, the flow is globally defined for all initial points  $x_0 \in \mathbb{R}^n$ .

**THEOREM 8.1.1 (Center Manifold Theorem).** *Let the constant  $M$  in (8.1.1) be sufficiently small. There exists a  $C^1$  function  $\eta$  with the following properties:*

- (i)  $\eta : E_c \rightarrow E_s + E_u$ ,  $\eta(0) = 0$ , and  $D\eta(0) = 0$ .
- (ii) The set  $W_c(0) = \{x_0 \in \mathbb{R}^n : P_s x_0 + P_u x_0 = \eta(P_c x_0)\}$  is invariant under the flow.

(iii) If  $0 < \alpha < \lambda$ ,

$$\|P_s x(t, x_0)\| \leq C e^{-\alpha t}, \quad \text{for all } t < 0,$$

and

$$\|P_u x(t, x_0)\| \leq C e^{\alpha t}, \quad \text{for all } t > 0,$$

then  $x_0 \in W_c(0)$ .

(iv) If  $x_0 \in W_c(0)$ , then  $w(t) = P_c x(t, x_0)$  solves

$$w' = Aw + P_c f(w + \eta(w)), \quad w(0) = P_c x_0.$$

Remark: It follows from (i), (ii) that  $W_c(0)$  is a center manifold.

PROOF. Fix  $0 < \varepsilon < \lambda$ . Let

$$X_\varepsilon = \{y \in C(\mathbb{R}, \mathbb{R}^n) : \|y\|_\varepsilon \equiv \sup_t e^{-\varepsilon|t|} \|y(t)\| < \infty\}.$$

$X_\varepsilon$  is a Banach space with the norm  $\|\cdot\|_\varepsilon$ . Define a mapping  $T : E_c \times X_\varepsilon \rightarrow X_\varepsilon$  by

$$\begin{aligned} T(y_0, y)(t) &= e^{At} y_0 + \int_0^t e^{A(t-\tau)} P_c f(y(\tau)) d\tau \\ &\quad + \int_{-\infty}^t e^{A(t-\tau)} P_s f(y(\tau)) d\tau \\ &\quad - \int_t^{\infty} e^{A(t-\tau)} P_u f(y(\tau)) d\tau. \end{aligned}$$

The following estimate shows that  $T(y_0, y)$  is a well-defined function in  $X_\varepsilon$ . Let  $y_0 \in E_c$  and  $y \in X_\varepsilon$ . Then

$$\begin{aligned} \|T(y_0, y)(t)\| &\leq C_0(1 + |t|^d) \|y_0\| \\ &\quad + M \left| \int_0^t (1 + |t - \tau|^d) d\tau \right| \\ &\quad + M \int_{-\infty}^t e^{-\lambda(t-\tau)} d\tau \\ &\quad + M \int_t^{\infty} e^{\lambda(t-\tau)} d\tau \\ &\leq C e^{\varepsilon|t|}. \end{aligned}$$

If  $M$  is small enough, then  $T$  is a uniform contraction on  $X_\varepsilon$ . Given  $y_0 \in E_c$  and  $y, z \in X_\varepsilon$ , we have

$$\begin{aligned} \|T(y_0, y)(t) - T(y_0, z)(t)\| &\leq M \left| \int_0^t (1 + |t - \tau|^d) \|y(\tau) - z(\tau)\| d\tau \right| \\ &\quad + M \int_{-\infty}^t e^{-\lambda(t-\tau)} \|y(\tau) - z(\tau)\| d\tau \\ &\quad + M \int_t^\infty e^{\lambda(t-\tau)} \|y(\tau) - z(\tau)\| d\tau \\ &\leq CM e^{\varepsilon|t|} \|y - z\|_\varepsilon. \end{aligned}$$

Thus,

$$\|T(y_0, y) - T(y_0, z)\|_\varepsilon \leq CM \|y - z\|_\varepsilon \leq (1/2) \|y - z\|_\varepsilon,$$

for  $M$  small.

It follows from the uniform contraction principle that for every  $y_0 \in E_c$  there is a unique fixed point  $\psi(y_0) \in X_\varepsilon$ :

$$T(y_0, \psi(y_0)) = \psi(y_0).$$

The assumptions on  $f(x)$  also imply that  $T$  is  $C^1$ , and so  $\psi : E_c \rightarrow X_\varepsilon$  is a  $C^1$  mapping. For notational convenience we will write  $y(t, y_0) = \psi(y_0)(t)$ . Note that  $y(t, y_0)$  is  $C^1$  in  $y_0$ . Since  $y(t, y_0)$  is a fixed point, we have explicitly

$$\begin{aligned} (8.1.2) \quad y(t, y_0) &= e^{At} y_0 + \int_0^t e^{A(t-\tau)} P_c f(y(\tau, y_0)) d\tau \\ &\quad + \int_{-\infty}^t e^{A(t-\tau)} P_s f(y(\tau, y_0)) d\tau \\ &\quad - \int_t^\infty e^{A(t-\tau)} P_u f(y(\tau, y_0)) d\tau. \end{aligned}$$

Now define

$$(8.1.3) \quad \eta(y_0) = (I - P_c)y(0, y_0) = \int_{-\infty}^0 e^{-A\tau} P_s f(y(\tau, y_0)) d\tau - \int_0^\infty e^{A\tau} P_u f(y(\tau, y_0)) d\tau.$$

$\eta : E_c \rightarrow E_s + E_u$  is  $C^1$ , and since  $y(t, 0) = 0$  we have that  $\eta(0) = 0$  and  $D\eta(0) = 0$ . Thus, the function  $\eta$  fulfills the requirements of (i).

Define  $W_c(0) = \{x_0 \in \mathbb{R}^n : (P_s + P_u)x_0 = \eta(P_c x_0)\}$ .

As a step towards proving invariance, we verify the property

$$(8.1.4) \quad y(t + s, y_0) = y(t, P_c y(s, y_0)).$$

Fix  $s \in \mathbb{R}$  and set  $z(t) = y(t + s, y_0)$ . Then from (8.1.2), we have

$$\begin{aligned}
z(t) &= y(t + s, y_0) \\
&= e^{A(t+s)}y_0 + \int_0^{t+s} e^{A(t+s-\tau)} P_c f(y(\tau, y_0)) d\tau \\
&\quad + \int_{-\infty}^{t+s} e^{A(t+s-\tau)} P_s f(y(\tau, y_0)) d\tau \\
&\quad - \int_{t+s}^{\infty} e^{A(t+s-\tau)} P_u f(y(\tau, y_0)) d\tau \\
&= e^{A(t+s)}y_0 \\
&\quad + \int_0^s e^{A(t+s-\tau)} P_c f(y(\tau, y_0)) d\tau \\
&\quad + \int_s^{t+s} e^{A(t+s-\tau)} P_c f(y(\tau, y_0)) d\tau \\
&\quad + \int_{-\infty}^{t+s} e^{A(t+s-\tau)} P_s f(y(\tau, y_0)) d\tau \\
&\quad - \int_{t+s}^{\infty} e^{A(t+s-\tau)} P_u f(y(\tau, y_0)) d\tau
\end{aligned}$$

Now factor  $e^{At}$  out of the first two terms, and make the change of variables  $\sigma = \tau - s$  in the last three integrals. This results in

$$\begin{aligned}
z(t) &= e^{At} \left[ e^{As} P_c y_0 + \int_0^s e^{A(s-\tau)} P_c f(y(\tau, y_0)) d\tau \right] \\
&\quad + \int_0^t e^{A(t-\sigma)} P_c f(y(\sigma + s, y_0)) d\sigma \\
&\quad + \int_{-\infty}^t e^{A(t-\sigma)} P_s f(y(\sigma + s, y_0)) d\sigma \\
&\quad - \int_t^{\infty} e^{A(t-\sigma)} P_u f(y(\sigma + s, y_0)) d\sigma \\
&= e^{At} P_c y(s, y_0) + \int_0^t e^{A(t-\sigma)} P_c f(z(\sigma)) d\sigma \\
&\quad + \int_{-\infty}^t e^{A(t-\sigma)} P_s f(z(\sigma)) d\sigma \\
&\quad - \int_t^{\infty} e^{A(t-\sigma)} P_u f(z(\sigma)) d\sigma \\
&= T(P_c y(s, y_0), z)(t).
\end{aligned}$$

By uniqueness of fixed points, (8.1.4) follows.

Notice that from (8.1.2) and (8.1.3), we have that

$$y(t, y_0) = e^{At}[y_0 + \eta(y_0)] + \int_0^t e^{A(t-\tau)} f(y(\tau, y_0)) d\tau = x(t, y_0 + \eta(y_0)).$$

Thus,  $x_0 \in W_c(0)$  if and only if  $y(t, P_c x_0) = x(t, x_0)$ .

Let  $x_0 \in W_c(0)$ . Then using (8.1.4) and then (8.1.3), we obtain

$$\begin{aligned} (P_s + P_u)x(t, x_0) &= (P_s + P_u)y(t, P_c x_0) \\ &= (P_s + P_u)y(0, P_c y(t, P_c x_0)) \\ &= (I - P_c)y(0, P_c y(t, P_c x_0)) \\ &= \eta(P_c y(t, P_c x_0)) \\ &= \eta(P_c x(t, x_0)). \end{aligned}$$

This proves invariance of the center manifold (ii).

Let  $x_0 \in \mathbb{R}^n$  be a point such that

$$(8.1.5) \quad \begin{aligned} \|P_s x(t, x_0)\| &\leq C e^{-\alpha t}, \quad \text{for all } t < 0, \\ \|P_u x(t, x_0)\| &\leq C e^{\alpha t}, \quad \text{for all } t > 0, \end{aligned}$$

for some  $\alpha < \lambda$ . From the linear estimates and (8.1.5), we get for  $t \leq 0$

$$\begin{aligned} \|e^{-At} P_s x(t, x_0)\| &\leq C_0 e^{\lambda t} \|P_s x(t, x_0)\| \\ &\leq C e^{(\lambda-\alpha)t} \end{aligned}$$

and for  $t \geq 0$

$$\begin{aligned} \|e^{-At} P_u x(t, x_0)\| &\leq C_0 e^{-\lambda t} \|P_u x(t, x_0)\| \\ &\leq C e^{(\alpha-\lambda)t}. \end{aligned}$$

Next, from the variation of parameters formula,

$$x(t, x_0) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} f(x(\tau, x_0)) d\tau,$$

it follows that

$$P_s x(t, x_0) = e^{At} \left[ P_s x_0 + \int_0^t e^{-A\tau} P_s f(x(\tau, x_0)) d\tau \right],$$

and

$$P_u x(t, x_0) = e^{At} \left[ P_u x_0 + \int_0^t e^{-A\tau} P_u f(x(\tau, x_0)) d\tau \right].$$

Combining these formulas with the previous estimates, we get

$$\left\| P_s x_0 + \int_0^t e^{-A\tau} P_s f(x(\tau, x_0)) d\tau \right\| \leq C e^{(\lambda-\alpha)t}, \quad \text{for all } t < 0,$$

and

$$\left\| P_u x_0 + \int_0^t e^{-A\tau} P_u f(x(\tau, x_0)) d\tau \right\| \leq C e^{(\alpha-\lambda)t}, \quad \text{for all } t > 0.$$

Hence, if we send  $t \rightarrow -\infty$  in the first inequality and  $t \rightarrow \infty$  in the second, we obtain

$$P_s x_0 + \int_0^{-\infty} e^{-A\tau} P_s f(x(\tau, x_0)) d\tau = 0,$$

and

$$P_u x_0 + \int_0^{\infty} e^{-A\tau} P_u f(x(\tau, x_0)) d\tau = 0.$$

It follows that  $x(t, x_0)$  solves the integral equation (8.1.2) with  $y_0 = P_c x_0$ . By uniqueness of fixed points, we have  $x(t, x_0) = y(t, P_c x_0)$ . This proves (iii) that  $x_0 \in W_c(0)$ .

Finally, let  $x_0 \in W_c(0)$  and set  $w(t) = P_c x(t, x_0)$ . Since the center manifold is invariant under the flow, we have that

$$x(t, x_0) = w(t) + \eta(w(t)).$$

Multiply the differential equation by  $P_c$  to get

$$\begin{aligned} w'(t) &= P_c x'(t, x_0) = P_c [Ax(t, x_0) + f(x(t, x_0))] \\ &= Aw(t) + P_c f(w(t) + \eta(w(t))), \end{aligned}$$

with initial condition

$$w(0) = P_c x(t, x_0) = P_c x_0.$$

This is (iv). □

Without the smallness restriction (8.1.1), we obtain the following weaker result:

**COROLLARY 8.1.1 (Local Center Manifold Theorem).** *Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $C^1$  with  $f(0) = 0$  and  $Df(0) = 0$ . There exists a  $C^1$  function  $\eta$  and a small neighborhood  $U = B_r(0) \subset \mathbb{R}^n$  with the following properties:*

- (i)  $\eta : E_c \rightarrow E_s + E_u$ ,  $\eta(0) = 0$ , and  $D\eta(0) = 0$ .
- (ii) The set  $W_c^{loc}(0) = \{x_0 \in U : P_s x_0 + P_u x_0 = \eta(P_c x_0)\}$  is invariant under the flow in the sense that if  $x_0 \in W_c^{loc}(0)$ , then  $x(t, x_0) \in W_c^{loc}(0)$  as long as  $x(t, x_0) \in U$ .
- (iii) If  $x_0 \in W_c(0)$ , then  $w(t) = P_c x(t, x_0)$  solves

$$w' = Aw + P_c f(w + \eta(w)), \quad w(0) = P_c x_0,$$

as long as  $x(t, x_0) \in U$ .

**DEFINITION 8.1.2.** A set  $W_c^{loc}(0)$  which satisfies (i) and (ii) is called a local center manifold.

**PROOF OF COROLLARY 8.1.1.** By choosing  $r > 0$  sufficiently small, we can find  $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $f(x) = \tilde{f}(x)$  for all  $x \in B_r(0) = U$  and (8.1.1) holds for  $\tilde{f}$ , with  $M$  as small as we please.

Let  $x(t, x_0)$ ,  $\tilde{x}(t, x_0)$  be the flows of  $Ax + f(x)$ ,  $Ax + \tilde{f}(x)$ , respectively. By uniqueness, we have that  $x(t, x_0) = \tilde{x}(t, x_0)$ , as long as  $x(t, x_0) \in U$ .

Fix the matrix  $A$ . Choose  $r$  (and hence  $M$ ) sufficiently small so that the Center Manifold Theorem applies for  $\tilde{x}(t, x_0)$ . The conclusions of the corollary follow immediately for  $x(t, x_0)$ .  $\square$

**Example: Nonuniqueness of the Center Manifold.** The smallness condition (8.1.1) lead to the intrinsic characterization of a center manifold given by (iii) in the center manifold theorem. However, the following example illustrates that there may exist other center manifolds.

Consider the  $2 \times 2$  system

$$x_1' = -x_1^3, \quad x_2' = -x_2; \quad x_1(0) = \alpha, \quad x_2(0) = \beta$$

which has the form

$$x' = Ax + f(x),$$

with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} -x_1^3 \\ 0 \end{bmatrix}.$$

The eigenvalues of  $A$  are  $\{0, -1\}$ , and  $A$  has one-dimensional center and stable subspaces spanned by the standard unit vectors  $e_1$  and  $e_2$ , respectively. The projections are given by

$$P_c x = P_c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \quad \text{and} \quad P_s x = P_s \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}.$$

Note that  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is  $C^1$  with  $f(0) = 0$  and  $Df(0) = 0$ .

This system is easily solved, of course, since it is uncoupled. We have

$$\begin{aligned} x_1(t, \alpha, \beta) &= \frac{\alpha}{\sqrt{1 + 2\alpha^2 t}}, \\ x_2(t, \alpha, \beta) &= \beta e^{-t}. \end{aligned}$$

Notice that

$$(8.1.6) \quad x_2 \exp\left(\frac{1}{2x_1^2}\right) = \beta \exp\left(\frac{1}{2\alpha^2}\right).$$

Let  $c_1, c_2 \in \mathbb{R}$  be arbitrary. Define the  $C^\infty$  function

$$\xi(s) = \begin{cases} c_1 \exp\left(-\frac{1}{2s^2}\right), & s < 0 \\ 0, & s = 0 \\ c_2 \exp\left(-\frac{1}{2s^2}\right), & s > 0. \end{cases}$$

Use the function  $\xi$  to obtain a mapping  $\eta : E_c \rightarrow E_s$  by

$$\eta(x_1 e_1) = \eta\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \xi(x_1) \end{bmatrix} = \xi(x_1) e_2.$$

Note that  $\eta$  is  $C^1$ ,  $\eta(0) = 0$ , and  $D\eta(0) = 0$ . Let

$$W_\xi = \{x \in \mathbb{R}^2 : P_s x = \eta(P_c x)\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \xi(x_1)\}.$$

Suppose that  $x(0) = (\alpha, \beta) \in W_c(0)$ . Then  $\beta = \xi(\alpha)$ . It follows from (8.1.6) that

$$x_2 \exp\left(\frac{1}{2x_1^2}\right) = \begin{cases} c_1, & \text{if } \alpha < 0 \\ c_2, & \text{if } \alpha > 0. \end{cases}$$

We have shown that  $W_\xi$  is invariant, and therefore it is a local center manifold.

**THEOREM 8.1.2** (Approximation of the Center Manifold). *Let  $U \subset E_c$  be a neighborhood of the origin. Let  $h : U \rightarrow E_s + E_u$  be a  $C^1$  mapping with  $h(0) = 0$  and  $Dh(0) = 0$ . If for  $x \in U$ ,*

$$Ah(x) + (P_s + P_u)f(x + h(x)) - Dh(x)[Ax + P_c f(x + h(x))] = O(\|x\|^k),$$

*as  $\|x\| \rightarrow 0$ , then there is a  $C^1$  mapping  $\eta : E_c \rightarrow E_s + E_u$  with  $\eta(0) = 0$  and  $D\eta(0) = 0$  such that*

$$\eta(x) - h(x) = O(\|x\|^k),$$

*as  $\|x\| \rightarrow 0$ , and*

$$\{x + \eta(x) : x \in U\}$$

*is a local center manifold.*

A proof of this result can be found in the book of Carr. The motivation for the result is the same as in the case of the stable manifold. If  $\eta$  defines a local center manifold, then

$$A\eta(x) + (P_s + P_u)f(x + \eta(x)) - D\eta(x)[Ax + P_c f(x + \eta(x))] = 0 \quad x \in U.$$

**Remark:** In the case where the local center manifold has a Taylor expansion near the origin, this formula can be used to compute coefficients. See Guckenheimer and Holmes (p. 133) for an example in which this does not work.

### 8.2. The Center Manifold as an Attractor

**DEFINITION 8.2.1.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set and let  $F : \Omega \rightarrow \mathbb{R}^n$  be an autonomous vector field with flow  $x(t, x_0)$ . A set  $A \subset \Omega$  is a local attractor for the flow if there exists an open set  $U$  with  $A \subset U \subset \Omega$  such that for all  $x_0 \in U$ ,  $x(t, x_0)$  is defined for all  $t \geq 0$  and  $\text{dist}(x(t, x_0), A) \rightarrow 0$  as  $t \rightarrow \infty$ .*

Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $C^1$  autonomous vector field with  $F(0) = 0$ . Write  $F(x) = Ax + f(x)$  with  $A = DF(0)$ . Suppose that the eigenvalues of  $A$  all satisfy  $\text{Re } \lambda \leq 0$ . Let

$$W_c^{loc}(0) = \{x \in B_\varepsilon(0) : P_c x \in B_\varepsilon(0) \cap E_c, P_s x = \eta(P_c x)\}$$

be a local center manifold for

$$(8.2.1) \quad x' = Ax + f(x).$$

As usual, denote by  $x(t, x_0)$  the (local) solution of (8.2.1) with initial data  $x(0, x_0) = x_0$ . Let  $w(t, w_0)$  be the (local) solution of the reduced flow on  $W_c^{loc}(0)$

$$(8.2.2) \quad w' = Aw + P_c f(w + \eta(w)),$$

with  $w(0, w_0) = w_0 \in B_\varepsilon(0) \cap E_c$ .

**THEOREM 8.2.1.** *Assume that  $w = 0$  is a stable fixed point for (8.2.2). There is a neighborhood of the origin  $V \subset B_\varepsilon(0) \subset \mathbb{R}^n$  with the property that for every  $x_0 \in V$ ,  $x(t, x_0)$  is defined and remains in  $B_\varepsilon(0)$ , for all  $t \geq 0$ , and there corresponds a unique  $w_0 \in V \cap E_c$  such that*

$$\|x(t, x_0) - w(t, w_0) - \eta(w(t, w_0))\| \leq C e^{-\beta t},$$

for all  $t \geq 0$ , where  $0 < \beta < \lambda_s$ .

Remarks:

- (1) The theorem implies, in particular, that  $x = 0$  is a stable fixed point for (8.2.1) and that  $W_c^{loc}(0)$  is a local attractor for (8.2.1).
- (2) The theorem holds with the word “stable” replaced by “asymptotically stable”. Corresponding “unstable” versions are also true as  $t \rightarrow -\infty$ .

**PROOF.** The proof works more smoothly if instead of the usual Euclidean norm on  $\mathbb{R}^n$ , we use the norm  $\|x\| = \|P_s x\| + \|P_c x\|$ . Then  $\|x\| < a$  implies that  $\|P_s x\| < a$  and  $\|P_c x\| < a$ . Alternatively, we could change coordinates so that  $E_s \perp E_c$ .

Thanks to the stability assumption, there is a  $\delta_1 < \varepsilon$  such that if  $w_0 \in B_{\delta_1}(0) \cap E_c$ , then  $w(t, w_0) \in B_{\varepsilon/4}(0)$ , for all  $t \geq 0$ . By continuity

of  $\eta$ , we may assume, moreover, that  $\delta_1$  has been chosen small enough so that  $\eta(w(t, w_0)) \in B_{\varepsilon/4}(0)$ , for all  $t \geq 0$ .

Let  $M/2 = \max\{\|Df(x)\| : x \in B_\varepsilon(0)\}$ . By changing  $f$  outside of  $B_\varepsilon(0)$ , we may assume that  $\|Df(x)\| \leq M$ , for all  $x \in \mathbb{R}^n$ . By uniqueness of solutions to the initial value problem, the modified flow is same as the original flow in  $B_\varepsilon(0)$ . Since  $W_c^{loc}(0) \subset B_\varepsilon(0)$ ,  $W_c^{loc}(0)$  remains a local center manifold for the modified flow.

Define the  $C^1$  function  $g : B_\varepsilon(0) \cap E_c \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$g(w, z) = f(z + w + \eta(w)) - f(w + \eta(w)).$$

Then for all  $w \in B_\varepsilon(0) \cap E_c$  and  $z \in \mathbb{R}^n$ ,

$$(8.2.3) \quad g(w, 0) = 0, \quad \|D_z g(w, z)\| \leq M, \quad \|g(w, z)\| \leq M\|z\|.$$

Given  $x_0 \in \mathbb{R}^n$  and  $w_0 \in B_{\delta_1}(0) \cap E_c$ , set

$$z(t) = x(t, x_0) - w(t, w_0) - \eta(w(t, w_0)).$$

Since both  $x(t, x_0)$  and  $w(t, w_0) + \eta(w(t, w_0))$  solve the differential equation (8.2.1),  $z(t)$  solves

$$(8.2.4) \quad z'(t) = Az(t) + g(w(t, w_0), z(t)), \quad w_0 \in B_{\delta_1}(0) \cap E_c$$

with data

$$z(0) = x_0 - w_0 - \eta(w_0).$$

Conversely, if  $z(t)$  is a solution of (8.2.4) in  $B_{\varepsilon/2}(0)$ , then

$$x(t) = z(t) + w(t, w_0) + \eta(w(t, w_0))$$

is a solution of (8.2.1) in  $B_\varepsilon(0)$  with data  $x(0) = z(0) + w_0 + \eta(w_0)$ .

The strategy will be to construct exponentially decaying solutions of (8.2.4) and then to jiggle the initial data appropriately.

Given  $0 < \beta < \lambda_s$ , define the set of functions

$$X = \{z \in C(\mathbb{R}^+, \mathbb{R}^n) : \|z\|_\beta \equiv \sup_{t \geq 0} e^{\beta t} \|z(t)\| < \infty\}.$$

$X$  is a Banach space with norm  $\|\cdot\|_\beta$ .

Now if  $z_0 \in B_{\delta_1}(0)$ , then  $w_0 = P_c z_0 \in B_{\delta_1}(0) \cap E_c$ . Thus,  $w(t, w_0) \in B_{\varepsilon/4}(0)$ ,  $\eta(w(t, w_0))$  is defined, and also remains in  $B_{\varepsilon/4}(0)$ . Next, if  $z \in X$ , then by the properties (8.2.3), we have that  $\|g(w(\tau, w_0), z(\tau))\| \leq M\|z\|_\beta e^{-\beta\tau}$ . It follows that

$$\begin{aligned} \left\| \int_0^t e^{A(t-\tau)} P_s g(w(\tau, w_0), z(\tau)) d\tau \right\| &\leq C\|z\|_\beta \int_0^t e^{-\lambda_s(t-\tau)} e^{-\beta\tau} d\tau \\ &\leq C e^{-\beta t} \|z\|_\beta, \end{aligned}$$

and

$$\begin{aligned}
\left\| \int_t^\infty e^{A(t-\tau)} P_c g(w(\tau, w_0), z(\tau)) d\tau \right\| &\leq C \|z\|_\beta \int_t^\infty (1 + \tau - t)^d e^{-\beta\tau} d\tau \\
&= C \|z\|_\beta \int_0^\infty (1 + \sigma)^d e^{-\beta(\sigma+t)} d\sigma \\
&= C e^{-\beta t} \|z\|_\beta.
\end{aligned}$$

From this we see that the mapping

$$\begin{aligned}
T(z_0, z)(t) &= z(t) - e^{At} P_s z_0 - \int_0^t e^{A(t-\tau)} P_s g(w(\tau, P_c z_0), z(\tau)) d\tau \\
&\quad + \int_t^\infty e^{A(t-\tau)} P_c g(w(\tau, P_c z_0), z(\tau)) d\tau.
\end{aligned}$$

is well-defined and that  $T : B_{\delta_1}(0) \times X \rightarrow X$ .  $T$  is a  $C^1$  mapping,  $T(0, 0) = 0$ , and  $D_z T(0, 0) = I$ . The implicit function theorem says that there exists  $\delta_2 < \delta_1$  and a  $C^1$  map  $\phi : B_{\delta_2}(0) \rightarrow X$  such that

$$\phi(0) = 0, \quad \text{and} \quad T(z_0, \phi(z_0)) = 0, \quad \text{for all } z_0 \in B_{\delta_2}(0).$$

Set  $z(t, z_0) = \phi(z_0)(t)$ . By continuity, we may assume that  $\delta_2$  is so small that  $\|z(t, z_0)\|_\beta < \varepsilon/2$ , for all  $z_0 \in B_{\delta_2}(0)$ .

The function  $z(t, z_0)$  is an exponentially decaying solution of (8.2.4) which remains in  $B_{\varepsilon/2}(0)$ . Thus, we have that

$$x(t) = z(t, z_0) + w(t, w_0) + \eta(w(t, w_0))$$

is a solution of (8.2.1) in  $B_\varepsilon(0)$  with initial data

$$F(z_0) = z(0, z_0) + P_c z_0 + \eta(P_c z_0).$$

It remains to show that given any  $x_0 \in \mathbb{R}^n$  close to the origin, we can find a  $z_0 \in B_{\delta_2}(0)$  such that  $x_0 = F(z_0)$ .

Notice that  $F : B_{\delta_2}(0) \rightarrow B_\varepsilon(0)$  is  $C^1$ ,  $F(0) = 0$ , and  $DF(0) = I$ . So by the inverse function theorem, there are neighborhoods of the origin  $V \subset B_{\delta_2}(0)$  and  $W \subset B_\varepsilon(0)$  such that  $F : V \rightarrow W$  is a diffeomorphism. Therefore, given any  $x_0 \in W$ , there is a  $z_0 = F^{-1}(x_0) \in V$ . Then  $w_0 = P_c z_0 \in B_{\delta_2}(0) \cap E_c$ , and

$$\|x(t) - w(t, w_0) - \eta(w(t, w_0))\| = \|z(t, z_0)\| < (\varepsilon/2)e^{-\beta t}.$$

□

**Example: The Lorentz Equation.** Let's take a look at the following system, known as the Lorentz equation,

$$\begin{aligned}x'_1 &= -\sigma x_1 + \sigma x_2 \\x'_2 &= x_1 - x_2 + x_1 x_3 \quad \beta, \sigma > 0 \\x'_3 &= -\beta x_3 + x_1 x_2.\end{aligned}$$

This can be expressed in the general form

$$x' = Ax + f(x)$$

with

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}, \quad \text{and} \quad f(x) = \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix}.$$

The eigenvalues of  $A$  are  $\{0, -(\sigma + 1), -\beta\}$ , and so  $A$  has a one-dimensional center subspace and a two-dimensional stable subspace. Since  $f(x)$  is a  $C^1$  map with  $f(0) = 0$  and  $Df(0) = 0$ , the center subspace goes over to a one-dimensional local center manifold  $W_c^{loc}(0)$  for the nonlinear equation. We are going to approximate the flow on  $W_c^{loc}(0)$ . We will show that the origin is an asymptotically stable critical point for the flow on  $W_c^{loc}(0)$ . It follows that  $W_c^{loc}(0)$  is a local attractor.

To simplify the analysis, we use the eigenvectors of  $A$  to change coordinates. Corresponding to the eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = -(\sigma + 1), \quad \lambda_3 = -\beta$$

the eigenvectors of  $A$  are:

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -\sigma \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If  $S$  is the matrix whose columns are  $u_1$ ,  $u_2$ , and  $u_3$ , then  $AS = SB$  with  $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ .

Make the change of variables  $x = Sy$ . Then

$$Sy' = x' = Ax + f(x) = ASy + f(Sy),$$

so that

$$y' = S^{-1}ASy + S^{-1}f(Sy) = By + g(y),$$

with

$$g(y) = S^{-1}f(Sy) = \begin{bmatrix} -\frac{\sigma}{1+\sigma}(y_1 - y_2)y_3 \\ -\frac{1}{1+\sigma}(y_1 - \sigma y_2)y_3 \\ (y_1 - \sigma y_2)(y_1 + y_2) \end{bmatrix}.$$

Of course we still have  $g(0) = 0$  and  $Dg(0) = 0$ . Since the equations are now diagonal, we have

$$\begin{aligned} E_c &= \text{span}\{e_1\} & E_s &= \text{span}\{e_2, e_3\} \\ P_c y &= y_1 e_1 & P_s y &= y_2 e_2 + y_3 e_3. \end{aligned}$$

This means that the local center manifold

$$W_c^{loc}(0) = \{y \in \mathbb{R}^n : P_c y \in U, P_s y = \eta(P_c y)\},$$

is determined by a function  $\eta : E_c \rightarrow E_s$  which must have the form

$$\eta(y) = \eta(y_1 e_1) = \eta_2(y_1) e_2 + \eta_3(y_1) e_3, \quad y = y_1 e_1 \in E_c.$$

To approximate  $\eta$ , use the equation

$$B\eta(y) + P_s g(y + \eta(y)) - D\eta(y)[By + P_c g(y + \eta(y))] = 0, \quad y \in E_c.$$

This is equivalent to the system

$$\begin{aligned} -(\sigma + 1)\eta_2(y_1) + g_2(y + \eta(y)) &= \eta_2'(y_1)g_1(y + \eta(y)) \\ (8.2.5) \quad -\beta\eta_3(y_1) + g_3(y + \eta(y)) &= \eta_3'(y_1)g_1(y + \eta(y)). \end{aligned}$$

Since  $\eta(0) = 0$  and  $D\eta(0) = 0$ , the first term in the approximation of  $\eta$  near 0 is of the form

$$\eta_2(y_1) = \alpha_2 y_1^2 + \cdots, \quad \eta_3(y_1) = \alpha_3 y_1^2 + \cdots.$$

Note that the right-hand side of (8.2.5) is of third order or higher. Therefore, using the form of  $\eta$  and  $G$ , we get

$$\begin{aligned} -(\sigma + 1)\eta_2(y_1) - \frac{1}{\sigma + 1}(y_1 - \sigma\eta_2(y_1))\eta_3(y_1) &= O(y_1^3), \\ -\beta\eta_3(y_1) + (y_1 - \sigma\eta_2(y_1))(y_1 + \eta_2(y_1)) &= O(y_1^3). \end{aligned}$$

There can be no quadratic terms on the left, so

$$\begin{aligned} -(\sigma + 1)\eta_2(y_1) &= 0, \\ -\beta\eta_3(y_1) + y_1^2 &= 0. \end{aligned}$$

This forces us to take  $\alpha_2 = 0$  and  $\alpha_3 = 1/\beta$ . In other words, we have

$$W_c^{loc}(0) = \{y \in \mathbb{R}^3 : y_1 \in U, y_2 = O(y_1^3), y_3 = y_1^2/\beta + O(y_1^3)\}.$$

The flow on  $W_c^{loc}(0)$  is governed by the equation

$$w' = Bw + P_c g(w + \eta(w)), \quad w(0) = w_0 \in E_c.$$

Since  $w = P_c w = w_1 e_1$ , this reduces to

$$\begin{aligned} w_1' &= -\frac{\sigma}{\sigma+1}(w_1 - \eta_2(w_1))\eta_3(w_1) \\ &\approx -\frac{\sigma}{\beta(\sigma+1)}w_1^3. \end{aligned}$$

Therefore, the origin is an asymptotically stable critical point for the reduced flow on  $W_c^{loc}(0)$ . A simple calculation shows that

$$w_1(t) = \frac{w_1(0)}{\left[1 + \left(\frac{2\sigma w_1^2(0)}{\beta(\sigma+1)}\right)t\right]^{1/2}} \left[1 + O\left(\frac{1}{1+t}\right)\right].$$

Now  $W_c^{loc}(0)$  is exponentially attracting, so

$$y(t) = w_1(t)e_1 + O(\exp(-Ct)).$$

It follows that all solutions in a neighborhood of the origin decay to 0 at a rate of  $t^{-1/2}$ , as  $t \rightarrow \infty$ .

### 8.3. Co-Dimension One Bifurcations

Consider the system  $x' = f(x, \mu)$  in which  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  is a  $C^\infty$  vector field depending on the parameter  $\mu \in \mathbb{R}$ . Suppose that  $f(0, 0) = 0$ ; i.e.  $x = 0$  is a critical point when  $\mu = 0$ . We wish to understand if and how the character of the flow near the origin changes when we vary the parameter  $\mu$ . Such a change is referred to as a *bifurcation*. Let  $A(\mu) = D_x f(0, \mu)$ . If  $A(0)$  hyperbolic, then by the implicit function theorem, there is a smooth curve of equilibria  $x(\mu)$  such that  $x(0) = 0$  and  $f(x(\mu), \mu) = 0$ , for  $\mu$  near 0. Moreover,  $x(\mu)$  is the only critical point of  $f(x, \mu)$  in a neighborhood of the origin. Again for  $\mu$  small, we see that by continuity  $A(\mu)$  will have stable and unstable subspaces of the same dimension as  $A(0)$ . By the Hartman-Grobman theorem 6.2.1, the flow of  $x' = f(x, \mu)$  is topologically equivalent for all  $\mu$  near 0. This argument shows that in order for a bifurcation to occur,  $A(0)$  can not be hyperbolic.

The simplest case is when  $A(0)$  has a single eigenvalue on the imaginary axis. Since  $A(0)$  is real, that eigenvalue must be zero and all other eigenvalues must have non-zero real part. This situation is called the *co-dimension one* bifurcation.

Bifurcation problems are conveniently studied by means of the so-called *suspended system*

$$(8.3.1) \quad x' = f(x, \mu) \quad \mu' = 0.$$

This is obviously equivalent to the original problem, except that now  $\mu$  is viewed as a dependent variable instead of a parameter. The orbits of

the suspended system lie on planes  $\mu = \text{Const.}$  in  $\mathbb{R}^{n+1}$ . The suspended system will have a center manifold of one dimension greater than the unperturbed equation  $x' = f(x, 0)$ . So in the co-dimension one case, it will be two-dimensional. We now proceed to reduce the flow of the suspended system to the center manifold.

Let  $A = A(0)$  have invariant subspaces  $E_c$ ,  $E_s$ , and  $E_u$ , with the projections  $P_c$ ,  $P_s$ , and  $P_u$ . We are assuming that  $E_c$  is one-dimensional, so let  $E_c$  be spanned by the vector  $v_1 \in \mathbb{R}^n$ . Let  $v_2, \dots, v_n \in \mathbb{R}^n$  span  $E_s + E_u$ , so that the set  $\{v_1, v_2, \dots, v_n\}$  is a basis for  $\mathbb{R}^n$ .

Define the vector  $C = \frac{\partial f}{\partial \mu}(0, 0) \in \mathbb{R}^n$ . Write  $C = C_1 + C_2$ , with  $C_1 = P_c C = \sigma v_1$  and  $C_2 = (P_s + P_u)C$ . Now  $A$  is an isomorphism on  $E_s + E_u$ , so there is a vector  $v_0 \in E_s + E_u$  such that  $Av_0 = -C_2$ .

The vectors

$$u_0 = \begin{bmatrix} v_0 \\ 1 \end{bmatrix}, \quad u_1 = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}, \quad \dots, \quad u_n = \begin{bmatrix} v_n \\ 0 \end{bmatrix},$$

form a basis for  $\mathbb{R}^n \times \mathbb{R}$ . Define the  $(n+1) \times (n+1)$  matrix

$$B = \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}.$$

Let  $E'_s, E'_c, E'_u$  be the stable, center, and unstable subspaces for  $B$  in  $\mathbb{R}^{n+1}$ . Note that

$$Bu_1 = 0 \quad \text{and} \quad Bu_0 = \begin{bmatrix} Av_0 + C \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma v_1 \\ 0 \end{bmatrix} = \sigma u_1;$$

i.e.  $u_0$  is an eigenvector and  $u_1$  is a generalized eigenvector for the eigenvalue 0. So the center subspace  $E'_c$  of  $B$  is (at least) two-dimensional. Since the restriction of  $B$  to the subspace spanned by the vectors  $\{u_2, \dots, u_n\}$  is the same as  $A$  on  $E_s + E_u$ , it follows that  $\{u_2, \dots, u_n\}$  spans  $E'_s + E'_u$ , the sum of the stable and unstable subspace of  $B$ . So it now also follows that  $E'_c$  is spanned by  $\{u_0, u_1\}$ .

Now returning to the suspended system, if we write

$$y = \begin{bmatrix} x \\ \mu \end{bmatrix}, \quad \text{and} \quad g(y) = \begin{bmatrix} f(x, \mu) \\ 0 \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R},$$

then (8.3.1) can be written as

$$y' = g(y).$$

Note that  $g(0) = 0$  and  $D_y g(0) = B$ .

There is a two-dimensional local center manifold described by a  $C^1$  function  $\eta : E'_c \rightarrow E'_s + E'_u$ , with  $\eta(0) = 0$ ,  $D\eta(0) = 0$ . The reduced flow is then

$$w' = P'_c g(w + \eta(w)), \quad w \in E'_c.$$

Given that  $E'_c$  is spanned by  $u_0, u_1$ , we can write

$$w = w_0u_0 + w_1u_1, \quad \text{and} \quad P'_c g(y) = g_0(y)u_0 + g_1(y)u_1.$$

Recall that the  $(n+1)^{\text{st}}$  component of  $g(y)$  is 0. This means that  $g_0(y) = 0$ .<sup>1</sup> In more explicit form, the reduced equation is

$$w'_0u_0 + w'_1u_1 = g_1(w_0u_0 + w_1u_1 + \eta(w_0u_0 + w_1u_1))u_1.$$

Define the scalar function

$$h(w_0, w_1) = g_1(w_0u_0 + w_1u_1 + \eta(w_0u_0 + w_1u_1)).$$

Compare the coefficients of the two components to get the system

$$w'_0 = 0, \quad w'_1 = h(w_0, w_1).$$

Looking at the definition of  $h$  and  $g_1$ , we see that  $h(0, 0) = 0$  and  $D_{w_1}h(0, 0) = 0$ , since

$$\begin{aligned} D_{w_1}h(0, 0)u_1 &= D_{w_1}[g_1(w_0u_0 + w_1u_1 + \eta(w_0u_0 + w_1u_1))]u_1|_{(w_1, w_2)=(0,0)} \\ &= D_{w_1}[P_c g(w_0u_0 + w_1u_1 + \eta(w_0u_0 + w_1u_1))]|_{(w_1, w_2)=(0,0)} \\ &= P_c Dg(0)[u_1 + D\eta(0)u_1] \\ &= Bu_1 \\ &= 0. \end{aligned}$$

Therefore, the co-dimension one bifurcation problem reduces to the study of a scalar equation

$$x' = f(x, \mu)$$

with  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  smooth and  $f(0, 0) = 0$ ,  $D_x f(0, 0) = 0$ .

In the following, we consider the three basic ways in which this type of bifurcation occurs.

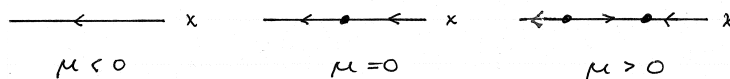
**Saddle-Node Bifurcation** The generic example is illustrated by the vector field

$$f(x, \mu) = \varepsilon_1\mu - \varepsilon_2x^2, \quad \varepsilon_i = \pm 1, \quad i = 1, 2.$$

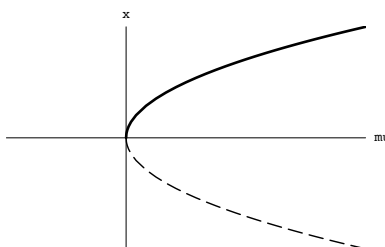
If, for example,  $\varepsilon_1 = \varepsilon_2 = 1$ , then  $f(x, \mu)$  has no critical points when  $\mu < 0$ , one critical point when  $\mu = 0$ , and two critical points when  $\mu > 0$ . Thus, we have three distinct phase portraits.

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<sup>1</sup>We note that  $P_c f(x, \mu) = f_1(x, \mu)v_1$  and  $f_1(x, \mu) = g_1(y)$ .



The situation is summarized in the bifurcation diagram:



The solid line indicates a branch of stable critical points, and the dashed line indicates unstable critical points. The picture is reflected in the  $\mu$ -axis when the sign of  $\varepsilon_1$  is negative and in the  $x$ -axis when  $\varepsilon_2$  changes sign.

The general saddle-node bifurcation occurs when  $f(\mu, x)$  satisfies

$$f(0, 0) = D_x f(0, 0) = 0, \quad \text{and} \quad D_\mu f(0, 0) \neq 0, \quad D_x^2 f(0, 0) \neq 0.$$

By the implicit function theorem, the equation

$$f(x, \mu) = 0$$

can be solved for  $\mu$  in terms of  $x$ . There is a smooth function  $\mu(x)$  defined in a neighborhood of  $x = 0$  such that

$$\mu(0) = 0, \quad f(x, \mu(x)) = 0,$$

and all zeros of  $f$  near  $(0, 0)$  lie on this curve. If we differentiate the equation  $f(x, \mu(x)) = 0$  and use the facts that  $D_x f(0, 0) = 0$  and  $D_\mu f(0, 0) \neq 0$  we find

$$\mu'(0) = 0 \quad \text{and} \quad \mu''(0) = -\frac{D_x^2 f(0, 0)}{D_\mu f(0, 0)} \neq 0.$$

We obtain the following bifurcation diagrams analogous to the model case.

**Transcritical Bifurcation** The basic example is

$$f(x, \mu) = \alpha x^2 + 2\beta\mu x + \gamma\mu^2.$$

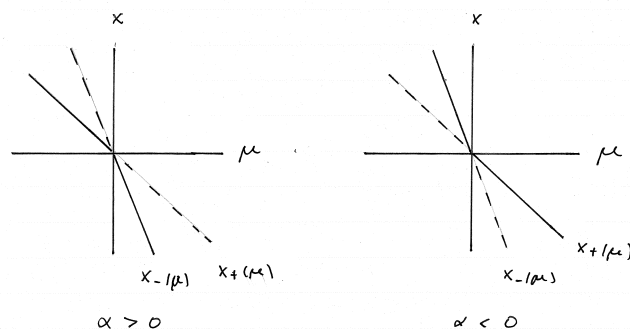
In order to have interesting dynamics for  $\mu \neq 0$ , there should be some critical points besides  $x = 0, \mu = 0$ . So we assume that  $\alpha \neq 0$  and  $\beta^2 - \alpha\gamma > 0$ . This yields two curves of critical points

$$x_{\pm}(\mu) = \mu r_{\pm}, \quad \text{with } r_{\pm} = [-\beta \pm \sqrt{\beta^2 - \alpha\gamma}]/\alpha,$$

and we can write

$$f(x, \mu) = \alpha(x - x_+(\mu))(x - x_-(\mu)).$$

For  $\mu \neq 0$ , there are a pair of critical points, one stable, the other unstable, and their stability is exchanged at  $\mu = 0$ .



Note that stability of both fixed points can be determined by the sign of  $\alpha$ , since  $x_- < -\beta\mu/\alpha < x_+$  and  $f(-\beta\mu/\alpha, \mu) = -[\beta^2 - \alpha\gamma]\mu^2/\alpha$ .

In general, a transcritical bifurcation occurs when

$$f(0,0) = D_x f(0,0) = D_\mu f(0,0) = 0, \quad \text{and } D_x^2 f(0,0) \neq 0.$$

In order to have critical points near  $(0,0)$ ,  $f$  cannot have a relative extremum at  $(0,0)$ . This is ruled out by assuming that the Hessian is negative. Let

$$\alpha = D_x^2 f(0,0) \quad \beta = D_x D_\mu f(0,0) \quad \gamma = D_\mu^2 f(0,0).$$

Then  $(0,0)$  is not a local extremum for  $f$  provided that

$$H = -\det \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \beta^2 - \alpha\gamma > 0.$$

Suppose that  $x(\mu)$  is a smooth curve of equilibria with  $x(0) = 0$ . From  $f(x(\mu), \mu) = 0$  and implicit differentiation, we get

$$D_x f(x(\mu), \mu)x'(\mu) + D_\mu f(x(\mu), \mu) = 0,$$

and

$$D_x^2 f(x(\mu), \mu)[x'(\mu)]^2 + 2D_x D_\mu f(x(\mu), \mu)x'(\mu) + D_\mu^2 f(x(\mu), \mu) + D_x f(x(\mu), \mu)x''(\mu) = 0.$$

If we set  $\mu = 0$ , we see that  $x'(0)$  is a root of

$$\alpha\xi^2 + 2\beta\xi + \gamma = 0.$$

The solvability condition is precisely  $H > 0$ . From this we can expect to find two curves

$$x_\pm(\mu) = \mu r_\pm + \dots \quad \text{with} \quad r_\pm = -[\beta \pm \sqrt{H}]/\alpha.$$

Now we proceed to construct the curves  $x_\pm(\mu)$ . By Taylor's theorem, we may write

$$f(x, \mu) = \frac{1}{2}\alpha x^2 + \beta\mu x + \frac{1}{2}\gamma\mu^2 + \delta(x, \mu),$$

in which the remainder has the form

$$\delta(x, \mu) = A(x, \mu)x^3 + B(x, \mu)x^2\mu + C(x, \mu)x\mu^2 + D(x, \mu)\mu^3,$$

with  $A, B, C, D$  smooth functions of  $(x, \mu)$ .

Next consider the function

$$g(y, \mu) = \mu^{-2}f(\mu(y + r_+), \mu).$$

Thanks to the above expansion, we have that

$$g(y, \mu) = \frac{1}{2}\alpha(y + r_+)^2 + \beta(y + r_+) + \frac{1}{2}\gamma + \mu F(y, \mu),$$

with  $F(y, \mu)$  a smooth function. It follows that  $g(y, \mu)$  is smooth,  $g(0, 0) = 0$ , and  $D_y g(0, 0) = \sqrt{H} > 0$ . If we apply the implicit function theorem, we get a smooth function  $y_+(\mu)$  such that

$$y_+(0) = 0, \quad g(y_+(\mu), \mu) = 0.$$

Set  $x_+(\mu) = \mu(y_+(\mu) + r_+)$ . Then

$$x_+(0) = 0, \quad x'_+(0) = r_+, \quad f(x_+(\mu), \mu) = 0.$$

In the same way we get a second curve  $x_-(\mu)$  with slope  $r_-$  at 0.

Since  $x_- < (x_+ + x_-)/2 < x_+$ , the stability can be determined by the sign of  $f((x_+ + x_-)/2, \mu)$  which a simple calculation shows is approximated by  $H\mu^2/\alpha$ , so we get diagrams similar to the one above.

**Pitchfork Bifurcation** The fundamental example is

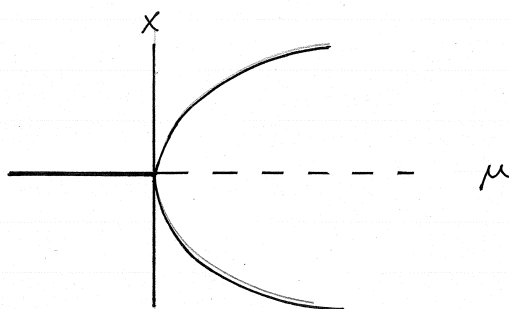
$$f(x, \mu) = \varepsilon_1\mu x - \varepsilon_2 x^3, \quad \varepsilon_i = \pm 1, \quad i = 1, 2.$$

For example, if  $\varepsilon_1 = \varepsilon_2 = 1$ , then for  $\mu < 0$  there is one stable critical point at  $x = 0$  and for  $\mu > 0$  two new critical points at  $x = \pm\sqrt{\mu}$  are

created. The stability of  $x = 0$  for  $\mu < 0$  is passed to the newly created pair for  $\mu > 0$ . The one-dimensional phase portraits look like:



and the bifurcation diagram is



The other cases are similar.

The bifurcation is said to be *supercritical* when  $\varepsilon_1\varepsilon_2 > 0$  since the new branches appear when  $\mu$  exceeds the bifurcation value  $\mu = 0$ . When  $\varepsilon_1\varepsilon_2 < 0$ , the bifurcation is *subcritical*.

The general case is identified by the following conditions

$$f(0, 0) = D_x f(0, 0) = D_\mu f(0, 0) = D_x^2 f(0, 0) = 0, \\ D_\mu D_x f(0, 0) \neq 0, \quad D_x^3 f(0, 0) \neq 0.$$

From the above analysis, we expect to find a pair of curves  $x(\mu) \approx c_1\mu$  and  $\mu(x) \approx \pm c_2x^2$  to describing the two branches of equilibria. If such curves exist, then implicit differentiation shows that a necessary condition is

$$x(0) = 0 \quad x'(0) = \sigma \equiv -D_\mu^2 f(0, 0)/D_\mu D_x f(0, 0)$$

and

$$\mu(0) = \mu'(0) = 0 \quad \mu''(0) = \rho = -D_x^3 f(0, 0)/3D_x D_\mu f(0, 0) \neq 0.$$

Therefore, the first curve can be found by applying the implicit function theorem to

$$g(y, \mu) = \mu^{-2} f(\mu(\sigma + y), \mu) = 0$$

to get  $y(\mu)$  satisfying

$$y(0) = 0, \quad g(y(\mu), \mu) = 0.$$

Then the curve is obtained by setting  $x(\mu) = \mu(\sigma + y(\mu))$ .

The second branch comes by considering the equation

$$h(x, \lambda) = x^{-2}f(x, x^2(\rho/2 + \lambda)) = 0.$$

The implicit function theorem yields a function  $\lambda(x)$  such that

$$\lambda(0) = 0, \quad g(x, \lambda(x)) = 0.$$

Finally, let  $\mu(x) = x^2(\rho/2 + \lambda(x))$ . The bifurcation is supercritical when  $\rho > 0$  and subcritical when  $\rho < 0$ .

**Example: The Lorentz Equation, II.** Let's have another look at the Lorentz equations considered in section 8.2. This time, however, we add a small bifurcation parameter:

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2 \\ x_2' &= (1 + \mu)x_1 - x_2 - x_1 x_3 \\ x_3' &= -\beta x_3 + x_1 x_2. \end{aligned}$$

Here  $\beta, \sigma > 0$  are regarded as being fixed, and  $\mu \in \mathbb{R}$  is small. The system has the form  $x' = f(x, \mu)$  with  $f(0, 0) = 0$ . Moreover, we have already seen that the matrix

$$D_x f(0, 0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix},$$

has a one-dimensional center subspace, so that for  $\mu \neq 0$  a bifurcation could occur. To study possible bifurcations near  $(x, \mu) = (0, 0)$ , we will consider the suspended system and reduce to the flow on the local center manifold. The suspended system is:

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2 \\ x_2' &= x_1 - x_2 + \mu x_1 - x_1 x_3 \\ x_3' &= -\beta x_3 + x_1 x_2 \\ \mu' &= 0. \end{aligned}$$

Notice that because  $\mu$  is being regarded as a dependent variable,  $\mu x_1$  is considered to be a nonlinear term. So we are going to approximate the flow on the two-dimensional center manifold of the equation

$$(8.3.2) \quad z' = Az + g(z),$$

where

$$(8.3.3) \quad z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \mu \end{bmatrix}, \quad A = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$g(z) = \begin{bmatrix} 0 \\ x_1(\mu - x_3) \\ x_1x_2 \\ 0 \end{bmatrix}.$$

By our earlier calculations, we have the following eigenvalues:

$$\lambda = 0, -(\sigma + 1), -\beta, 0$$

and corresponding eigenvectors:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\sigma \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

for the matrix  $A$ . Thus, the matrix

$$S = \begin{bmatrix} 1 & -\sigma & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

can be used to diagonalize  $A$ :

$$AS = SB, \quad \text{with } B = \text{diag}(0, -(\sigma + 1), -\beta, 0).$$

If we make the change of coordinates  $z = Sy$ , then (8.3.2), (8.3.3) becomes

$$y' = By + h(y),$$

in which  $h(y) = S^{-1}g(Sy)$ . More explicitly, we have

$$S^{-1} = \begin{bmatrix} \frac{1}{\sigma+1} & \frac{\sigma}{\sigma+1} & 0 & 0 \\ -\frac{1}{\sigma+1} & \frac{\sigma}{\sigma+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and hence

$$h(y) = \begin{bmatrix} \frac{\sigma}{\sigma+1}(y_1 - \sigma y_2)(y_4 - y_3) \\ \frac{1}{\sigma+1}(y_1 - \sigma y_2)(y_4 - y_3) \\ (y_1 - \sigma y_2)(y_1 + y_2) \\ 0 \end{bmatrix}.$$

Tracing back through the substitutions, we have  $y_4 = z_4 = \mu$ .

The local center manifold is described as

$$W_c^{loc}(0) = \{y \in \mathbb{R}^4 : P_s y = \eta(P_c y)\},$$

with  $\eta : E_c \rightarrow E_s$  a  $C^1$  map such that  $\eta(0) = 0$ ,  $D\eta(0) = 0$ . Since  $B$  is diagonal, it follows that  $E_c = \text{span}\{e_1, e_4\}$  and  $E_s = \text{span}\{e_2, e_3\}$ . Given the form of  $E_c$  and  $E_s$ , we can write

$$W_c^{loc}(0) = \{y \in \mathbb{R}^4 : \eta_1(y_1, y_4) = \eta_4(y_1, y_4) = 0, \\ y_2 = \eta_2(y_1, y_4), y_3 = \eta_3(y_1, y_4)\}.$$

Using theorem 8.1.2, we can approximate  $\eta$  to third order. We find that if  $H(y)$  satisfies

$$-(\sigma + 1)H_2(y_1, y_4) + \frac{1}{\sigma + 1}y_1 y_4 = O(|y|^3) \\ -\beta H_3(y_1, y_4) + y_1^2 = O(|y|^3),$$

then  $\eta(y) = H(y) + O(|y|^3)$ . This shows that

$$\eta_2(y_1, y_4) = (\sigma + 1)^{-2}y_1 y_4 + O(|y|^3), \quad \eta_3(y_1, y_4) = \beta^{-1}y_1^2 + O(|y|^3).$$

By part (iii) of corollary 8.1.1, the flow on  $W_c^{loc}(0)$  is governed by

$$w' = P_c h(w + \eta(w)), \quad w = P_c y,$$

which, in this case, is equivalent to

$$y_1' = \frac{\sigma}{\sigma + 1}(y_1 - \sigma\eta_2(y_1, y_4))(y_4 - \eta_3(y_1, y_4)) \\ y_4' = 0,$$

with initial conditions  $y_1(0)$  and  $y_4(0) = \mu$ . Since  $y_4 = \text{Const.} = \mu$ , we end up with a scalar equation for  $u = y_1$  of the form

$$u' = F(u, \mu),$$

with

$$F(u, \mu) = \frac{\sigma}{\sigma + 1}(u - \sigma\eta_2(u, \mu))(\mu - \eta_3(u, \mu)) \\ \approx \frac{\sigma}{\sigma + 1} \left[ u - \frac{\sigma}{(\sigma + 1)^2}\mu u \right] \left[ \mu - \frac{1}{\beta}u^2 \right] + \dots \\ = \frac{\sigma}{\sigma + 1} \left[ \mu u - \frac{1}{\beta}u^3 - \frac{\sigma}{(\sigma + 1)^2}\mu^2 u \right] + \dots$$

Notice that

$$F(0, 0) = D_u F(0, 0) = D_\mu F(0, 0) = D_u^2 F(0, 0) = 0,$$

and

$$D_u D_\mu F(0, 0) = \frac{\sigma}{\sigma + 1} > 0, \quad D_u^3 F(0, 0) = \frac{6\sigma}{\beta(\sigma + 1)} > 0.$$

Thus we have a supercritical pitchfork bifurcation. For  $\mu \leq 0$  there is a single asymptotically stable fixed point at  $x = 0$ . If  $\mu < 0$ , then all solutions decay towards the origin exponentially fast. For  $\mu > 0$  there is an unstable fixed point at  $x = 0$ . It has a two-dimensional stable manifold and a one-dimensional unstable manifold. Additionally, there are a pair of asymptotically stable fixed points. These three critical points lie on the invariant curve obtained by intersecting the center manifold of the suspended equation with the plane  $\mu = \text{Const}$ . This curve contains the unstable manifold of  $x = 0$ .

#### 8.4. Poincaré Normal Forms

The analysis of the reduced flow on the center manifold can be a difficult task in more than one dimension. In 1D, we have seen how the isolation of certain terms of the Taylor expansion of the vector field is essential in understanding the nature of a bifurcation. The *reduction to normal form* is a systematic procedure for eliminating all inessential terms.

Consider the flow of

$$(8.4.1) \quad x' = Ax + f(x)$$

where  $A$  is an  $n \times n$  real matrix and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $C^\infty$  with  $f(0) = 0$ ,  $Df(0) = 0$ . Suppose we make a change of variables

$$(8.4.2) \quad x = y + h(y) = \Phi(y),$$

with  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  in  $C^\infty$  and  $h(0) = 0$ ,  $Dh(0) = 0$ . The inverse function theorem ensures us that  $\Phi$  is a local diffeomorphism near the origin. That is, there exist neighborhoods of the origin  $U, V \subset \mathbb{R}^n$  such that  $\Phi : U \rightarrow V$  is a diffeomorphism.

Let's see what happens to the system under this change of coordinates. Suppose that  $x(t)$  is a solution of (8.4.1) in  $V$  and define  $y(t)$  through (8.4.2). By the chain rule, we have

$$x' = y' + Dh(y)y' = [I + Dh(y)]y'.$$

Then using the fact that  $x$  is a solution and then the invertibility of  $D\Phi$ , we get

$$\begin{aligned} y' &= [I + Dh(y)]^{-1}x' \\ &= [I + Dh(y)]^{-1}[Ax + f(x)] \\ &= [I + Dh(y)]^{-1}x' \\ (8.4.3) \quad &= [I + Dh(y)]^{-1}[A(y + h(y)) + f(y + h(y))]. \end{aligned}$$

This isn't as bad as the averaging lemma where we studied the nonautonomous analog of this computation!

Suppose that the terms of the Taylor expansion of  $f(x)$  at  $x = 0$  all have degree  $r \geq 2$ . Write

$$f(x) = f_r(x) + O(|x|^{r+1}),$$

where  $f_r(x)$  is a vector field all of whose components are homogeneous polynomials of degree  $r$  and  $O(|x|^{r+1})$  stands for a smooth function whose Taylor expansion at  $x = 0$  starts with terms of degree at least  $r + 1$ . Take the function  $h(y)$  in the coordinate change (8.4.2) to have the same form as  $f_r(x)$ , that is, suppose that  $h(y)$  is a vector-valued homogeneous polynomial of degree  $r$ . Then since

$$[I + Dh(y)]^{-1} = I - Dh(y) + Dh(y)^2 - \dots = I - Dh(y) + O(|y|^{2(r-1)}),$$

we have when we go back to (8.4.3)

$$y' = Ay + Ah(y) - Dh(y)Ay + f_r(y) + O(|y|^{r+1}).$$

Then expression

$$Ah(y) - Dh(y)Ay + f_r(y)$$

is homogeneous of degree  $r$ . We can attempt to kill these terms by choosing the function  $h(y)$  so that

$$(8.4.4) \quad L_A h(y) \equiv Dh(y)Ay - Ah(y) = f_r(y).$$

The expression  $L_A$  is known as the *Poisson* or *Lie bracket* of the vector fields  $Ax$  and  $h(x)$ .

Equation (8.4.4) is really just a linear algebra problem. The set of all homogeneous vector-valued polynomials of degree  $r$  in  $\mathbb{R}^n$  is a finite-dimensional vector space, call it  $\mathcal{H}_r^n$ , and  $L_A \mathcal{H}_r^n \rightarrow \mathcal{H}_r^n$  is a linear map. (The dimension of  $\mathcal{H}_r^n$  is  $n \binom{r+n-1}{r}$ .) So we are looking for a solution of the linear system  $L_A h = f_r$  in  $\mathcal{H}_r^n$ .

When does (8.4.4) have solutions? In order to more easily examine our linear system in the vector space  $\mathcal{H}_r^n$ , we need to introduce a bit of notation. An  $n$ -tuple of nonnegative integers

$$m = (m_1, m_2, \dots, m_n)$$

is called a multi-index. Its order or degree is defined to be

$$|m| = m_1 + m_2 + \dots + m_n.$$

Given an  $n$ -tuple  $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$  and a multi-index  $m$  of degree  $r$ , monomials of degree  $r$  are conveniently notated as

$$y^m = y_1^{m_1} y_2^{m_2} \dots y_n^{m_n}.$$

Now if  $\{v_k\}_{k=1}^n$  is a basis for  $\mathbb{R}^n$ , then

$$\{y^m v_k : k = 1, \dots, n; |m| = r\}$$

is a basis for  $\mathcal{H}_r^n$ .

LEMMA 8.4.1. *Let  $A$  be an  $n \times n$  matrix, and let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be its  $n$ -tuple of eigenvalues. Then the eigenvalues of the linear transformation  $L_A$  on  $\mathcal{H}_r^n$  are given by*

$$\lambda \cdot m - \lambda_j$$

where  $|m| = r$  is an multi-index of degree  $r$  and  $j = 1, \dots, n$ .

REMARKS ON THE PROOF. Suppose that  $A$  has a basis of eigenvectors  $v_1, \dots, v_n \in \mathbb{C}^n$  with corresponding eigenvalues  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ . Suppose that  $S$  is the matrix whose columns are formed by the eigenvectors of  $A$ . Set  $x = S^{-1}y$ . Then the set of homogeneous polynomials  $\{x^m v_j : j = 1, \dots, n; |m| = r\}$  forms a basis for (complex)  $\mathcal{H}_r^n$  as well as being a set of eigenvectors for  $L_A$ . A direct calculation shows that  $L_A x^m v_j = (\lambda \cdot m - \lambda_j)x^m v_j$ . This confirms the statement about the eigenvalues, and it also shows that, in the case that  $A$  is diagonalizable, the range of  $L_A$  is spanned by  $x^m v_j$  for  $m$  and  $j$  such that  $\lambda \cdot m - \lambda_j \neq 0$ . A real basis for the range of  $L_A$  is obtained by taking real and imaginary parts. If  $A$  is not diagonalizable, things get a bit more complicated.  $\square$

DEFINITION 8.4.1. *A monomial  $y^m$  of degree  $r$  is resonant for  $A$  if for some  $k$ ,  $\lambda \cdot m - \lambda_k = 0$ .*

Thus, if  $A$  has no resonant monomials of degree  $r$ , then  $L_A$  is invertible on  $\mathcal{H}_r^n$  and equation (8.4.4) has a unique solution  $h \in \mathcal{H}_r^n$  for every  $f_r \in \mathcal{H}_r^n$ .

Example: Let's compute the resonances of degree two and three for the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of  $A$  are  $\lambda = (i, -i)$ . First, we list  $\lambda \cdot m$  for the three multi-indices  $m$  of degree two:

$$(2, 0) \cdot (i, -i) = 2i, \quad (1, 1) \cdot (i, -i) = 0, \quad (0, 2) \cdot (i, -i) = -2i.$$

None of these numbers is an eigenvalue for  $A$ , so there are no resonances of order 2.

Now we do the same thing for  $r = 3$ . There are 4 monomials of degree 3:

$$\begin{aligned} (3, 0) \cdot (i, -i) &= 3i, & (2, 1) \cdot (i, -i) &= i = \lambda_1, \\ (1, 2) \cdot (i, -i) &= -i = \lambda_2, & (0, 3) \cdot (i, -i) &= -3i. \end{aligned}$$

We find two resonant monomials of degree 3, namely  $x_1^2x_2$  and  $x_1x_2^2$ .

**DEFINITION 8.4.2.** *The convex hull of a set  $\Omega \subset \mathbb{R}^n$  is the smallest convex set which contains  $\Omega$ . The convex hull is denoted by  $\text{co } \Omega$ .*

**LEMMA 8.4.2.** *Let  $A$  be an  $n \times n$  matrix, and let  $\sigma(A)$  be its spectrum, i.e. the set containing its eigenvalues. If  $0 \notin \text{co } \sigma(A)$ , then  $A$  has at most finitely many resonances.*

**PROOF.** Since  $\sigma(A)$  is finite,  $\text{co } \sigma(A)$  is closed. Therefore, since  $0 \notin \text{co } \sigma(A)$ , there is a  $\delta > 0$  such that  $|z| \geq \delta$  for all  $z \in \text{co } \sigma(A)$ .

Let  $m$  be a multi-index with  $|m| = r$ . Then  $\lambda \cdot m/r \in \text{co } \sigma(A)$ , so  $|\lambda \cdot m/r| \geq \delta$ .

Let  $M > 0$  bound the eigenvalues of  $A$ :  $|\lambda_k| \leq M$ ,  $k = 1, \dots, n$ . If  $r \geq (M+1)/\delta$ , then  $|\lambda \cdot m| \geq r\delta \geq M+1 > \lambda_k$ ,  $k = 1, \dots, m$ . Thus,  $x^m$  is nonresonant if  $|m| \geq (M+1)/\delta$ .  $\square$

**Remark:** If  $A$  is real, then since complex eigenvalues come in conjugate pairs,  $0 \notin \text{co } \sigma(A)$  if and only if  $\sigma(A)$  lies on one side of the imaginary axis.

**THEOREM 8.4.1.** *Let  $Ax + f(x)$  be a smooth vector field with  $f(0) = 0$  and  $Df(0) = 0$ . Consider the Taylor expansion of  $f(x)$  near  $x = 0$ :*

$$f(x) = f_2(x) + f_3(x) + \dots + f_p(x) + Rf_{p+1}(x),$$

in which  $f_r(x) \in \mathcal{H}_r^n$ ,  $r = 2, \dots, p$  and the remainder  $Rf_{p+1}$  is smooth and  $O(|x|^{p+1})$ .

For each  $r = 1, \dots, p$  decompose  $\mathcal{H}_r^n$  as  $\mathcal{H}_r^n = R_r(L_A) + G_r(L_A)$ , where  $R_r(L_A)$  is the range of  $L_A$  on  $\mathcal{H}_r^n$  and  $G_r(L_A)$  is a complementary subspace. There exists a polynomial change of coordinates,  $x = \Phi(y)$ , which is invertible in a neighborhood of the origin such that the equation

$$x' = Ax + f(x)$$

is transformed near the origin into

$$y' = Ay + g_2(y) + \dots + g_p(y) + Rg_{p+1}(y),$$

with  $g_r \in G_r$  and the remainder  $Rg_{p+1}$  smooth and  $O(|y|^{p+1})$  near  $y = 0$ .

**PROOF.** We will proceed inductively, treating terms of order  $r = 2, \dots, p$  in succession. First, write

$$f_2(x) = f_2(x) - g_2(x) + g_2(x),$$

with  $f_2(x) - g_2(x) \in R(L_A)$  and  $g_2(x) \in G_2$ . Let  $h \in \mathcal{H}_2^n$  be a solution of

$$L_A h_2 = f_2 - g_2.$$

Then  $x = y + h_2(y)$  transforms  $x' = Ax + f(x)$  into

$$y' = Ay + g_2(y) + \tilde{f}_3(y) + \cdots + \tilde{f}_p(y) + O(|y|^{p+1}).$$

If we continue in the same manner to eliminate terms of order 3, then we will have a coordinate change of the form  $y = z + h_3(z)$  with  $h_3 \in \mathcal{H}_3^n$ . The point is that this change of coordinates will not affect terms of lower order. Thus, we can proceed to cancel terms of successively higher order.  $\square$

Example: Consider the system  $x' = Ax + f(x)$  with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and  $f(0) = 0$ ,  $Df(0) = 0$ . Since  $A$  has the double eigenvalue 0, all monomials are resonant! Nevertheless,  $L_A$  is nonzero, so we can remove some terms. Let's study  $L_A$  on  $\mathcal{H}_2^2$  and determine the normal form of degree 2.

First, for an arbitrary function  $H(x) = \begin{bmatrix} H_1(x) \\ H_2(x) \end{bmatrix}$ , we have

$$L_A H(x) = DH(x)Ax - AH(x) = \begin{bmatrix} x_2 \partial_1 H_1(x) - H_2(x) \\ x_2 \partial_1 H_2(x) \end{bmatrix}.$$

Choose a basis for  $\mathcal{H}_2^2$ :

$$\begin{aligned} h_1(x) &= \frac{1}{2}x_1^2 e_1 & h_4(x) &= \frac{1}{2}x_1^2 e_2 \\ h_2(x) &= x_1 x_2 e_1 & h_5(x) &= x_1 x_2 e_2 \\ h_3(x) &= \frac{1}{2}x_2^2 e_1 & h_6(x) &= \frac{1}{2}x_2^2 e_2 \end{aligned}$$

We have used the standard basis vectors here because they are generalized eigenvectors for  $A$ . After a little computation, we find that

$$\begin{aligned} L_A h_1 &= h_2 & L_A h_2 &= 2h_3 \\ L_A h_3 &= 0 & L_A h_4 &= -h_1 + h_5 \\ L_A h_5 &= -h_2 + 2h_6 & L_A h_6 &= -h_3 \end{aligned}$$

Thus, the range of  $L_A$  is given by

$$R(L_A) = \text{span} \{h_2, h_3, -h_1 + h_5, h_6\}.$$

Now using Taylor's theorem, expand  $f(x)$  to second order near  $x = 0$ :

$$\begin{aligned} f(x) &= f_2(x) + O(|x|^3) \\ &= \sum_{i=1}^6 \alpha_i h_i(x) + O(|x|^3) \\ &= [\alpha_2 h_2(x) + \alpha_3 h_3(x) + \alpha_5(-h_1(x) + h_5(x)) + \alpha_6 h_6(x)] \\ &\quad + [(\alpha_1 + \alpha_5)h_1(x) + \alpha_4 h_4(x)] + O(|x|^3) \\ &\equiv \bar{f}_2(x) + g_2(x) + O(|x|^3), \end{aligned}$$

with  $f_2 \in R_2(L_A)$  and  $g_2 \in G_2(L_A)$ . Set

$$H(x) = \alpha_2 h_1(x) + \frac{1}{2} \alpha_3 h_2(x) + \alpha_5 h_4(x) + \frac{1}{2} \alpha_6 (h_1(x) - h_5(x)).$$

Then

$$L_A H(x) = \bar{f}_2(x),$$

and using the transformation  $x = y + H(y)$ , we can achieve the normal form

$$y' = Ay + g_2(y) + O(|y|^3).$$

To complete this section we state a general theorem about reduction to normal form.

**THEOREM 8.4.2 (Sternberg's Theorem).** *Let  $A$  be an  $n \times n$  matrix with at most a finite number of resonances. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be  $C^\infty$  with  $f(0) = 0$ ,  $Df(0) = 0$ . For any positive integer  $k$ , there exists a  $C^k$  function  $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $h(0) = 0$ ,  $Dh(0) = 0$  such that the change of variables  $x = y + h(y)$  transforms  $x' = Ax + f(x)$  in a neighborhood of the origin into*

$$y' = Ay + g_2(y) + \cdots + g_p(y),$$

where  $g_r \in \mathcal{H}_r^n \setminus R(L_A)$ ,  $r = 2, \dots, p$ , and where  $p$  is the maximum order of any resonance.

Remarks:

- (1) The difficult proof can be found in the book of Hartman, 2nd ed. See theorem 12.3.
- (2) The result is the smooth analogue of the Hartman-Grobman theorem 6.2.1. It says, in particular, that if  $A$  has no resonances, then the flow is  $C^k$  conjugate to the linear flow.
- (3) The hypothesis that  $A$  has finitely many resonances implies that  $A$  is hyperbolic.

- (4) In general, the size of the neighborhood in which the transformation is invertible shrinks as  $k$  increases. Nothing is being said here about the existence of a  $C^\infty$  change of coordinates. Theorems of that sort have been established in the analytic case.

### 8.5. The Hopf Bifurcation

The Hopf bifurcation is the most common type of bifurcation. It occurs when a pair of distinct complex conjugate eigenvalues of an equilibrium point cross the imaginary axis as the bifurcation parameter is varied. At the critical bifurcation value, there are two (nonzero) eigenvalues on the imaginary axis. So this is an example of a codimension two bifurcation. As the bifurcation parameter crosses the critical value, a periodic solution is created.

The general case will be reduced to the following simple planar autonomous system, with higher order perturbative terms. For now, the illustrative paradigm is

$$(8.5.1) \quad \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (x_1^2 + x_2^2) \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Here, as usual,  $\mu$  denotes the bifurcation parameter. The remaining constants  $a, b, \omega$  are fixed with  $a \neq 0, \omega > 0$ . Notice that  $x = 0$  is a fixed point, for all  $\mu \in \mathbb{R}$ .

The corresponding linear problem

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

has eigenvalues  $\mu \pm i\omega$ . Thus, when  $\mu < 0$ , the critical point  $x = 0$  is asymptotically stable for (8.5.1), and when  $\mu > 0$ , it is unstable. When  $\mu = 0$ , the origin is a center, since the eigenvalues lie on the imaginary axis.

In order to find the periodic solution of (8.5.1), we change to polar coordinates. Let

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta.$$

The equations transform into

$$\begin{aligned} r' &= \mu r + ar^3 \\ \theta' &= \omega + br^2. \end{aligned}$$

The interesting feature here is that the first equation is independent of  $\theta$ . In fact, we recognize the equation for  $r$  as the basic example of a pitchfork bifurcation. If  $a < 0$  the bifurcation is supercritical, and if  $a > 0$  it is subcritical.

Suppose that  $a < 0$ . Then the asymptotically stable critical point which appears at  $r = (-\mu/a)^{1/2}$  when  $\mu > 0$  corresponds to an asymptotically orbitally stable periodic orbit for (8.5.1). If we set  $\alpha = 1/\sqrt{-a}$  and  $\beta = b/(-a)$ , then this periodic solution is explicitly represented by

$$\begin{aligned}x_1(t) &= \alpha\sqrt{\mu} \cos[(\omega + \beta\mu)t] \\x_2(t) &= \alpha\sqrt{\mu} \sin[(\omega + \beta\mu)t],\end{aligned}$$

The amplitude is  $\alpha\sqrt{\mu}$  and the period is  $2\pi/(\omega + \beta\mu)$ .

Simple as this example is, surprisingly it contains the essential information necessary for understanding the general case. However, in order to see this it will be necessary to make several natural changes of variables.

We begin with a planar autonomous equation depending on a parameter

$$(8.5.2) \quad x' = f(x, \mu),$$

with a critical point at the origin when  $\mu = 0$ , i.e.  $f(0, 0) = 0$ . Think of this equation as the result of reducing a higher dimensional system with a critical point at the origin when  $\mu = 0$  to its two-dimensional center manifold. If  $D_x f(0, 0)$  is invertible, then by the implicit function theorem there exists a smooth curve of equilibria  $x(\mu)$  for  $\mu$  near 0. If we set  $g(x, \mu) = f(x + x(\mu), \mu)$ , then  $g(0, \mu) = 0$ . Therefore, we will consider vector fields in this form. Altogether we will assume that

- A1.  $f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$  is  $C^\infty$ ,
- A2.  $f(0, \mu) = 0$ ,
- A3.  $D_x f(0, 0)$  has eigenvalues  $\pm i\omega$ , with  $\omega > 0$ , and
- A4.  $\tau(\mu) = \text{tr } D_x f(0, \mu)$  satisfies  $\tau'(0) \neq 0$ .

Condition [A3] implies that  $D_x f(0, \mu)$  has eigenvalues of the form  $\xi(\mu) \pm i\eta(\mu)$  with  $\xi(0) = 0$  and  $\eta(0) = \omega$ . Assumption [A4] says that  $\xi'(0) \neq 0$ , since  $\tau(\mu) = 2\xi(\mu)$ . It means that as the bifurcation parameter passes through the origin, the eigenvalues cross the imaginary axis with nonzero speed.

The first step will be to show:

**LEMMA 8.5.1.** *Let  $f(x, \mu)$  be a vector field which satisfies the assumptions [A1]–[A4] above. For  $|\mu|$  small, there is a smooth change of coordinates  $(t, x, \mu) \mapsto (s, y, \lambda)$  which transforms solutions  $x(t, \mu)$  of (8.5.2) to solutions  $y(s, \lambda)$  of*

$$(8.5.3) \quad y' = A_\lambda y + g(y, \lambda),$$

in which  $g(y, \lambda)$  is smooth and

$$A_\lambda = \begin{bmatrix} \lambda & -\omega \\ \omega & \lambda \end{bmatrix}, \quad g(0, \lambda) = 0, \quad D_y g(0, \lambda) = 0.$$

PROOF. Let  $\xi(\mu) \pm i\eta(\mu)$  denote the eigenvalues of  $A_\mu = D_x f(0, \mu)$ . Choose a basis which reduces  $A_\mu$  to canonical form:

$$S_\mu^{-1} A_\mu S_\mu = \begin{bmatrix} \xi(\mu) & -\eta(\mu) \\ \eta(\mu) & \xi(\mu) \end{bmatrix}.$$

Then the transformation  $x(t, \mu) = S_\mu y(t, \mu)$  transforms (8.5.2) into  $y' = g(y, \mu)$  in which  $g(y, \mu) = S_\mu^{-1} f(S_\mu y, \mu)$  still satisfies [A1]–[A4] and in addition

$$D_y g(0, \mu) = \begin{bmatrix} \xi(\mu) & -\eta(\mu) \\ \eta(\mu) & \xi(\mu) \end{bmatrix}.$$

Next we rescale time through the change of variables  $t = (\omega/\eta(\mu))s$ . Then  $y((\omega/\eta(\mu))s, \mu)$  solves

$$\frac{d}{ds} y = h(y, \mu),$$

with

$$h(y, \mu) = (\omega/\eta(\mu))g(y, \mu).$$

Again  $h(y, \mu)$  satisfies [A1]–[A4], and also

$$D_y h(0, \mu) = \begin{bmatrix} \omega \frac{\xi(\mu)}{\eta(\mu)} & -\omega \\ \omega & \omega \frac{\xi(\mu)}{\eta(\mu)} \end{bmatrix}.$$

Finally, because of our assumptions, the function  $\psi(\mu) = \omega\xi(\mu)/\eta(\mu)$  has  $\psi'(0) \neq 0$  and is therefore locally invertible. Set  $\lambda = \psi(\mu)$ . Then  $y(s, \lambda)$  is a solution of  $y' = h(y, \lambda)$ , and

$$D_y h(0, \lambda) = \begin{bmatrix} \lambda & -\omega \\ \omega & \lambda \end{bmatrix}.$$

This completes the proof of the lemma.  $\square$

We continue from (8.5.3) with the variables renamed. We are now going to apply the normal form technique to the suspended system

$$x' = A_\mu x + g(x, \mu), \quad \mu' = 0.$$

From this point of view, we should write this as

$$x' = A_0 x + \mu x + g(x, \mu), \quad \mu' = 0,$$

since  $\mu x$  is a quadratic term. The eigenvalues for the linearized problem

$$x' = A_0 x, \quad \mu' = 0$$

are  $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (i\omega, -i\omega, 0)$ .

The resonances of order  $|m| = 2$  occur when

$$m = (1, 1, 0), (0, 1, 1), (1, 0, 1), (0, 0, 2).$$

The monomial  $x_1 x_2$  is nonresonant for  $A_0$ , so such a term can be eliminated unless it occurs in the third equation, which it does not. The term  $\mu^2$  is also resonant only for the third equation where it does not appear. The only resonances of order 2 are the terms  $\mu x_1$  in the first equation and  $\mu x_2$  in the second. So by a quadratic change of variables in  $x$  only we can transform the system to

$$x' = A_0 x + \mu x + \tilde{g}(x, \mu), \quad \mu' = 0,$$

in which  $\tilde{g}$  is at least cubic in  $x, \mu$  and still with  $\tilde{g}(0, \mu) = 0$  and  $D_x \tilde{g}(0, \mu) = 0$ .

The resonances of order  $|m| = 3$  are given by

$$m = (2, 1, 0), (1, 2, 0), (1, 1, 1), (0, 1, 2), (1, 0, 2), (0, 0, 3).$$

The monomials  $x_1 x_2 \mu$  and  $\mu^3$  correspond to resonances in the third equation, where they do not occur. The monomials  $x_1 \mu^2$  and  $x_2 \mu^2$  are resonant for the first two equations, but they do not appear since  $\tilde{g}$  is at least quadratic in  $x_1, x_2$ . Thus, the normal form of degree three is spanned by two vectors

$$\|x\|^2 \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \quad \text{and} \quad \|x\|^2 \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}.$$

Thus, we get using a cubic change of variables

$$(8.5.4) \quad x' = A_0 x + \mu x + \|x\|^2 B x + \tilde{\tilde{g}}(x, \mu), \quad \mu' = 0,$$

with

$$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and with  $\tilde{\tilde{g}}$  of fourth order in  $x, \mu$ . Moreover we also still have  $\tilde{\tilde{g}}(0, \mu) = 0$  and  $D_x \tilde{\tilde{g}}(0, \mu) = 0$ , so  $\tilde{\tilde{g}}$  is at least quadratic in  $x_1, x_2$ .

Recognize that our normal form (8.5.4) is, of course, just a perturbation of the model case (8.5.1). Does the periodic solution survive the perturbation?

We know that the sign of the constant  $a$  is important. According to Guckenheimer and Holmes

$$\begin{aligned} 16a = & [\partial_1^3 \tilde{g}_1 + \partial_1 \partial_2^2 \tilde{g}_1 + \partial_1^2 \partial_2 \tilde{g}_2 + \partial_2^3 \tilde{g}_2] \\ & + \frac{1}{\omega} [\partial_1 \partial_2 \tilde{g}_1 (\partial_1^2 \tilde{g}_1 + \partial_2^2 \tilde{g}_1) \\ & - \partial_1 \partial_2 \tilde{g}_2 (\partial_1^2 \tilde{g}_2 + \partial_2^2 \tilde{g}_2) \\ & - \partial_1^2 \tilde{g}_1 \partial_2^2 \tilde{g}_2 + \partial_2^2 \tilde{g}_1 \partial_1^2 \tilde{g}_2], \end{aligned}$$

where

$$\tilde{g}(x, \mu) = \begin{bmatrix} \tilde{g}_1(x, \mu) \\ \tilde{g}_2(x, \mu) \end{bmatrix},$$

and all derivatives are evaluated at  $(x, \mu) = (0, 0)$ .

We will look at the supercritical case when  $a < 0$ . In the subcritical case, an analysis similar to what follows can be performed. When  $a = 0$ , there is a so-called generalized Hopf bifurcation to a periodic solution, but its properties depend on higher order terms.

Suppose that  $a < 0$ . We will show that there is a  $\mu_0 > 0$  such that for all  $0 < \mu < \mu_0$ , the annulus

$$(8.5.5) \quad \mathcal{A} = \{x \in \mathbb{R}^2 : (-\mu/a)^{1/2} - \mu^{3/4} \leq \|x\| \leq (-\mu/a)^{1/2} + \mu^{3/4}\}$$

is positively invariant under the flow of (8.5.4). Since  $\mathcal{A}$  contains no critical points, it follows by the Poincaré-Bendixson theorem ?? that  $\mathcal{A}$  contains a periodic orbit.

Let's check the invariance. We will show that the flow of (8.5.4) goes into  $\mathcal{A}$ . Let  $x(t)$  be a solution of (8.5.4), let  $\alpha = 1/(-a)^{1/2}$ , and set  $r(t) = \|x(t)\|$ . Multiply (8.5.4) by  $x/r$ :

$$r' = \mu r + ar^3 + \frac{x}{r} \cdot \tilde{g}(x, \mu).$$

We have that

$$\left| \frac{x}{r} \cdot \tilde{g}(x, \mu) \right| \leq \|\tilde{g}(x, \mu)\| \leq C_0 r^2 (r^2 + \mu^2),$$

and so for  $x \in \mathcal{A}$  we have  $\frac{x}{r} \cdot \tilde{g}(x, \mu) = O(\mu^2)$ . Now consider  $r = \alpha\mu^{1/2} + \mu^{3/4}$  on the outer boundary of  $\mathcal{A}$ . Then

$$r' = \mu(\alpha\mu^{1/2} + \mu^{3/4}) + a(\alpha\mu^{1/2} + \mu^{3/4})^3 + O(\mu^2) = -2\mu^{7/4} + O(\mu^2) < 0,$$

for  $0 < \mu \leq \mu_0$ , sufficiently small. Similarly, if  $r = \alpha\mu^{1/2} - \mu^{3/4}$ , then

$$r' = 2\mu^{7/4} + O(\mu^2) > 0,$$

for  $0 < \mu \leq \mu_0$ . This proves invariance.

Within the region  $\mathcal{A}$ , it is easy to show that the trace of the Jacobian of the vector field is everywhere negative. This means that all periodic

solutions inside  $\mathcal{A}$  have their nontrivial Floquet multiplier inside the unit circle and are therefore orbitally stable. It follows that there can be only one periodic solution within  $\mathcal{A}$ , and it encircles the origin.

Define the ball

$$(8.5.6) \quad B = \{x \in \mathbb{R}^2 : \|x\| < |\mu/a|^{1/2} + |\mu|^{3/4}\}.$$

It is immediate to check that  $r' > 0$  in  $B \setminus \mathcal{A}$ , so that the origin is asymptotically unstable, and there are no periodic orbits in  $B \setminus \mathcal{A}$ .

On the other hand, if  $a < 0$ ,  $\mu < 0$ , then it is straightforward to show that  $r = (x_1^2 + x_2^2)^{1/2}$  is a strict Liapunov function for (8.5.4) in the ball  $B$ . Thus, the origin is asymptotically stable, and there are no periodic orbits in  $B$ .

Similar arguments can be made when  $a > 0$ .

The results can be summarized in the following:

**THEOREM 8.5.1.** *Suppose that  $a \neq 0$  and that  $\mu_0 > 0$  is sufficiently small. Let  $\mathcal{A}$  and  $B$  be defined as in (8.5.5), (8.5.6).*

*Supercritical case,  $a < 0$ : If  $0 < \mu < \mu_0$ , then (8.5.4) has an asymptotically orbitally stable periodic orbit inside  $\mathcal{A}$ . It is the unique periodic orbit in  $B$ . The origin is asymptotically unstable. If  $-\mu_0 < \mu < 0$ , then the origin is asymptotically stable for (8.5.4), and there are no periodic orbits in  $B$ .*

*Subcritical case,  $a > 0$ : If  $-\mu_0 < \mu < 0$ , then (8.5.4) has an asymptotically orbitally unstable periodic orbit inside  $\mathcal{A}$ . It is the unique periodic orbit in  $B$ . The origin is asymptotically stable. If  $0 < \mu < \mu_0$ , then the origin is asymptotically unstable for (8.5.4), and there are no periodic orbits in  $B$ .*

There are a few questions that remain open. Does the periodic solution depend continuously on  $\mu$ ? How does the period depend on  $\mu$ ? In the next sections, we take a new approach to study these questions.

## 8.6. The Liapunov-Schmidt Method

Let  $X$ ,  $Y$ , and  $\Lambda$  be Banach spaces. Suppose that  $A : X \rightarrow Y$  is a bounded linear map and that  $N : X \times \Lambda \rightarrow Y$  is a  $C^1$  map such that

$$N(0, 0) = 0, \quad D_x N(0, 0) = 0.$$

We are interested in finding nontrivial solutions  $x \in X$  of the nonlinear equation

$$(8.6.1) \quad Ax = N(x, \lambda).$$

If  $A$  is an isomorphism, that is,  $A : X \rightarrow Y$  is one-to-one and onto and  $A^{-1} : Y \rightarrow X$  is bounded, then the implicit function theorem ensures

the existence of a  $C^1$  function  $\phi : U \rightarrow V$  from a neighborhood of the origin in  $U \subset \Lambda$  to a neighborhood of the origin in  $V \subset X$  such that

$$A\phi(\lambda) = N(\phi(\lambda), \lambda), \quad \lambda \in U.$$

The Liapunov-Schmidt technique deals with the situation where  $A$  is **not** an isomorphism.

Let  $K \subset X$  be the nullspace of  $A$  and let  $R \subset Y$  be the range. Assume that

- (i) There exists a closed subspace  $M \subset X$  such that  $X = M + K$ , and
- (ii)  $R$  is closed.

It follows from the closed graph theorem the the restriction of  $A$  to  $M$  is an isomorphism onto  $R$ . These assumptions are satisfied, for example, when  $K$  is finite dimensional and  $R$  has finite co-dimension. Such an operator is called *Fredholm*.

Suppose that  $x$  is some solution of (8.6.1). Write  $x = x_1 + x_2 \in M + K$ . Write  $Y = R + S$ , and let  $P_R$  be the projection of  $Y$  onto  $R$  along  $S$ . Then

$$(8.6.2) \quad \begin{aligned} Ax_1 &= P_R N(x_1 + x_2, \lambda) = F(x_1, x_2, \lambda) \\ (I - P_R)N(x_1 + x_2, \lambda) &= 0. \end{aligned}$$

Turn this around. By the implicit function theorem, there exists a solution  $x_1(x_2, \lambda) \in M$  of (8.6.2), for  $(x_2, \lambda)$  in a neighborhood of the origin in  $K \times \Lambda$ . So now we want to solve

$$(8.6.3) \quad (I - P_R)N(x_1(x_2, \lambda) + x_2, \lambda) = 0.$$

This is called the *bifurcation equation*. We can then attempt to solve (8.6.3) for  $\lambda$  in terms of  $x_2$ . If this is possible, then we get a family of solutions of (8.6.1) in the form

$$x(x_2) = x_1(x_2, \lambda(x_2)) + x_2.$$

A typical situation might be the case where  $A$  is Fredholm and that the parameter space  $\Lambda$  is finite dimensional. Then (8.6.3) is a system of finitely many equations in a finite number of unknowns. We can then attempt to solve it using the standard implicit function theorem.

### 8.7. Hopf Bifurcation via Liapunov-Schmidt

Let's return to the reduced problem given at the end of section 8.5 after the normal form transformation. For notational convenience, we will drop the double tilde from  $\tilde{g}$ .

$$(8.7.1) \quad x' = A_0 x + \mu x + \|x\|^2 Bx + g(x, \mu),$$

with

$$(8.7.2) \quad A_0 = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and

$$(8.7.3) \quad g(0, \mu) = 0, \quad D_x g(x, \mu) = 0, \quad g(x, \mu) = O(\|x\|^2(\|x\|^2 + \mu^2)).$$

We are looking for a family of periodic solutions  $x(t, \mu)$  with period  $\approx T_0 = 2\pi/\omega$  and amplitude  $\approx \mu^{1/2}$ . We shall prove:

**THEOREM 8.7.1.** *There are smooth functions  $\mu(s)$ ,  $\lambda(s)$ ,  $x(s)$ , defined for  $|s| \leq s_0$ , such that*

$$\mu(s) \approx -as^2 + \dots, \quad \lambda(s) \approx -bs^2 + \dots,$$

*and when  $\mu = \mu(s)$ ,  $x(s)$  is a periodic solution of (8.7.1) of period  $2\pi(1 + \lambda(s))/\omega$  satisfying  $\|x(s)\| \leq C|s|$ , for some constant  $C$  which is independent of  $s$ .*

We remark that this theorem allows  $a = 0$ , however in this case we know less about the nature of periodic solutions since theorem 8.5.1 is not available. When  $a \neq 0$ , theorems 8.5.1 and 8.7.1 show that the Hopf bifurcation for the perturbed case is close to the unperturbed case.

**PROOF.** We will account for variations in the period by introducing a second parameter. Set

$$y(t, \lambda, \mu) = x((1 + \lambda)t, \mu), \quad \lambda \approx 0.$$

Then  $y$  is  $T_0$ -periodic if and only if  $x$  is  $T_0(1 + \lambda)$ -periodic. Now, a simple calculation gives

$$y' = A_0 y + \lambda A_0 y + (1 + \lambda)\mu y + (1 + \lambda)g(y, \mu).$$

Viewed as a function of  $(y, \lambda, \mu)$ , the right-hand side is quadratic (or higher) except for the term  $A_0 y$ . To be able to apply the Liapunov-Schmidt technique, we will eliminate this term by introducing rotating coordinates (cf. sections 7.2, 7.4). Let

$$U(t) = e^{A_0 t} = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix},$$

and set

$$y(t) = U(t)z(t).$$

Another routine calculation shows that

$$(8.7.4) \quad z' = \lambda A_0 z + (1 + \lambda)\mu z + (1 + \lambda)U(-t)g(U(t)z, \mu).$$

We are going to construct  $T_0$ -periodic solutions of this equation.

Here comes the set-up for Liapunov-Schmidt. Let

$$X = \{z \in C^1(\mathbb{R}, \mathbb{R}^2) : z(t + T_0) = z(t)\},$$

with the norm

$$\|z\|_X = \max_{0 \leq t \leq T_0} \|z(t)\| + \max_{0 \leq t \leq T_0} \|z'(t)\|.$$

Also, let

$$Y = \{h \in C^0(\mathbb{R}, \mathbb{R}^2) : h(t + T_0) = h(t)\},$$

with the norm

$$\|h\|_Y = \max_{0 \leq t \leq T_0} \|h(t)\|$$

If we define  $L = \frac{d}{dt}$ , then  $L : X \rightarrow Y$  is a bounded linear map. It is clear that the null space of  $L$  is

$$K = \{z \in X : z(t) = v = \text{Const.}\},$$

and that the range of  $L$  is

$$R = \{h \in Y : \int_0^{T_0} h(t) dt = 0\}.$$

So we have

$$X = M + K, \quad \text{and} \quad Y = R + S,$$

with

$$M = \{z \in X : \int_0^{T_0} z(t) dt = 0\},$$

and

$$S = \{h \in Y : h(t) = \text{Const.}\}.$$

$M$  is a closed subspace of  $X$  and  $R$  is a closed subspace of  $Y$ , so  $L : M \rightarrow R$  is an isomorphism by the closed graph theorem. (It is easy to verify this directly.) Note that  $L$  is Fredholm, since  $K$  and  $S$  are finite (two) dimensional. The projection of  $Y$  onto  $R$  along  $S$  is given by

$$P_R h(t) = h(t) - \frac{1}{T_0} \int_0^{T_0} h(s) ds.$$

Now our problem (8.7.4) can be encoded as

$$Lz = N(z, \lambda, \mu),$$

in which

$$\begin{aligned} N(z, \lambda, \mu) = & \lambda A_0 z + (1 + \lambda) \mu z \\ & + (1 + \lambda) \|z\|^2 U(-t) B U(t) z + (1 + \lambda) U(-t) g(U(t) z, \mu). \end{aligned}$$

Note that  $N : X \times \mathbb{R}^2 \rightarrow Y$  and

$$(8.7.5) \quad N(0, \lambda, \mu) = 0, \quad D_z N(0, \lambda, \mu) = 0.$$

All of the conditions for Liapunov-Schmidt are fulfilled, so we proceed to analyze the problem

$$\begin{aligned} Lu &= P_R N(u + v, \lambda, \mu), \\ (I - P_R)N(u + v, \lambda, \mu) &= 0, \end{aligned}$$

with  $u \in M$  and  $v \in K = \mathbb{R}^2$ .

By the implicit function theorem, we get a  $C^1$  mapping  $\phi(v, \lambda, \mu)$  from a neighborhood of the origin in  $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$  into  $M$  such that

$$\phi(0, 0, 0) = 0 \quad \text{and} \quad L\phi(v, \lambda, \mu) = P_R N(\phi(v, \lambda, \mu) + v, \lambda, \mu).$$

By the uniqueness part of the implicit function theorem, we get that  $\phi(0, \lambda, \mu) = 0$ . Next, from

$$LD_v \phi(v, \lambda, \mu) = P_R [D_z N(\phi(v, \lambda, \mu) + v, \lambda, \mu)(D_v \phi(v, \lambda, \mu) + I)],$$

we see using (8.7.5) that

$$LD_v \phi(0, \lambda, \mu) = P_R [D_z N(0, \lambda, \mu)(D_v \phi(0, \lambda, \mu) + I)] = 0.$$

It follows that  $D_v \phi(0, \lambda, \mu) = 0$ , since  $L$  is an isomorphism.

Set  $u(t, v, \lambda, \mu) = \phi(v, \lambda, \mu)(t)$ . Then

$$u(t, 0, \lambda, \mu) = 0, \quad \text{and} \quad D_v u(t, 0, \lambda, \mu) = 0.$$

We can therefore write

$$\begin{aligned} u(t, v, \lambda, \mu) &= \int_0^1 \frac{d}{d\sigma} u(t, \sigma v, \lambda, \mu) d\sigma \\ &= \int_0^1 D_v u(t, \sigma v, \lambda, \mu) v d\sigma \\ &= J(t, v, \lambda, \mu) v, \end{aligned}$$

with  $J(t, 0, \lambda, \mu) = 0$ , and  $J(t, v, \lambda, \mu)$  smoothly dependent on  $v, \lambda, \mu$ .

We now need to study the bifurcation function

$$\begin{aligned}
\mathcal{B}(v, \lambda, \mu) &= (I - P_R)N(\phi(v, \lambda, \mu) + v, \lambda, \mu) \\
&= \frac{1}{T_0} \int_0^{T_0} (I - P_R)N(u(t, v, \lambda, \mu) + v, \lambda, \mu) dt \\
&= [\lambda A_0 + (1 + \lambda)\mu]v \\
&\quad + \frac{1 + \lambda}{T_0} \int_0^{T_0} \|u(t, v, \lambda, \mu) + v\|^2 U(-t)BU(t)[u(t, v, \lambda, \mu) + v] dt \\
&\quad + \frac{1 + \lambda}{T_0} \int_0^{T_0} U(-t)g(U(t)[u(t, v, \lambda, \mu) + v], \mu) dt.
\end{aligned}$$

Consider  $v$  in the form  $v = (s, 0) = se_1$ . Define  $\mathcal{B}_0(s, \lambda, \mu) = \frac{1}{s}\mathcal{B}(se_1, \lambda, \mu)$ . Since  $u(t, se_1, \lambda, \mu) + se_1 = s[J(t, se_1, \lambda, \mu) + I]e_1$ , we have

$$\mathcal{B}_0(s, \lambda, \mu) = [\lambda A_0 + (1 + \lambda)\mu I]e_1 + s^2\mathcal{B}_1(s, \lambda, \mu) + \mathcal{B}_2(s, \lambda, \mu),$$

with

$$\begin{aligned}
\mathcal{B}_1(s, \lambda, \mu) &= \\
&\quad \frac{1 + \lambda}{T_0} \int_0^{T_0} \|[J(t, v, \lambda, \mu) + I]e_1\|^2 U(-t)BU(t)[J(t, v, \lambda, \mu) + I]e_1 dt
\end{aligned}$$

and

$$\mathcal{B}_2(s, \lambda, \mu) = \frac{1 + \lambda}{sT_0} \int_0^{T_0} U(-t)g(U(t)[u(t, se_1, \lambda, \mu) + se_1], \mu) dt.$$

A straightforward calculation gives,

$$\mathcal{B}_1(0, 0, 0) = \frac{1 + \lambda}{T_0} \int_0^{T_0} U(-t)BU(t)v dt = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Thanks to the properties of  $g(x, \mu)$ , we have that

$$\mathcal{B}_2(0, 0, 0) = \frac{\partial}{\partial s}\mathcal{B}_2(0, 0, 0) = \frac{\partial^2}{\partial s^2}\mathcal{B}_2(0, 0, 0) = 0.$$

Now we solve the bifurcation equation.  $\mathcal{B}_0 : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2$  is  $C^1$ ,  $\mathcal{B}_0(0, 0, 0) = 0$ , and

$$D_{\lambda, \mu}\mathcal{B}_0(0, 0, 0) = \begin{bmatrix} 0 & 1 \\ \omega & 0 \end{bmatrix},$$

is invertible. By the implicit function theorem, there are smooth curves  $\lambda(s)$ ,  $\mu(s)$  such that  $\lambda(0) = 0$ ,  $\mu(0) = 0$ , and  $\mathcal{B}_0(s, \lambda(s), \mu(s)) = 0$ .

This construction yields a unique (up to rotation) family of periodic solutions

$$x(t, s) = U(t)[u(t/(1 + \lambda(s)), se_1, \lambda(s), \mu(s)) + se_1].$$

This family depends continuously on the parameter  $s$ . The amplitude is  $\approx s$ , and the period is  $T_0(1 + \lambda(s))$ .

Differentiation of the bifurcation equation with respect to  $s$  gives  $\lambda'(0) = \mu'(0) = 0$ , and

$$\begin{bmatrix} 0 & 1 \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \lambda''(0) \\ \mu''(0) \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix}.$$

□