1. Since $H(t,z) \in C^1$ and $H(t,0) = 0$, we may write

$$H(t,z) = \int_0^1 \frac{d}{d\sigma} H(t,\sigma z)\, d\sigma$$

$$= \int_0^1 D_z H(t,\sigma z)\, d\sigma \cdot z$$

$D_z H(t,z)$ is continuous, and hence it is uniformly continuous on any compact set,

$K = [0,T] \times B_1(0)$, for example.

So for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $(t_1, z_1), (t_2, z_2) \in K$ and $\|(t_1,z_1) - (t_2,z_2)\| < \delta$ implies that $\|D_z H(t_1, z_1) - D_z H(t_2, z_2)\| < \varepsilon$.

Since $D_z H(t,0) = 0$, we have that for any $\varepsilon > 0$ there exists $\delta > 0$ s.t. $\|z\| = \|(t,z) - (t,0)\|$ $(0 \le t \le T)$ implies $\|D_z H(t,z)\| < \varepsilon$. It follows that

$\|z\| < \delta$ implies $\|H(t,z)\| \le \varepsilon \|z\|$, $0 \le t \le T$.

Since H is T-periodic, this holds for all $t \in \mathbb{R}$. The rest of the proof is the same as Thm 2.9.2.

2a) By Cor 3.6.1,

$$W_s^{loc}(0) = \{x_0 \in \mathbb{R}^n : P_s x_0 \in \bar{U}, \ \sup_{t \geq 0} \|\Phi_t(x_0)\| < r\}.$$

If $x_0 \in W_s(0)$, then $\Phi_t(x_0) \to 0$ as $t \to \infty$.

Thus, there exists $t_0 > 0$ s.t.

$$\Phi_{t_0}(x_0) \in \bar{U} \ \text{ \& } \ \|\Phi_t(x_0)\| < r \ \text{ for } t > t_0.$$

This means $\Phi_{t_0}(x_0) \in W_s^{loc}(0)$; i.e. $x_0 \in \Phi_{-t_0}(x_0)$.

On the other hance, if $x_0 \in \bigcup_{t \leq 0} \Phi_t(W_s^{loc}(0))$,

then $\Phi_{t_0}(x_0) \in W_s^{loc}(0)$ for some $t_0 \leq 0$.

It follows that $\lim_{t \to \infty} \Phi_{t+t_0}(x_0) =$

$\lim_{t \to \infty} \Phi_t(\Phi_{t_0}(x_0)) = 0$; i.e. $\Phi_{t_0}(x_0) \in W_s(0)$.

But then $x_0 = \Phi_{-t_0}(\Phi_{t_0}(x_0)) \in W_s(0)$ since

$W_s(0)$ is invariant.

2b) Write

$$W_s^{loc}(0) = \{ x : P_s x \in U \quad P_u x = \eta(P_s x) \}$$

$$\eta(0) = 0 \qquad D\eta(0) = 0.$$

and

$$W_u^{loc}(0) = \{ x : P_u x \in V,\ P_s x = \xi(P_u x) \}$$

$$\xi(0) = 0 \qquad D\xi(0) = 0.$$

By the properties of η, ξ, given $\varepsilon > 0$ there is a $\delta > 0$ s.t. $\|x\| < \delta$ implies

$$\|\eta(P_s x)\| < \varepsilon \|P_s x\| \qquad \|\xi(P_u x)\| < \varepsilon \|P_u x\|$$

Let $\varepsilon = \dfrac{\|P_s\| + \|P_u\|}{100}$, and choose $\delta > 0$ accordingly. Then

$$x \in B_\delta(0) \cap W_s^{loc}(0) \cap W_u^{loc}(0)$$

implies

$$\|x\| = \|P_s x + P_u x\| = \|\xi(P_u x) + \eta(P_s x)\|$$

$$\leq \varepsilon \|P_u x\| + \varepsilon \|P_s x\| \leq \frac{1}{100} \|x\|,$$

Thus, $\|x\| = 0$.

3. If $x \in \mathbb{X}$, then $F \circ x$ is a bdd cont function, i.e. $T(x) \in \mathbb{X}$.

Since $F \in C^1$, $x \mapsto DF(x)$ is continuous, and hence uniformly continuous on any compact set, K. For any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $x_1, x_2 \in K$ $\|x_1 - x_2\| < \delta$ implies

$$\|DF(x_1) - DF(x_2)\| < \varepsilon.$$

For $x_1, x_2 \in B_r(0) \equiv K$, write

$$F(x_1) - F(x_2) = \int_0^1 D_x F(\sigma x_1 + (1-\sigma)x_2)\, d\sigma \cdot (x_1 - x_2)$$

Then

$$\|F(x_1) - F(x_2)\| \leq \max_{B_r(0)} \|D_x F\| \; \|x_1 - x_2\|$$

and

$$\|F(x_1) - F(x_2) - DF(x_2)(x_1 - x_2)\|$$

$$\leq \max_{0 \leq \sigma \leq 1} \|DF(\sigma x_1 + (1-\sigma)x_2) - DF(x_2)\| \; \|x_1 - x_2\|$$

$$\leq \varepsilon \; \|x_1 - x_2\|.$$

Fix $x_0 \in \mathbb{X}$. Let $r > \|x_0\|_{\mathbb{X}}$. Fix $\varepsilon > 0$

Choose $\delta > 0$ as above. For any $x \in \mathbb{X}$ with $\|x - x_0\|_{\mathbb{X}} < \delta$ and $\|x\|_{\mathbb{X}} < r$ we have

$$\| F(x(t)) - F(x_0(t))\| \leq \max_{B_r(0)} \|D_x F\| \; \|x_1(t) - x_2(t)\|$$

$$\leq C \; \|x_1 - x_2\|_{\mathbb{X}}$$

Thus,

$$\| T(x) - T(x_1) \|_{\mathbb{X}} \leq C \; \|x_1 - x_2\|_{\mathbb{X}}$$

which proves that T is continuous.

Also

$$\| F(x(t)) - F(x_0(t)) - DF(x_0(t))\,(x(t) - x_0(t))\|$$

$$\leq \varepsilon \; \|x(t) - x_0(t)\|$$

$$\leq \varepsilon \; \|x - x_0\|_{\mathbb{X}}$$

Thus,

$$\| T(x) - T(x_0) - DT(x_0)(x - x_0)\|_{\mathbb{X}} \leq \varepsilon \|x - x_0\|$$

with $DT(x_0)(t) = F(x_0(t))$.

This proves that T is Fréchet differentiable at x_0 and $DT(x_0) = DF \circ x_0$.

Finally, again by uniform continuity for x, x_0 as above

$$\| DT(x) - DT(x_0) \|_{\mathcal{L}(\mathbb{X}, \mathbb{X})}$$

$$= \sup_t \| DF(x(t)) - DF(x_0(t)) \|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)}$$

$$\leq \varepsilon ,$$

this proves that $x \mapsto DT(x)$ is continuous.

thus $T: \underline{\mathbb{X}} \to \underline{\mathbb{X}}$ is C^1.

4. Since $D_x f_0(0) = A$ is hyperbolic its e-values λ satisfy $\lambda T \notin 2\pi i \mathbb{Z}$. Thus, Thm 7.1.1 applies. There is a nghd U of $0 \in \mathbb{R}^n$ and $\varepsilon_0 > 0$ s.t. for $|\varepsilon| < \varepsilon_0$ there is a unique point $p(\varepsilon) \in U$ for which $x(t, p(\varepsilon), \varepsilon)$ is a T-periodic soln. of

$$x' = f(t, x, \varepsilon).$$

Set $\varphi_\varepsilon(t) = x(t, p(\varepsilon), \varepsilon)$.
Let $\Psi(t, \varepsilon)$ be the fundamental matrix for $A(t, \varepsilon) = D_x f(t, \varphi_\varepsilon(t), \varepsilon)$. The Floquet multipliers of $\varphi_\varepsilon(t)$ are the e-values of $\Psi(T, \varepsilon)$. By continuous dependence, $\Psi(T, \varepsilon) \approx \Psi(T, 0) = \exp(TA)$. Since A is hyperbolic, the e-values of $\exp(TA)$ lie off the unit circle. The same holds for $\Psi(T, \varepsilon)$ for ε small. Thus, $\varphi_\varepsilon(t)$ is hyperbolic.

5. a)

$$x(t,p,\varepsilon) = \begin{bmatrix} (p_1 + \frac{1}{3}p_2^2 + \varepsilon)e^{t} + \frac{-1}{3}p_2^2 e^{-2t} + \varepsilon(\sin t - \cos t) \\ p_2 e^{-t} \end{bmatrix}$$

For $\varepsilon = 0$,

b) the linearized problem is $y' = Ay$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

the solution of which is $y(t,p) = \exp At\, p$

$= \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} p$. Define $\Lambda(p) = \begin{bmatrix} p_1 + \frac{1}{3}p_2^2 \\ p_2 \end{bmatrix}$.

Then $\Lambda(x(t,p,0)) = y(t, \Lambda(p))$. Λ is

C^∞ and $\Lambda^{-1}(q) = \begin{bmatrix} q_1 - \frac{1}{3}q_2^2 \\ q_2 \end{bmatrix}$ is C^∞.

Thus, Λ is a homeomorphism with conjugates the flow.

c). For $\varepsilon = 0$, $x(t,p,0) \to 0$ as $t \to \infty$ iff

$p \in \{p : p_1 + \frac{1}{3}p_2^2 = 0\}$, and $x(t,p,0) \to 0$ as

$t \to -\infty$ iff $p_2 = 0$. Thus

$W_s(0) = \{p : p_1 + \frac{1}{3}p_2^2 = 0\}$

$W_u(0) = \{p : p_2 = 0\}$.

d) $x(t,p,\varepsilon)$ is 2π-periodic iff $p_2 = 0$ and $p_1 + \varepsilon = 0$.

Set $p(\varepsilon) = (-\varepsilon, 0)$. then

$$\varphi_\varepsilon(t) = x(t, p(\varepsilon), \varepsilon) = \varepsilon(\sin t - \cos t)$$

is the unique 2π-periodic solution.

e) The Floquet multipliers of $\varphi_\varepsilon(t)$ are the e-values of $\underset{\underset{2\pi}{\parallel}}{\underline{\Psi}(T, \varepsilon)}$, where $\underline{\Psi}(t, \varepsilon)$ is the Fundamental matrix of $A(t, \varepsilon) = D_x \left[\begin{matrix} x_1 + x_2^2 + \varepsilon\cos t \\ -x_2 \end{matrix} \right] \Bigg|_{x=\varphi_\varepsilon}$

$$= \begin{bmatrix} 1 & 2x_2 \\ 0 & -1 \end{bmatrix} \Bigg|_{x=\varphi_\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad \text{Thus,}$$

$$\underline{\Psi}(t, \varepsilon) = \exp \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T = \begin{bmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}.$$

The Floquet mult. are $e^{2\pi}$, $e^{-2\pi}$.

The periodic solution φ_ε is hyperbolic, consistent with problem # 4.

f) To answer the final part, we need to solve the system with initial data at time τ.

$$x(t,\tau,p) = \begin{bmatrix} \left[p_1 + \frac{p_2^2}{3} + \frac{\varepsilon}{2}(\cos\tau - \sin\tau)\right]e^{(t-\tau)} - \frac{p_2^2}{3}e^{-2(t-\tau)} + \frac{\varepsilon}{2}(\sin t - \cos t) \\ p_2\, e^{-(t-\tau)} \end{bmatrix}$$

$$W_s(0) = \left\{ (\tau,p) : \quad p_1 + \frac{p_2^2}{3} + \frac{\varepsilon}{2}(\cos\tau - \sin\tau) = 0 \right\}$$

$$W_u(0) = \left\{ (\tau,p) : \quad p_2 = 0 \right\}$$

$$(\tau,p) \in W_s(0) \Rightarrow x(t,\tau,p) \to \begin{bmatrix} \frac{\varepsilon}{2}(\cos t - \sin t) \\ 0 \end{bmatrix}, \quad t \to \infty$$

$$(\tau,p) \in W_u(0) \Rightarrow x(t,\tau,p) \longrightarrow \quad '' \quad , \quad t \to -\infty$$