

## Analysis Crib Sheet

### IMPLICIT FUNCTION THEOREM

**Theorem 1.** Let  $f(x, y)$  be a  $C^1$  mapping of an open set  $\Omega \subset \mathbb{R}^{n+m}$  into  $\mathbb{R}^n$ , such that  $f(x_0, y_0) = 0$  for some point  $(x_0, y_0) \in \Omega$ .

Set  $A = D_x f(x_0, y_0)$ ,  $B = D_y f(x_0, y_0)$ , and assume that  $A = D_x f(x_0, y_0)$  is invertible.

Then there exist open sets  $U \subset \mathbb{R}^{n+m}$  and  $W \subset \mathbb{R}^m$ , with  $(x_0, y_0) \in U$  and  $y_0 \in W$ , and a  $C^1$  function  $g(y)$  from  $W$  to  $\mathbb{R}^n$  having the property that for every  $y \in W$ ,  $(y, g(y))$  is the unique point in  $U$  such that

$$f(g(y), y) = 0.$$

Moreover, it holds that

$$D_y g(y_0) = -A^{-1}B.$$

### INVERSE FUNCTION THEOREM

**Theorem 2.** Suppose that  $f(x)$  is a  $C^1$  mapping of an open set  $\Omega \subset \mathbb{R}^n$  to  $\mathbb{R}^n$ ,  $D_x f(x_0)$  is invertible for some  $x_0 \in \Omega$ , and  $y_0 = f(x_0)$ . Then (a) there exist open sets  $U$  and  $V$  in  $\mathbb{R}^n$  such that  $x_0 \in U$ ,  $y_0 \in V$ ,  $f$  is one-to-one on  $U$ , and  $f(U) = V$ ; and (b) if  $g = f^{-1}$  (which exists by (a)), then  $g$  is a  $C^1$  map from  $V$  to  $U$ .

### CONTRACTION MAPPING PRINCIPLE

Let  $(X, d)$  be a metric space. A mapping  $T : X \rightarrow X$  is called a *contraction* if there exists a constant  $\alpha$ ,  $0 \leq \alpha < 1$ , such that  $D(T(x), T(y)) \leq \alpha d(x, y)$ , for all  $x, y \in X$ . A point  $x \in X$  is called a *fixed point* of  $T$  if  $x = T(x)$ .

**Theorem 3.** If  $T$  is a contraction on a complete metric space, then  $T$  has a unique fixed point.