Consider the matrix $A = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$ 1

I) Compute its eigenvalues & eigenvectors

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2-\lambda & -2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$\iff (2-\lambda)(3-\lambda) - 2 = 0 \iff \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$$

EIGENVALUES $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 4 \end{cases}$

Notice that $\det A = 4 = \lambda_1 \lambda_2$

Trace of $A = 5 = \lambda_1 + \lambda_2$

Eigenvector for $\lambda_1 = 1$ $\vec{V}_1 =$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} 2x - 2y = x \\ -x + 3y = y \end{cases} \iff \begin{cases} x - 2y = 0 \\ -x + 2y = 0 \end{cases}$$

$$\iff x = 2y \qquad \vec{V}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

EIGENVECTOR for $\lambda_2 = 4$ $\vec{V}_2$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} 2x - 2y = 4x \\ -x + 3y = 4y \end{cases} \iff \begin{cases} -2y = 2x \\ -x = y \end{cases}$$

$$\iff y = -x \qquad \vec{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So A is DIAGONALIZABLE.

So

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \iff \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A\vec{v}_2 = \lambda_2 \vec{v}_2 \iff \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So $A\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

or $AV = VD$ $\qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$V = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}.$

V is invertible since ① $\det V = -3 \neq 0$

or ② its columns are eigenvectors corresponding to different eigenvalues so they are Lin. indep

Let us compute V^{-1}

I) $\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & -3/2 & -1/2 & 1 \end{array}\right)$

$\to \left(\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1/3 & -2/3 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 1/3 & -2/3 \end{array}\right)$

So $V^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}$

(II) $$V^{-1} = \frac{1}{\det V}\left(M_{ij}\right)^T = -\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

so $$\boxed{A = VDV^{-1}}$$

Problem 1 Find two matrices B_1, B_2 such that

$$B_1^2 = B_2^2 = A \quad ; \text{ i.e. } B_j = \sqrt{A} \quad j=1,2.$$

(This can be done because the eigenvalues are real, positive and A diagonalizable).

ANSWER
$$B_1 = V + D^{1/2} V^{-1} = V \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} V^{-1}$$

$$B_2 = V - D^{1/2} V^{-1} = V \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} V^{-1}$$

so $$B_1 = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 4/3 & -2/3 \\ -1/3 & 5/3 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}$$

$$= \begin{pmatrix} -4/3 & 2/3 \\ 1/3 & -5/3 \end{pmatrix}$$

<u>check</u> $B_1 B_1 = \begin{pmatrix} 4/3 & -2/3 \\ -1/3 & 5/3 \end{pmatrix} \begin{pmatrix} 4/3 & -2/3 \\ -1/3 & 5/3 \end{pmatrix}$

$$= \frac{1}{9} \begin{pmatrix} 4 & -2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -1 & 5 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 18 & -18 \\ -9 & 27 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

qed.

$B_2 B_2 = B_1 B_1$ qed.

<u>Problem 2</u> Find A^{100} since $A = VDV^{-1}$

<u>Answ</u> $A^2 = VDV^{-1}VDV^{-1} = VD^2V^{-1}$

so $A^{100} = VD^{100}V^{-1} = V \begin{pmatrix} 1 & 0 \\ 0 & 4^{100} \end{pmatrix} V^{-1}$.

<u>Problem 3</u> Compute $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k (VDV^{-1})^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{t^k V D^k V^{-1}}{k} = V \left(\sum_{k=0}^{\infty} \frac{t^k D^k}{k!} \right) V^{-1}$$

$$= V \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^k 4^k}{k!} \end{pmatrix} V^{-1} = V \begin{pmatrix} e^t & 0 \\ 0 & e^{4t} \end{pmatrix} V^{-1}$$

$$= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2e^t & e^{4t} \\ e^t & -e^{4t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2e^t + e^{4t} & 2e^t - 2e^{4t} \\ e^t - e^{4t} & e^t + 2e^{4t} \end{pmatrix}$$

<u>Problem 4</u> <u>check</u> that $e^{tA} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

is <u>the solution</u> of

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$$

<u>or</u>

$$\begin{cases} \dfrac{dx}{dt} = 2x - 2y & x(0) = 1 \\ \dfrac{dy}{dt} = -x + 3y & y(0) = 1. \end{cases}$$

$$e^{tA}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2e^t + e^{4t} & 2e^t - 2e^{4t} \\ e^t - e^{4t} & e^t + 2e^{4t} \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 4e^t - e^{4t} \\ 2e^t + e^{4t} \end{pmatrix} \quad \text{so} \quad \begin{cases} x(t) = \dfrac{4e^t - e^{4t}}{3} \\ y(t) = \dfrac{2e^t + e^{4t}}{3} \end{cases}$$

<u>check</u> ① $x(0) = 1$
$y(0) = 1$

$$x'(t) = \frac{4}{3}\left(e^t - e^{4t}\right)$$

$$2x - 2y = 2\left(\frac{4e^t - e^{4t}}{3}\right) - 2\left(\frac{2e^{2t} + e^{4t}}{3}\right)$$

$$= e^t\left(\frac{8}{3} - \frac{4}{3}\right) + e^{4t}\left(-\frac{2}{3} - \frac{2}{3}\right)$$

$$= e^t \frac{4}{3} - \frac{4}{3}e^{4t} \checkmark$$

SAME FOR $\boxed{y'(t) = -x + 3y}$

Notice that

$$\det(A-\lambda I)=0 \iff (2-\lambda)(3-\lambda)-2=0$$

$$(\lambda-4)(\lambda-1)=0.$$

Consider the polynomial $P(A)=(A-4I)(A-1I)$

$$= A^2-5A+4I.$$

<u>Observe that</u> $P(A)\begin{pmatrix}2\\1\end{pmatrix} = P(A)\vec{v}_1 = (A-4I)(A-1I)\begin{pmatrix}2\\1\end{pmatrix} \equiv 0$

$$P(A)\begin{pmatrix}1\\-1\end{pmatrix} = P(A)\vec{v}_2 = (A-1I)(A-4I)\begin{pmatrix}1\\-1\end{pmatrix} \equiv 0$$

They commute.

So the matrix $P(A)$ vanishes at $\underbrace{\begin{pmatrix}2\\1\end{pmatrix}\ \begin{pmatrix}1\\-1\end{pmatrix}}_{\text{basis}}$ so $P(A)\equiv 0$

$$A^2 = \begin{pmatrix}2 & -2\\-1 & 3\end{pmatrix}\begin{pmatrix}2 & -2\\-1 & 3\end{pmatrix} = \begin{pmatrix}6 & -10\\-5 & 11\end{pmatrix}$$

$$-5A = -5\begin{pmatrix}2 & -2\\-1 & 3\end{pmatrix} = \begin{pmatrix}-10 & 10\\5 & -15\end{pmatrix} \quad +$$

$$4I = 4\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$$

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$