MATH 108 B FALL 2011.

PRACTICE PROBLEMS 1^est MIDTERM.

-1) Let $A = (a_{jk})_{j,k=1,\dots,n}$ be an $n\times n$ matrix

Let $\lambda_1 \dots \lambda_n$ be its eigenvalues, i.e. the roots of the polynomial

$$p(\lambda) = \det(A - \lambda I_{n\times n}) = 0$$

Prove:

ⓐ $\det A = \prod_{j=1}^{n} \lambda_j$

ⓑ $\mathrm{Trace}(A) = a_{11} + \dots + a_{nn} = \lambda_1 + \dots + \lambda_n$

-2) Given $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ find V invertible and D diagonal such that

$$A = VDV^{-1} \qquad \text{(diagonalize } A\text{).}$$

-3) Given $A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$

ⓐ Find its eigenvalues

ⓑ Is A diagonalizable? If yes, find V such that

$A = VDV^{-1}$.

-4) Let E be a vector space $\dim E = n$.
Let $T: E \to E$ be a linear map.

Prove that if $\vec{v}_1, \ldots, \vec{v}_k$ $(k \le n)$ are eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent

-5) Given $A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$.

(a) Compute $B \equiv (A - 5I)(A + 2I)$.

(b) Explain your answer.

-6) For A as in problem 3

(a) Compute $B \equiv (A - 3I)^2 (A - 5I)$

(b) Explain your answer.

-7) Find 2×2 matrices B_1, B_2 such that

$$B_1^2 = B_2^2 = A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$ (see problem 2)

-8) Find a 2×2 matrix A whose eigenvalues are $\lambda_1 = 1$ $\lambda_2 = 4$ and the corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

-9) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+y-2z \\ 2x+3y-4z \\ x+y-z \end{pmatrix}$$

Is T diagonalizable?

-10) Is the matrix $A = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ diagonalizable?

-11) Let $B = \begin{pmatrix} 8 & 12 & 0 \\ 0 & 8 & 12 \\ 0 & 0 & 8 \end{pmatrix}$. Find a real matrix A

such that $B = A^3$.

-12) Given $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Find B^{10^6}.

-13) Let A be an $n\times n$ matrix such that

$A^2 = A$. Find all the possible eigenvalues.
Find all possible diagonal forms of A.

-14) Find A a 3×3 real matrix such that
$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 9$ are its eigenvalues and

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ the corresponding eigenvectors.

– 15) Let A be an $n \times n$ matrix for which $A^k \equiv 0$ for some $k > n$. Show that $A^n \equiv 0$.

– 16) Let $T : E \to E$ be a linear operator (E vector space). Assume that $T(\vec{v}) = \lambda \vec{v}$ for some $\begin{cases} \vec{v} \in E & \vec{v} \neq 0 \\ \lambda \in \mathbb{C} \end{cases}$

Prove: ⓐ $\forall n > 0$ $\vec{v}$ eigenvector of T^n with eigenvalue λ^n

ⓑ for all polynomial $p(x) = a_0 + a_1 x + \dots + a_m x^m$ $\vec{v}$ is an eigenvector for $p(T)$ with eigenvalue $p(\lambda)$.

– 17) Let A, B be $n \times n$ matrices. Suppose that $\lambda \neq 0$ is an eigenvalue of $A \circ B$. Prove that λ is also an eigenvalue of $B \circ A$.

– 18) Is $A = \begin{pmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{pmatrix}$ diagonalizable?

-19) Let A be an $n \times n$ matrix. A IS SAID TO BE HERMITIAN or (self-adjoint) if

$$A^* = \bar{A}^t = A \quad \text{i.e. } A = (a_{jk}); \ \overline{a_{kj}} = a_{jk}$$

Prove (a) If A is HERMITIAN then all its eigenvalues are real.

(b) If A is HERMITIAN then eigenvectors corresponding to different eigenvalues are perpendicular.

-20) Let B be an $n \times n$ matrix. B IS SAID TO BE SKEW-HERMITIAN if

$$B^* = \bar{B}^t = -B. \quad \text{i.e. } B = (b_{jk}); \ \bar{b}_{kj} = -b_{jk}$$

Prove that if B is skew hermitian then all its eigenvalues are pure imaginaries.

-21) Prove that any $n \times n$ matrix C can be written as the sum of A hermitian and B skew hermitian

$$C = A + B.$$

Is this representation unique?