MATH 5-A FALL 2011 FINAL EXAM

NAME :
Perm Number :
TA :
Diss. Section Time :

Answer in BOXES . Write clearly. Show your work. Otherwise no partial credit.

Q 1. (10 points) ____ __

Q 2. (6+4 points) ____ __

Q 3. (12 points) ____ __

Q 4. (8 points) ____ __

Q 5. (10 points) ____ __

FINAL EXAM ____ _____

Midterm ____ ____

HW. ____ ______

DISS. ____ ______

FINAAL GRADE ____ __

(1) Find a 2×2 matrix A having $\lambda_1 = 2,\ \lambda_2 = 3$ as eigenvalues and

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as corresponding eigenvectors (Suggestion: verify your answer).

$$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

$\det A = 6$

$\text{Trace}(A) = 5$

$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$AV = VD \qquad V = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Find: V^{-1}.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \qquad V^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$A = VDV^{-1} = V\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

(2) (a) Given the function $x(t) = c_1 e^t + c_2 e^{-t} + \cos(t) + t$, find a second order linear constant coefficients differential equation for which this function is the general solution (Suggestion: verify your answer).

(b) If $-\cos(2t) - \sin(2t) = A\cos(2(t - \delta/2))$. Find A and δ (Suggestion: verify your answer).

(a) Roots ± 1 polynomial $(r-1)(r+1) = r^2 - 1$

equation $x'' - x = \_\_\_$

$f(t) = \cos(t) + t$ $(\cos t + t)'' - (\cos t - t) = -\cos t - \cos t - t$

so $\boxed{x'' - x = -2\cos(t) - t}$

(b) $A\cos(2(t - \delta/2)) = A\cos(2t - \delta) = A\cos(2t)\cos(\delta) + A\sin(2t)\sin(\delta)$
$= -\cos(2t) - \sin(2t)$

so $\begin{cases} A\cos(\delta) = -1 \\ A\sin(\delta) = -1 \end{cases}$ $\Rightarrow A = \sqrt{2}$

$\tan(\delta) = 1$ with $\sin, \cos < 0$

Third quadrant

$\delta = \pi/4 + \pi.$

$\cos\left(\frac{\pi}{4} + \pi\right) = \sin\left(\frac{\pi}{4} + \pi\right) = -\frac{\sqrt{2}}{2}$

(a) $x'' - x = -2\cos(t) - t$

(b) $A = \sqrt{2}$, $\delta = \pi + \pi/4 = \frac{5\pi}{4}$

(3) Consider the three systems

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A_j \begin{pmatrix} x \\ y \end{pmatrix}, \quad j = 1, 2, 3,$$

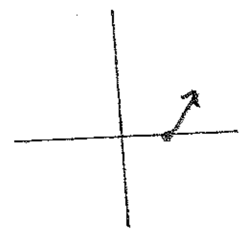
with

$$A_1 = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}.$$

In each case $j = 1, 2, 3$, classify the origin $(0, 0)$ (the isolated equilibrium solution $(x(t), y(t)) \equiv (0, 0)$) as a saddle, source (repelling node), sink (attracting source), spiral sink (attracting spiral), repelling spiral (spiral source), or a center. In the case of a spiral or a center decide if the solutions move clock-wise or counter clock-wise.

① A_1 $\lambda_1 = 5 + 2i$, $\lambda_2 = 5 - 2i$ $\lambda_1 + \lambda_2 = 10$, $\lambda_1 \cdot \lambda_2 = 29$. spiral source ✓

at $(1,0)$ $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (6, 5)$



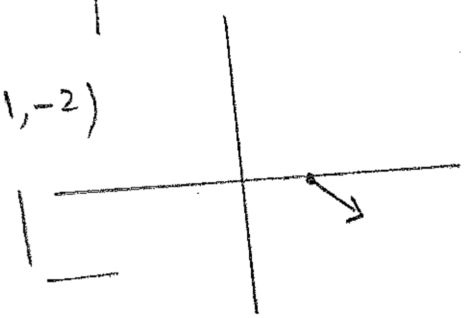
counter-clock wise

② A_2 $\lambda_1 + \lambda_2 = 0$, $\lambda_1 \cdot \lambda_2 = 1$ $\Rightarrow$ $\lambda_1 = i$, $\lambda_2 = -i$ ✓

Center

clockwise

at $(0,0)$ $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (1, -2)$



$\det(A - \lambda I)$

③ A_3 $\lambda_1 + \lambda_2 = 4$, $\lambda_1 \cdot \lambda_2 = -12$

$(1-\lambda)(3-\lambda) - 15 = 0$

$\lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2) = 0$ ✓ $\lambda_1 = -2$, $\lambda_2 = 6$.

for A_1 SPIRAL SOURCE / COUNTER CLOCK WISE
for A_2 CENTER / clockwise
for A_3 Saddle.

(4) For the in-homogemeous system

$$\begin{cases} x' = -3x + y + 8t, \\ y' = \ \ x - 3y + 2, \end{cases}$$

given the solution of the (associated) homogeneous system

$$\begin{pmatrix} x_h \\ y_h \end{pmatrix}(t) = c_1\, e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\, e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

find a particular solution (Suggestion: verify your answer).

TRY $\begin{cases} x_p(t) = at + b \\ y_p(t) = ct + d \end{cases}$ so $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 8t \\ 2 \end{pmatrix}$

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -3(at+b) + ct + d + 8t \\ at + b - 3(ct+d) + 2 \end{pmatrix}$$

$$\begin{cases} ② & 0 = -3a + c + 8 \\ ③ & a = -3b + d \\ ① & 0 = a - 3c \\ ③' & c = b - 3d + 2. \end{cases}$$

① $a = 3c$
↓
② $-9c + c + 8 = 0 \Rightarrow c = 1$
$\Rightarrow a = 3$

$$\begin{cases} -3b + d = 3 \\ b - 3d = -1 \end{cases}$$

$\Rightarrow 8d = 0$

$d = 0 \Rightarrow b = -1$

$a = 3 \quad c = 1, \quad b = -1, \quad d = 0$

$x(t) = 3t - 1$
$y(t) = t$

(5) For the system of equations

$$\begin{cases} x' = -6x + 5y, \\ y' = -5x + 4y. \end{cases}$$

(a) Find the solution satisfying the initial values

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(Suggestion: verify your answer).
(b) Determine if the origin (equilibrium solution) is stable or unstable.

(a) $A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$ det $A = -24 + 25 = 1$

trace $A = -2$

So $\lambda_1 = \lambda_2 = -1$ eigenvector $\vec{v}_1$ $\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ generalized eigenvector $\vec{v}_2$ $(A - \lambda_1 I)\vec{v}_2 = \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= A + I = \begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}$ $\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $-5x + 5y = 1$ $x = -1/5$ $y = 0$

$A - \lambda I$

$\vec{v}_2 = \begin{pmatrix} -1/5 \\ 0 \end{pmatrix}$ so general solution

$$c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/5 \\ 0 \end{pmatrix}\right]$$

at $t=0$ $c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$c_1 = 0$ $c_2 = -5$

solution

$$-5e^{-t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/5 \\ 0 \end{pmatrix}\right]$$

(a) $x(t) = e^{-t}(1 - 5t)$

$y(t) = e^{-t}(-5t)$

(b) STABLE $\lambda_1 = \lambda_2 = -1$.

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Q 2. (6+4 points) _____ __

Q 3. (12 points) _____ __

Q 4. (8 points) _____ __

Q 5. (10 points) _____ __

FINAL EXAM ____ ______

Midterm ____ ____

HW. ____ ______

DISS. ____ ______

(b) FINAL GRADE _____ __

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

(1) Find a 2×2 matrix A having $\lambda_1 = -3,\ \lambda_2 = 2$ as eigenvalues and

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$D = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$$

as corresponding eigenvectors (Suggestion: verify your answer).

$AV = VD \qquad V = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ Find V^{-1}

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -4 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 5 & 4 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/5 & 1/5 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 1/5 & -1/5 \\ 0 & 1 & 4/5 & 1/5 \end{array}\right) \quad \text{so} \quad V^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$$

$$A = VDV^{-1} = \frac{1}{5} V \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} -3 & 3 \\ 8 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 5 & 5 \\ 20 & -10 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

(2) (a) Given the function $x(t) = c_1 e^{-t} + c_2 e^{t} + t^2 + t$, find a second order linear constant coefficients differential equation for which this function is the general solution (Suggestion: verify your answer).
(b) If $-\cos(2t) - \sin(2t) = A\cos(2(t - \delta/2))$. Find A and δ (Suggestion: verify your answer).

(a) roots ± 1 polynomial $(r-1)(r+1) = r^2 - 1$

equation $x'' - x = f(t)$

with $f(t) = (t^2+t)'' - (t^2+t) = 2 - t - t^2$.

$$x'' - x = 2 - t - t^2$$

(b) $A\cos(2t - \delta) = A\cos(2t)\cos\delta + A\sin(2t)\sin(\delta)$

$= -\cos(2t) - \sin(2t)$

$\begin{cases} A\cos(\delta) = -1 \\ A\sin(\delta) = -1 \end{cases} \Rightarrow A = \sqrt{2}$ $\tan(\delta) = 1$ with sin, cos < 0

Third quadrant

$\delta = \pi/4 + \pi$

$\cos(\pi + \pi/4) = \sin(\pi + \pi/4) = -\frac{\sqrt{2}}{2}$

(a) $x'' - x = 2 - t - t^2$

(b) $A = \sqrt{2}$, $\delta = \pi + \pi/4$.

(3) Consider the three systems

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A_j \begin{pmatrix} x \\ y \end{pmatrix}, \quad j = 1, 2, 3,$$

with

$$A_1 = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} \quad A_3 = \begin{pmatrix} -6 & 1 \\ -5 & -4 \end{pmatrix}.$$

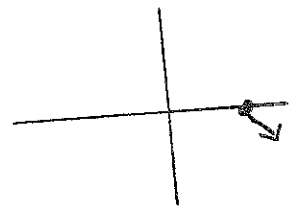
In each case $j = 1, 2, 3$, classify the origin $(0,0)$ (the isolated equilibrium solution $(x(t), y(t)) \equiv (0,0)$) as a saddle, source (repelling node), sink (attracting source), spiral sink (attracting spiral), repelling spiral (spiral source), or a center. In the case of a spiral or a center decide if the solutions move clock-wise or counter clock-wise.

① A_1 eigenvalues $(1-\lambda)(3-\lambda) - 15 = 0 \Leftrightarrow \lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2) = 0$.

$\lambda_1 = -2 \quad \lambda_2 = 6$. Saddle ✓

② $\lambda_1 = 2i \quad \lambda_2 = -2i$ center at $(x,y) = (1,0)$ $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (2,-1)$

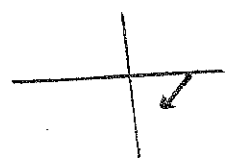
clock wise ✓



③ $\lambda_1 = -5 + 2i$
$\lambda_2 = -5 - 2i$ spiral sink at $(1,0)$ $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (-6,-5)$

clockwise

✓



for A_1 SADDLE

for A_2 CENTER / CLOCK-WISE

for A_3 SPIRAL SINK / CLOCK-WISE

(4) For the in-homogemeous system

$$\begin{cases} x' = -3x + y + 8t, \\ y' = \ \ x - 3y + 2, \end{cases}$$

given the solution of the (associated) homogeneous system

$$\begin{pmatrix} x_h \\ y_h \end{pmatrix}(t) = c_1\, e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\, e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

find a particular solution (Suggestion: verify your answer).

TRY $\begin{cases} x_p(t) = at + b \\ y_p(t) = ct + d \end{cases}$ So $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 8t \\ 2 \end{pmatrix}$

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -3(at+b) + ct + d + 8t \\ at + b - 3(ct+d) + 2 \end{pmatrix}$$

(2) $0 = -3a + c + 8$
(3) $a = -3b + d$
(1) $0 = a - 3c$
(3)' $c = b - 3d + 2$.

(1) $a = 3c$
↓
(2) $-9c + c + 8 = 0 \Rightarrow c = 1$
$\Rightarrow a = 3$

$\begin{cases} -3b + d = 3 \\ b - 3d = -1 \end{cases}$
$\Rightarrow 8d = 0$

$d = 0 \Rightarrow b = -1$

$a = 3 \quad c = 1, \quad b = -1, \quad d = 0$

$x(t) = 3t - 1$

$y(t) = t$

(5) For the system of equations

$$\begin{cases} x' = 3x - 18y, \\ y' = 2x - \ \ 9y. \end{cases}$$

(a) Find the solution satisfying the initial values

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

(Suggestion: verify your answer).
(b) Determine if the origin (equilibrium solution) is stable or unstable.

(a) eigenvalues $\lambda_1 = \lambda_2 = -3$

$A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$ trace$(A) = -6$

det$(A) = -27 + 36 = 9$.

eigenvector $\vec{v}_1$ $\begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$ $\begin{cases} 3x - 18y = -3x \\ 2x - 9y = -3y \end{cases}$

$\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\begin{cases} 6x - 18y = 0 \\ 2x - 6y = 0 \end{cases} \iff 3y = x$

$\vec{v}_2$: generalized eigenvector $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

$(A + 3I)\vec{v}_2 = \vec{v}_1$ $\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ TAKE $\vec{v}_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

GENERAL SOLUTION $c_1 e^{-3t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-3t}\left[t\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right]$

at $t=0$ $c_1\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ $c_2 = 1$, $c_1 = 0$

OUR SOLUTION $e^{-3t}\left[t\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}\right]$

(a) $x(t) = e^{-3t}(3t + 1/2)$

$y(t) = e^{-3t}t$

(b) STABLE $\lambda_1 = \lambda_2 < 0$

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Q 2. (6+4 points) ______

Q 3. (12 points) ______

Q 4. (8 points) ______

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FINAL EXAM ________

Midterm ______

HW. ________

DISS. ________

FINAAL GRADE ______

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

(1) Find a 2×2 matrix A having $\lambda_1 = 2,\ \lambda_2 = 3$ as eigenvalues and

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as corresponding eigenvectors (Suggestion: verify your answer).

$$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \qquad \det A = 6 \qquad \text{Trace}(A) = 5$$

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$AV = VD \qquad V = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

Find: V^{-1}.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right) \qquad V^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$A = VDV^{-1} = V\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

(2) (a) Given the function $x(t) = c_1 e^t + c_2 e^{-t} + \cos(t) + t$, find a second order linear constant coefficients differential equation for which this function is the general solution (Suggestion: verify your answer).

(b) If $-\cos(2t) - \sin(2t) = A\cos(2(t - \delta/2))$. Find A and δ (Suggestion: verify your answer).

(a) Roots ± 1 polynomial $(r-1)(r+1) = r^2 - 1$

equation $x'' - x = \_\_\_$

$f(t) = \cos(t) + t$ $(\cos t + t)'' - (\cos t - t) = -\cos t - \cos t - t$

so $\boxed{x'' - x = -2\cos(t) - t}$

(b) $A\cos(2(t - \delta/2)) = A\cos(2t - \delta) = A\cos(2t)\cos(\delta) + A\sin(2t)\sin(\delta)$
$= -\cos(2t) - \sin(2t)$

so $\begin{cases} A\cos(\delta) = -1 \\ A\sin(\delta) = -1 \end{cases}$ $\Rightarrow A = \sqrt{2}$

$\tan(\delta) = 1$ with $\sin, \cos < 0$

Third quadrant

$\delta = \pi/4 + \pi$.

$\cos\left(\frac{\pi}{4} + \pi\right) = \sin\left(\frac{\pi}{4} + \pi\right) = -\frac{\sqrt{2}}{2}$

(a) $x'' - x = -2\cos(t) - t$

(b) $A = \sqrt{2}$, $\delta = \pi + \pi/4 = \frac{5\pi}{4}$

(3) Consider the three systems

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A_j \begin{pmatrix} x \\ y \end{pmatrix}, \quad j = 1, 2, 3,$$

with

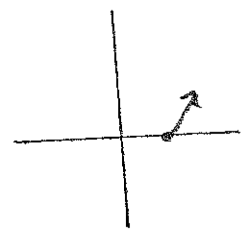
$$A_1 = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}.$$

In each case $j = 1, 2, 3$, classify the origin $(0,0)$ (the isolated equilibrium solution $(x(t), y(t)) \equiv (0,0)$) as a saddle, source (repelling node), sink (attracting source), spiral sink (attracting spiral), repelling spiral (spiral source), or a center. In the case of a spiral or a center decide if the solutions move clock-wise or counter clock-wise.

① A_1 $\lambda_1 = 5 + 2i$ $\lambda_1 + \lambda_2 = 10$

$\lambda_2 = 5 - 2i$ $\lambda_1 \cdot \lambda_2 = 29.$ spiral source

at $(1,0)$ $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (6,5)$



counter-clock wise

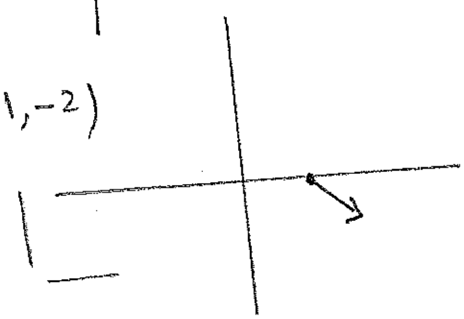
② A_2 $\lambda_1 + \lambda_2 = 0$

$\lambda_1 \cdot \lambda_2 = 1 \Rightarrow \lambda_1 = i$

$\lambda_2 = -i$

Center at $(0,0)$ $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (1,-2)$

clockwise



$\det(A - \lambda I$

③ A_3 $\lambda_1 + \lambda_2 = 4$ $(1-\lambda)(3-\lambda) - 15 = 0$

$\lambda_1 \cdot \lambda_2 = -12$ $\lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2) = 0$ $\lambda_1 = -2$

$\lambda_2 = 6.$

for A_1	SPIRAL SOURCE / COUNTER CLOCKWISE
for A_2	CENTER / clockwise
for A_3	Saddle.

(4) For the in-homogemeous system

$$\begin{cases} x' = -3x + y + 8t, \\ y' = \ \ x - 3y + 2, \end{cases}$$

given the solution of the (associated) homogeneous system

$$\begin{pmatrix} x_h \\ y_h \end{pmatrix}(t) = c_1\, e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\, e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

find a particular solution (Suggestion: verify your answer).

TRY $\begin{cases} x_p(t) = at + b \\ y_p(t) = ct + d \end{cases}$ So $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 8t \\ 2 \end{pmatrix}$

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -3(at+b) + ct + d + 8t \\ at + b - 3(ct+d) + 2 \end{pmatrix}$$

(2) $0 = -3a + c + 8$
(3) $a = -3b + d$
(1) $0 = a - 3c$
(3)' $c = b - 3d + 2$.

$a = 3 \quad c = 1, \quad b = -1, \quad d = 0$

(1) $a = 3c$
↓
(2) $-9c + c + 8 = 0 \Rightarrow c = 1$
$\Rightarrow a = 3$

$\begin{cases} -3b + d = 3 \\ b - 3d = -1 \end{cases}$

$\Rightarrow 8d = 0$

$d = 0 \Rightarrow b = -1$

$x(t) = 3t - 1$

$y(t) = t$

(5) For the system of equations

$$\begin{cases} x' = -6x + 5y, \\ y' = -5x + 4y. \end{cases}$$

(a) Find the solution satisfying the initial values

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(Suggestion: verify your answer).
(b) Determine if the origin (equilibrium solution) is stable or unstable.

(a) $A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$ $\quad \det A = -24 + 25 = 1$

trace $A = -2$

So $\lambda_1 = \lambda_2 = -1$ eigenvalue $\vec{v}_1$ $\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ generalized eigenvector $\vec{v}_2$ $(A - \lambda_1 I)\vec{v}_2 = \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= A + I = \begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}$ $\quad \begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad -5x + 5y = 1$ $\quad x = -1/5 \quad y = 0$

$A - \lambda I$

$\vec{v}_2 = \begin{pmatrix} -1/5 \\ 0 \end{pmatrix}$ so general solution

$$c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/5 \\ 0 \end{pmatrix}\right]$$

at $t=0$ $\quad c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$c_1 = 0 \quad c_2 = -5$

solution

$$-5e^{-t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/5 \\ 0 \end{pmatrix}\right]$$

(a) $x(t) = e^{-t}(1 - 5t)$

$y(t) = e^{-t}(-5t)$

(b) STABLE $\quad \lambda_1 = \lambda_2 = -1.$