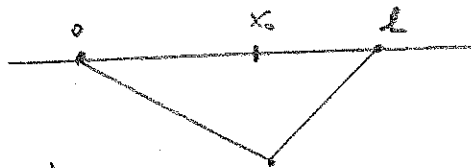


We want  $G(x, x_0)$  st.

①  $G$  continuous (smooth for  $x \neq x_0$ )

②  $\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0) = 1$  so  $\partial_x^2 G(x, x_0) = \delta(x - x_0)$

③  $G(x, x_0) = 0$  if  $x = 0$  or  $l$ .



$$G(x, x_0) = \begin{cases} ax & \text{for } 0 < x < x_0 < l \\ b(x-l) & \text{for } x_0 < x < l. \end{cases}$$

$$\text{st } \begin{cases} \text{① } ax_0 = b(x_0 - l) \\ \text{② } b - a = 1 \end{cases} \Leftrightarrow \begin{cases} (b-a)x_0 = bl \\ b-a = 1 \end{cases}$$

Hence  $b = \frac{x_0}{l}$      $a = \frac{x_0 - l}{l}$

and  $G(x, x_0) = \begin{cases} \frac{(x_0 - l)x}{l} & 0 < x \leq x_0 \leq l \\ \frac{x_0}{l}(x - l) & x_0 \leq x \leq l. \end{cases}$

$G(x, x_0)$  satisfies ①, ② & ③. Finally let us see that

$G(x, x_0) - \frac{1}{2}|x - x_0|$  is harmonic

$$G(x, x_0) - \frac{1}{2}|x - x_0| = \begin{cases} \frac{(x_0 - l)x}{l} - \frac{(x_0 - x)}{2}, & x < x_0 \\ \frac{x_0}{l}(x - l) - \frac{(x - x_0)}{2}, & x_0 < x \end{cases}$$

(2)

Since  $G(x, x_0) - \frac{1}{2}|x-x_0|$  is piecewise linear  
it suffices to see that is  $C^2$  at  $x=x_0$

$$G(x_0^\pm, x_0) - \frac{1}{2}(x_0^\pm, x_0) = 0$$

$$\partial_x \left( G(x, x_0) - \frac{1}{2}|x-x_0| \right) \Big|_{x=x_0^\pm} = \begin{cases} \frac{x_0}{c} - \frac{1}{2} & \text{at } x_0^+ \\ \frac{x_0}{c} - 1 + \frac{1}{2} & \text{at } x_0^- \end{cases}$$

and

$$\partial_x^2 \left( G(x, x_0) - \frac{1}{2}|x-x_0| \right) \equiv 0.$$

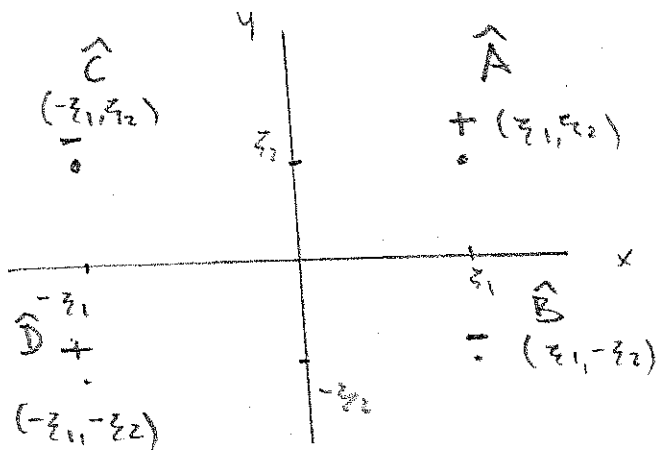
q.d.

Chapter 7 EXERCISE 17 W. STRAUSS' book.

(a) A fundamental solution of the Laplacian  $\Delta$  in  $\mathbb{R}^2$

is 
$$\frac{\log |\vec{x} - \vec{\xi}|}{2\pi} \quad \vec{x} = (x, y) \quad \vec{\xi} = (\xi_1, \xi_2)$$

s.t. 
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \Delta \left( \frac{\log |\vec{x} - \vec{\xi}|}{2\pi} \right) = \delta(\vec{x} - \vec{\xi}).$$



on  $Q = \{(x, y) \mid x, y > 0\}$  we have (by symmetry).

$$\begin{aligned} G((x, y), (\xi_1, \xi_2)) &= \frac{1}{2\pi} \left[ \log \left( (x - \xi_1)^2 + (y - \xi_2)^2 \right)^{1/2} - \log \left( (x - \xi_1)^2 + (y + \xi_2)^2 \right)^{1/2} \right. \\ &\quad \left. - \log \left( (x + \xi_1)^2 + (y - \xi_2)^2 \right)^{1/2} + \log \left( (x + \xi_1)^2 + (y + \xi_2)^2 \right)^{1/2} \right] \\ &= \frac{1}{2\pi} \log \left[ \frac{\left( (x - \xi_1)^2 + (y - \xi_2)^2 \right) \cdot \left( (x + \xi_1)^2 + (y + \xi_2)^2 \right)}{\left( (x + \xi_1)^2 + (y - \xi_2)^2 \right) \cdot \left( (x - \xi_1)^2 + (y + \xi_2)^2 \right)} \right]^{1/2} = (*) \end{aligned}$$

$$A = (x - \xi_1)^2 + (y - \xi_2)^2 \quad B = (x - \xi_1)^2 + (y + \xi_2)^2$$

$$C = (x + \xi_1)^2 + (y - \xi_2)^2 \quad D = (x + \xi_1)^2 + (y + \xi_2)^2$$

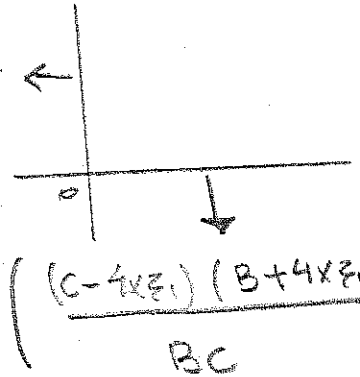
So

$$\textcircled{*} = \frac{1}{4\pi} \log \left[ \frac{AD}{BC} \right]$$

$$\text{but } A = B - 4y\zeta_2 = C - 4x\zeta_1$$

$$D = C + 4y\zeta_2 = B + 4x\zeta_1$$

$$= \frac{1}{4\pi} \log \left( \frac{(B - 4y\zeta_2)(C + 4y\zeta_2)}{BC} \right) = \frac{1}{4\pi} \log \left( \frac{(C - 4x\zeta_1)(B + 4x\zeta_1)}{BC} \right)$$



$$\textcircled{b} \text{ Solve the equations } \begin{cases} \Delta u = 0 \text{ in } Q \\ u(0, y) = g(y) \quad y > 0 \\ u(x, 0) = h(x) \quad x > 0 \end{cases}$$

$$u(x, y) = \int_{\partial\Omega} \partial_n G((x, y), (\zeta_1, \zeta_2)) \text{ data } dS_{\zeta}$$

$$= \int_0^{\infty} -\partial_{\zeta_2} G((x, y), (\zeta_1, \zeta_2)) \Big|_{\zeta_2=0} h(\zeta_1) d\zeta_1$$

$$+ \int_0^{\infty} -\partial_{\zeta_1} G((x, y), (\zeta_1, \zeta_2)) \Big|_{\zeta_1=0} g(\zeta_2) d\zeta_2$$

$$G(x, y), (z_1, z_2) = \frac{1}{4\pi} \log \left( 1 - \frac{4y z_2}{B} + \frac{4y z_2}{C} - \frac{16y^2 z_2^2}{BC} \right)$$

$$\partial_{z_2} G(x, y), (z_1, z_2) = \frac{1}{4\pi} \left( \frac{1}{1 - \frac{4y z_2}{B} + \frac{4y z_2}{C} - \frac{16y^2 z_2^2}{BC}} \right) \cdot$$

$$\cdot \left( -\partial_{z_2} \left( \frac{4y z_2}{B} \right) + \partial_{z_2} \left( \frac{4y z_2}{C} \right) - \partial_{z_2} \left( \frac{16y^2 z_2^2}{BC} \right) \right)$$

$$\partial_{z_2} \left( \frac{4y z_2}{B} \right) = \frac{4y}{B} - \frac{4y z_2 \cdot 2(y + z_2)}{B^2}$$

$$\partial_{z_2} \left( \frac{4y z_2}{C} \right) = \frac{4y}{C} - \frac{4y z_2 (-2)(y - z_2)}{C^2}$$

$$\partial_{z_2} \left( \frac{16y^2 z_2^2}{BC} \right) = \frac{32y^2 z_2}{BC} - \frac{(16y^2 z_2) [B(-2)(y - z_2) + C \cdot 2(y + z_2)]}{B^2 C^2}$$

Hence

$$-\partial_{z_2} G(x, y), (z_1, z_2) \Big|_{z_2=0} = \frac{-1}{4\pi} \left[ -\frac{4y}{(x-z_1)^2 + y^2} + \frac{4y}{(x+z_1)^2 + y^2} \right]$$

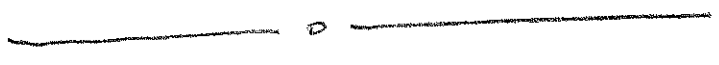
by symmetry

$$-\partial_{z_1} G(x, y), (z_1, z_2) \Big|_{z_1=0} = -\frac{1}{4\pi} \left[ -\frac{4x}{x^2 + (y-z_2)^2} + \frac{4x}{x^2 + (y+z_2)^2} \right]$$

Hence.

$$u(x, y) = \int_0^{\infty} \left( \frac{x}{x^2 + (y - \xi_2)^2} - \frac{x}{x^2 + (y + \xi_2)^2} \right) \frac{g(\xi_2)}{\pi} d\xi_2$$

$$+ \int_0^{\infty} \left( \frac{y}{(x - \xi_1)^2 + y^2} - \frac{y}{(x + \xi_1)^2 + y^2} \right) \frac{h(\xi_1)}{\pi} d\xi_1$$



# F. John Cap 4.

GUSTAVO POUCE  
Section 1

(2)

Let  $L = \Delta + c$  in  $n=3$  dimensions, where  $c = \text{constant}$ .

(a) Find all solutions of  $Lu=0$  with spherical symmetry.

Proof

$$L = \Delta + c = [\partial_x^2 + \partial_y^2 + \partial_z^2 + c]$$

we consider

$$u(x, y, z) = \psi(r)$$

$$r = \sqrt{(x-z_1)^2 + (y-z_2)^2 + (z-z_3)^2}$$

$$= \|(x, y, z) - (z_1, z_2, z_3)\|$$

$$z = (z_1, z_2, z_3)$$

$$\partial_x \psi(r) = \psi'(r) r_x$$

$$r_x = \frac{x-z_1}{r}$$

$$r_{xx} = \frac{r - (x-z_1)^2/r}{r^2} = \frac{r^2 - (x-z_1)^2}{r^3}$$

$$\partial_x^2 \psi(r) = \psi''(r) (r_x)^2 + \psi'(r) r_{xx}$$

(equal for  $y, z$  by symmetry).

Then

$$(\Delta + c) \psi(r) = \psi'' + \frac{2}{r} \psi' + c \psi = 0$$

changed variable.

$$\psi = \frac{\varphi}{r}$$

$$\text{or } \varphi = \psi r$$

$$\left[ \psi' = \frac{\varphi' r - \varphi}{r^2} \right]$$

$$\left[ \psi'' = \frac{\varphi'' r^2 - 2\varphi' r + 2\varphi}{r^3} \right] + \text{simplifying}$$

$$\text{then } \psi'' + \frac{2}{r} \psi' + c \psi = \frac{1}{r} (\varphi'' + c\varphi) = 0$$

then

$$\varphi(t) = A \cos(\sqrt{c}t) + B \sin(\sqrt{c}t)$$

and

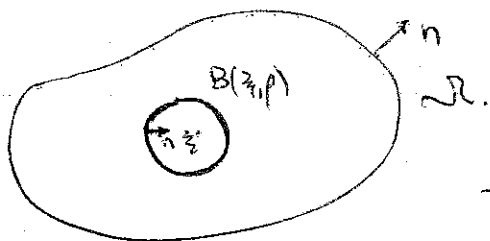
$$\psi(r) = \frac{1}{r} [A \cos(\sqrt{c}r) + B \sin(\sqrt{c}r)]$$

(b) Prove that  $k(x, z) = -\frac{\cos(\sqrt{z}r)}{4\pi r}$   $r = \|x - z\|$   
 is a fundamental solution for  $\Delta$  with pole  $z$ .

Proof

we have the special case of Green's identity,

$$\int_{\Omega} [v(\Delta u + cu) - u(\Delta v + cv)] dx = \int_{\partial\Omega} (v \frac{du}{dn} - u \frac{dv}{dn}) ds.$$



$\Omega_p = \Omega - B(z, \rho)$   
 Taking  $v = \psi(r)$  part (a)  
 $L(v) = 0$

$$\int_{\Omega_p} v(\Delta u + cu) dx = \int_{\partial\Omega} + \int_{S(z, \rho)} (v \frac{du}{dn} - u \frac{dv}{dn}) ds.$$

Since  $v = \psi(r) = \psi(r)$   $\frac{dv}{dn} = -\psi'(r)$  on  $S(z, \rho)$ .

then  $\int_{S(z, \rho)} v \frac{du}{dn} ds = \psi(\rho) \int_{S(z, \rho)} \frac{du}{dn} ds$

but  $\psi(\rho) = O(1/\rho)$  and  $\int_{S(z, \rho)} \frac{du}{dn} ds = O(\rho^2)$ .

then  $\psi(\rho) \int_{S(z, \rho)} \frac{du}{dn} ds \rightarrow 0$  as  $\rho \rightarrow 0$ .

Taking  $A = -\frac{1}{4\pi}$   $B = 0$ . (We can take other  $B$ .)

we have the solution

$$\psi(r) = -\frac{1}{4\pi r} \cos \sqrt{\epsilon} r.$$

then. (Take limit  $\rho \rightarrow 0$ ). ✓

$$\int_{\Sigma} \psi(\Delta u + cu) dx = \int_{\Sigma} \left( \psi \frac{\partial u}{\partial n} - u \frac{\partial \psi}{\partial n} \right) + u(\xi).$$

Taking the notation  $K(x, \xi) = \psi(r) = \psi(|x|) = \psi(\|x - \xi\|)$ .

we have. 
$$u(\xi) = \int_{\Sigma} K(x, \xi) (\Delta u + cu) dx -$$

$$\int_{\Sigma} \left( K(x, \xi) \frac{\partial u}{\partial n} - u \frac{\partial K(x, \xi)}{\partial n} \right) dS. \quad (**)$$

let  $\phi \in C^\infty(\Sigma)$ . putting this function in the place we have.

$$\tilde{u}(\xi) = \int_{\Sigma} K(x, \xi) (\Delta \phi + c\phi) dx - \int_{\Sigma} \left( K(x, \xi) \frac{\partial \phi}{\partial n} - \phi \frac{\partial K(x, \xi)}{\partial n} \right) dS. =$$

$$= \int_{\Sigma} K(x, \xi) \cdot L\phi \, dx = \psi \cdot [L\phi] = \partial_{\xi} \phi.$$

the other side.

$$\int_{S(\frac{3}{2}p)} u \frac{d\sigma}{dn} ds = - \left[ \frac{A \cos(\sqrt{c}r) + B \sin(\sqrt{c}r)}{r} \right]_{r=p} \cdot \int_{S(\frac{3}{2}p)} u ds.$$

$$= - \frac{[-\sqrt{c} A \sin(\sqrt{c}r) + B \sqrt{c} \cos(\sqrt{c}r)]r - [A \cos(\sqrt{c}r) + B \sin(\sqrt{c}r)]}{r^2} \Big|_{r=p} \int_{S(\frac{3}{2}p)} u ds$$

Taking limit  
 $r=p \rightarrow 0$

$$\underbrace{\frac{[-\sqrt{c} A \sin(\sqrt{c}r) + B \sqrt{c} \cos(\sqrt{c}r)]}{r}}_{\alpha(r)} \Big|_{r=p} \int_{S(\frac{3}{2}p)} u ds \rightarrow 0$$

$r=p$

because.

$$\alpha(r) = O\left(\frac{1}{r}\right)$$

$$\int_{S(\frac{3}{2}p)} u ds = O(p^2).$$

then we have.

$$\frac{A \cos(\sqrt{c}r) - B \sin(\sqrt{c}r)}{r^2} \Big|_{r=p} \int_{S(\frac{3}{2}p)} u ds.$$

taking limit  
 $p \rightarrow 0$

$$\frac{A \cos(\sqrt{c}p)}{p^2} \int_{S(\frac{3}{2}p)} u ds \rightarrow A \cdot 4\pi \cdot u\left(\frac{3}{2}\right).$$

(Theorem of Lebesgue).

$$- \frac{B \sin(\sqrt{c}p)}{p^2} \int_{S(\frac{3}{2}p)} u ds \rightarrow 0$$

If we choose  $k_0$  s.t.  $G|_{\partial\Omega} = 0$ .

This is

$$r=p \quad -\frac{\cos(\sqrt{c}p)}{4\pi p} + k_0 \frac{\sin(\sqrt{c}p)}{p} = 0 \Rightarrow k_0 = \frac{\cos(\sqrt{c}p)}{4\pi \sin(\sqrt{c}p)}$$

(assuming  $p \neq 0, \pi \text{ or } 2\pi$  is  $c \geq 0$ ).

Then.

$$u(\xi) = \int_{\partial\Omega} (u(x) \frac{dG(x, \xi)}{dn_x}) \cdot dS_x$$

Now we calculate  $\frac{dG(x, \xi)}{dn_x}$

$$G(x, \xi) = -\frac{\cos(\sqrt{c}r)}{4\pi r} + \frac{k_0 \sin(\sqrt{c}r)}{r}$$

$$\frac{dG}{dn_x}(x, \xi) = -\frac{dG}{dr} \Big|_{r=p} \quad \text{on } \partial\Omega = \{ \|x - \xi\| = p \}$$

$$= \frac{4\pi \sin(\sqrt{c}r) \sqrt{c} + 4\pi \cos(\sqrt{c}r)}{(4\pi r)^2} \Big|_{r=p} + k_0 \left[ \frac{\sqrt{c} \cos(\sqrt{c}r) \cdot r - \sin(\sqrt{c}r)}{r^2} \right] \Big|_{r=p}$$

$$= \frac{4\pi p \sqrt{c} \sin(\sqrt{c}p) + 4\pi \cos(\sqrt{c}p) + 4\pi \sqrt{c} p \frac{\cos(\sqrt{c}p)}{\sin(\sqrt{c}p)} - 4\pi \cos(\sqrt{c}p)}{(4\pi p)^2}$$

$$= \frac{4\pi p \sqrt{c} \sin^2(\sqrt{c}p) + 4\pi p \sqrt{c} \cos^2(\sqrt{c}p)}{(4\pi p)^2 \sin(\sqrt{c}p)} = \frac{4\pi p \sqrt{c} (\cos^2(\sqrt{c}p) + \sin^2(\sqrt{c}p))}{(4\pi p)^2 \sin(\sqrt{c}p)}$$

Then  $L\sigma = \delta z_1$  is a fundamental solution with pole at  $z_1$ .

(Consider  $L = \Delta^2$  self adjoint).?

— (c) Show that a solution of  $Lu = 0$  in the ball  $|x - z_1| \leq \rho$  has the modified mean value property

$$u(z_1) = \frac{\sqrt{c} \cdot \rho}{\text{seu}(\sqrt{c} \rho)} \cdot \frac{1}{4\pi \rho^2} \int_{|x - z_1| = \rho} u(x) dS.$$

Proof

We consider the more general solution with spherical symmetry, which is also a fundamental solution for  $L$  with pole  $z_1$ .

( $k_0$  constant)

$$G(x, z_1) = k(x, z_1) + \frac{k_0 \text{seu}(\sqrt{c} r)}{r} = -\frac{\omega(\sqrt{c} r)}{4\pi r} + \frac{k_0 \text{seu}(\sqrt{c} r)}{r}$$

replacing  $k$  by  $G$  in the formula. and  $Lu = 0$  on  $(**)$

$$u(z_1) = \int_{\partial \Omega} u(x, z_1) (\Delta u + cu) dx - \int_{\partial \Omega} \left( k(x, z_1) \frac{du}{dn} - u(x) \frac{dk(x, z_1)}{dn} \right) dS$$

we have

$$u(z_1) = - \int_{\partial \Omega} \left( G(x, z_1) \frac{du}{dn} - u(x) \frac{dG(x, z_1)}{dn} \right) dS.$$

$$= \frac{\sqrt{c}}{\sqrt{c} \sqrt{\rho}} \cdot \frac{1}{4\pi\rho^2} = \frac{dG(x, \xi)}{dx}$$

then.

$$u(\xi) = \frac{\sqrt{c}}{\sqrt{c} \sqrt{\rho}} \frac{1}{4\pi\rho^2} \int_{|x-\xi|=\rho} u(x) dS_x. \quad \checkmark$$

ed

— (d) Try  $u \equiv \phi(x) = \phi(r) = \frac{\sin(\sqrt{c}r)}{r}$  for  $c > 0$

for  $c < 0$  use the maximum principle to show that such  $u$  does not exist!

There is no solution for  $c < 0$

solutions for  $\sqrt{c} = k\pi \quad k \in \mathbb{Z}^+$

$$c = (k\pi)^2 \dots$$