MATH 108 B FALL 2011 EXTRA-CREDIT PROBLEMS

SHOW YOUR WORK CLEARLY.

–1) Let A be a $n \times n$ skew-Hermitian matrix ($A^* = \overline{A}^T = -A$). Prove:

(a) The eigenvalues of A are pure imaginary, and eigenvectors corresponding to different eigenvalues are orthogonal. (we recall that if x, $y \in \mathbb{C}^n$, then $\langle x, y\rangle = x^T \overline{y} = \sum_{j=1}^n x_j \, \overline{y}_j$)

(b) Prove that $(I - A)$ and $(I + A)$ are invertible.

(c) Prove that e^A defined below is an unitary matrix (i.e. U unitary if $U^*U = I$).

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

(d) Prove that $Q = (I - A)(I + A)^{-1}$ is an unitary matrix.

(e) If

$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

compute Q as in part (d).

–2) If the vectors x_1 and x_2 are the columns of S, what are the eigenvalues and eigenvectors of

$$B = S \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} S^{-1}, \qquad C = S \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} S^{-1} \text{ ?}$$

–3) (a) Show that if B is unitary and $\lambda \in \mathbb{C}$ is an eigenvalue of B, then $|\lambda| = 1$.

(b) Show that if A is normal (i.e. $A^*A = AA^*$) and invertible, then $B = A^*A^{-1}$ is unitary.

–4) Show that

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

has no square root, i.e. there is no B such that $B^2 = A$.