MATH 108 B FALL 2011 EXTRA-CREDIT PROBLEMS

SHOW YOUR WORK CLEARLY.

–1) Let A be a $n \times n$ skew-Hermitian matrix ($A^* = \overline{A}^T = -A$). Prove:

(a) The eigenvalues of A are pure imaginary, and eigenvectors corresponding to different eigenvalues are orthogonal. (we recall that if $x, y \in \mathbb{C}^n$, then $\langle x, y \rangle = x^T \overline{y} = \sum_{j=1}^{n} x_j \overline{y}_j$)

(i) Suppose that $A\vec{x} = \lambda \vec{x}$, $\vec{x} \neq 0$ then $\lambda \langle \vec{x}, \vec{x} \rangle = \langle \lambda \vec{x}, \vec{x} \rangle =$

$= \langle A\vec{x}, \vec{x} \rangle = \langle \vec{x}, A^*\vec{x} \rangle = \langle \vec{x}, -A\vec{x} \rangle = \langle \vec{x}, -\lambda \vec{x} \rangle = -\overline{\lambda} \langle \vec{x}, \vec{x} \rangle$

so $\lambda = -\overline{\lambda}$ if $\lambda = \alpha + i\beta$

$\overline{\lambda} = \alpha - i\beta \Rightarrow -\overline{\lambda} = -\alpha + i\beta$

Hence $\lambda = -\overline{\lambda} \Rightarrow \lambda = i\beta$ (pure imaginary)

(ii) Suppose $A\vec{x}_j = \lambda_j \vec{x}_j$, $j = 1, 2$, $\lambda_1 \neq \lambda_2$ Then $\lambda_1 \langle \vec{x}_1, \vec{x}_2 \rangle = \langle A\vec{x}_1, \vec{x}_2 \rangle$

$= \langle \vec{x}_1, A^* x_2 \rangle = \langle \vec{x}_1, -A\vec{x}_2 \rangle = \langle \vec{x}_1, -\lambda_2 \vec{x}_2 \rangle = -\overline{\lambda}_2 \langle \vec{x}_1, \vec{x}_2 \rangle$ (*)

Since λ_1, λ_2 are pure imaginary $\lambda_1 \neq \lambda_2 = -\overline{\lambda}_2$

(*) $\Rightarrow \langle \vec{x}_1, \vec{x}_2 \rangle = 0$. qed.

(b) Prove that $(I - A)$ and $(I + A)$ are invertible.

In (a) we have shown that the eigenvalues of A (skew Hermitian) are pure imaginary, so ± 1 are not eigenvalues of A, therefore $I - A = -(A - 1I)$, $I + A = A + 1I$ are invertible (definition of eigenvalue)

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(c) Prove that e^A defined below is an unitary matrix (i.e. U unitary if $U^*U = I$).

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

$$(e^A)^* = \left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right)^* = \sum_{n=0}^{\infty} \frac{(A^n)^*}{n!} = \sum_{n=0}^{\infty} \frac{(A^*)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-A)^n}{n!} = e^{-A}.$$

Hence $e^{A^*} e^A = e^{-A} e^A = e^0 = I$.

(d) Prove that $Q = (I - A)(I + A)^{-1}$ is an unitary matrix.

$$Q^*Q = \left((I-A)(I+A)^{-1}\right)^* (I-A)(I+A)^{-1}$$
$$= \left[(I+A)^{-1}\right]^* (I-A)^* (I-A)(I+A)^{-1}$$
$$= \left[(I+A)^*\right]^{-1} (I-A)\ (I+A)(I+A)^{-1} = (I-A)^{-1}(I-A)\, I = I$$

Notice That

① $(I-A)^* = I^* - A^* = (I+A)$

② $(I+A)$, $(I-A)$ commute

③ $(I+A)^* = I - A$

④ $\left((I+A)^{-1}\right)^* = \left((I+A)^*\right)^{-1}$

(e) If

$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

compute Q as in part (d).

$$I - A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$I + A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$(I+A)^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/5 & 1/5 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1/5 & -2/5 \\ 0 & 1 & 2/5 & 1/5 \end{array}\right)$$

So $Q = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$$\boxed{Q = \begin{pmatrix} -3/5 & -4/5 \\ 4/5 & -3/5 \end{pmatrix}}$$

–2) If the vectors x_1 and x_2 are the columns of S, what are the eigenvalues and eigenvectors of

$$B = S\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} S^{-1}, \qquad C = S\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} S^{-1}?$$

For B $\quad S = \begin{pmatrix} | & | \\ x_1 & x_2 \\ | & | \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad B = SDS^{-1} \Rightarrow BS = SD$

$\Rightarrow \begin{cases} Bx_1 = 2x_1 \\ Bx_2 = x_2 \end{cases} \Rightarrow$ $x_1\ x_2$ eigenvectors corresp to eigenvalues $\lambda_1 = 2 \quad \lambda_2 = 1$

For C We diagonalize $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. eigenvalues $\lambda_1 = 2 \ \lambda_2 = 1$

eigenvector for $\lambda_1 = 2$ $\quad \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\vec{v}_1 = 2\vec{v}_1 \Leftrightarrow \begin{cases} 2x + 3y = 2x \\ y = 2y \end{cases} \Rightarrow y = 0 \quad x = 1$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

eigenvector for $\lambda_2 = 1$ $\quad \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}\vec{v}_2 = \vec{v}_2 \Leftrightarrow \begin{cases} 2x + 3y = x \\ y = y \end{cases} \Rightarrow x + 3y = 0 \quad \begin{matrix} x = 3 \\ y = -1 \end{matrix}$

$\vec{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

where $V = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} \quad V^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$

so $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} V = VD$

Hence $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = VDV^{-1}$. Inserting this in C gives

$C = S\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}S^{-1} = SVDV^{-1}S^{-1} = (SV)D(SV)^{-1}$ so

The eigenvalues of C are $\lambda_1 = 2 \ \lambda_2 = 1 \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

the eigenvector of C are the columns of $SV = \begin{pmatrix} | & | \\ x_1 & x_2 \\ | & | \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} | & | \\ x_1 & 3x_1 - x_2 \\ | & | \end{pmatrix}$ i.e. "x_1" and "$3x_1 - x_2$" are the eigenvectors of C.

-3) (a) Show that if B is unitary and $\lambda \in \mathbb{C}$ is an eigenvalue of B, then $|\lambda| = 1$.

Suppose $B\vec{x} = \lambda\vec{x}$, $\vec{x} \neq 0$. So

$\langle B\vec{x}, B\vec{x}\rangle = \langle \lambda\vec{x}, \lambda\vec{x}\rangle = \lambda\bar{\lambda}\langle\vec{x},\vec{x}\rangle$

$= \langle B^*B\vec{x}, \vec{x}\rangle$

$= \langle \vec{x}, \vec{x}\rangle$

So $\lambda\bar{\lambda} = |\lambda|^2 = 1 \Rightarrow |\lambda| = 1$.

(b) Show that if A is normal (i.e. $A^*A = AA^*$) and invertible, then $B = A^*A^{-1}$ is unitary.

$B^*B = (A^*A^{-1})^* A^*A^{-1} = (A^{-1})^* A^{**} A^* A^{-1}$

$= (A^*)^{-1} A A^* A^{-1} = (A^*)^{-1} A^* A A^{-1} = I$. qed.

(we have used that $(A^{-1})^* = (A^*)^{-1}$, see this below:

$(A^{-1})^* A^* = (A A^{-1})^* = I^* = I$

so $(A^{-1})^* = (A^*)^{-1}$.

–4) Show that

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

has no square root, i.e. there is no B such that $B^2 = A$.

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Suppose $B^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

So $\begin{cases} a^2 + bc = 0 \\ ab + bd = 1 \\ ac + cd = 0 \\ bc + d^2 = 0 \end{cases}$ $a^2 = -bc = d^2$

$\longrightarrow b(a+d) = 1$

$\longrightarrow \boxed{c(a+d) = 0}$ Two cases.

Cases ① $c = 0 \Rightarrow a^2 = d^2 = 0 \Rightarrow a = d = 0$

So $b(a+d) = 0 \neq 1$ contradiction

② $a + d = 0 \Rightarrow a = -d$

but $(a+d)b = 1$ contradiction!