MATH 5-A FALL 2011 FIRST MID TERM

NAME :
TA :
SECTION TIME :

Put your final answer in BOXES (except graphs). Write clearly. Show your work. Otherwise no partial credit.

Q 1. (8 points) ______

Q 2. (6 points) ______

Q 3. (6 points) ______

TOTAL ________

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1) Consider equation

$$\frac{d^2x}{dt^2} + 4\,x = 4\,\cos(2t).$$

(a) Find the solution satisfying the initial conditions $x(0) = 0, \quad x'(0) = 0.$
(b) Sketch the graph of the solution for $0 \le t \le 2\pi$.

(a) STEP 1 HOMOGENEOUS EQ $x'' + 4x = 0$ $\quad r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$$\boxed{x_H(t) = C_1 \cos(2t) + C_2 \sin(2t).}$$

– STEP 2 PARTICULAR SOLUTION of $x_p'' + 4x_p = 4\cos(2t)$

we try $\boxed{x_p(t) = At\cos(2t) + Bt\sin(2t)}$, so

$$x_p''(t) = -4A\sin(2t) - \cancel{4At\cos(2t)} + 4B\cos(2t) - \cancel{4Bt\sin(2t)}$$

$$+ \quad 4x_p(t) = \qquad \cancel{4At\cos(2t)} \qquad + \cancel{4Bt\sin(2t)}$$

$$4\cos(2t) = -4A\sin(2t) + 4B\cos(2t)$$

so $A = 0 \quad B = 1$

and the particular solution $x_p(t) = t\sin(2t)$.

general solution $x_g(t) = x_H(t) + x_p(t)$

$$\boxed{x_g(t) = C_1\cos(2t) + C_2\sin(2t) + t\sin(2t)}$$

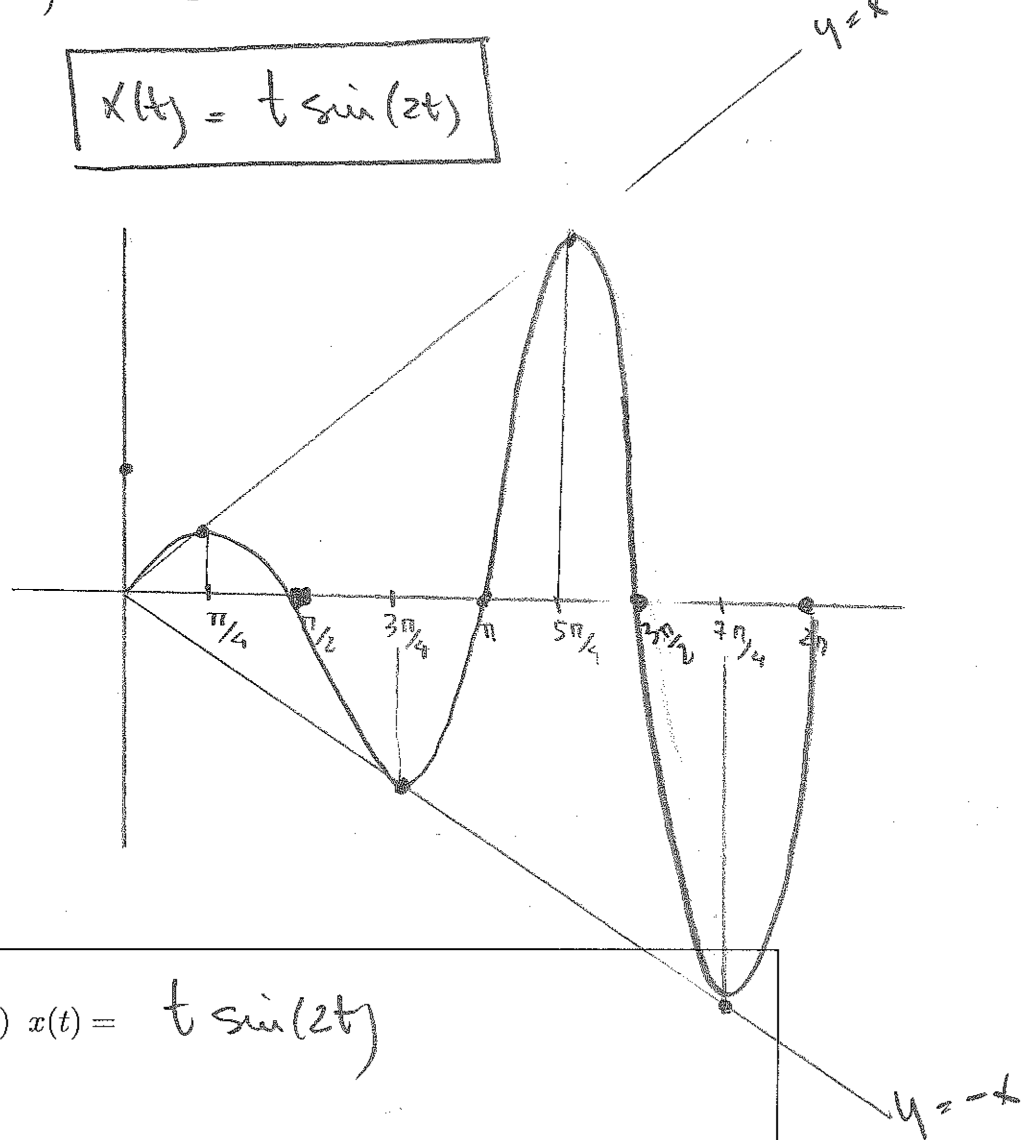
STEP 3 choose C_1, C_2 to satisfy the initial conditions

$x(0) = C_1 = 0 \Rightarrow x(t) = C_2\sin(2t) + t\sin(2t)$

$x'(t) = 2C_2\cos(2t) + \sin(2t) + 2t\cos(2t)$

$$x'(0) = 2C_2 = 0. \Rightarrow C_2 = 0$$

So $\boxed{x(t) = t\sin(2t)}$



(a) $x(t) =$ $t\sin(2t)$

(b)

2) Given the function

$$x(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + t + e^{3t},$$

find a second order linear constant coefficients differential equation for which this function represents the general solution. Check your answer.

The homogeneous solution is $c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$

So roots $= 1 \pm 2i$ polynomial $(r - r_1)(r - r_2)$

$r_1 = 1 + 2i$

$r_2 = 1 - 2i$

$= (r - 1 - 2i)(r - 1 + 2i)$

$= (r-1)^2 + 4$

$= r^2 - 2r + 1 + 4$

$= r^2 - 2r + 5.$

So the homogeneous eq. is

$x'' - 2x' + 5x = 0.$

Now $t + e^{3t} = x_p(t)$ = particular solution of

$x'' - 2x' + 5x = f(t)$

so $f(t) = (t + e^{3t})'' - 2(t + e^{3t})' + 5(t + e^{3t})$

$= 9e^{3t} - 2 - 6e^{3t} + 5t + 5e^{3t}$

$= 8e^{3t} + 5t - 2,$

Equation $\boxed{x'' - 2x' + 5x = 8e^{3t} + 5t - 2.}$

3) Consider the list of matrices

$$A_1 = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 2 & 2 & -2 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 3 & 0 & 3 \\ 0 & -2 & -2 \\ -3 & 2 & -1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) Identify those whose $Image(A)$ is the plane $x + y + z = 0$.

(b) identify those whose $Ker(T)$ is spanned by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(a) The image of A is the space of all linear combinations of the columns of A.

for A_1 $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ is not in the plane

for A_2 $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ " " " " "

for A_3 all columns are multiple of $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ (line)

so its Image has range $\equiv 1$

for A_4 $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is not in the plane.

✓ (only) A_5 is OK. ① all columns in the plane

② $\begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$ are linearly independent

③ $\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$ combination of $\begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$

A_6 $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ is not in the plane

(b) we just need to check

① $A\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\qquad A\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

② and $\dim \mathrm{Ker}(A)=2$ or $\dim \mathrm{Im}(A)=1$

i.e. A has 1 linearly independent column.

A_1 does not satisfy ①

A_2 " " " ①

A_3 " " " ①

A_4 " " " ①

A_5 " " " ①

✓ A_6 satisfies ① and all its columns only are multiple of $\begin{pmatrix}1\\1\\0\end{pmatrix}$ (line)

so it holds that $\dim \mathrm{Im}(A)=1$

$\Rightarrow \dim \mathrm{Ker}(A)=2$

(a) A_5

(b) A_6