MATH 201-A FALL 2009 MIDTERM

Problem 1 Let $f_k : \mathbb{R} \to \mathbb{R}$ be the sequence of function defined as

$$f_k(x) = \frac{1}{k}\,\chi_{[0,k]}(x).$$

(a) Compute

$$(i)\ \lim_{k\to\infty} f_k(x), \qquad (ii)\ \int_{\mathbb{R}} \lim_{k\to\infty} f_k(x)dx, \qquad (iii)\ \lim_{k\to\infty} \int_{\mathbb{R}} f_k(x)dx.$$

(b) Why does not the monotone convergence theorem hold? Explain.
(c) Why does not the Lebegue dominated convergence theorem hold ? Explain.
(d) Does the Fatou's lemma apply? Explain.

(a) (i) $f_k(x) \le \frac{1}{k}$ Hence $\lim_{k\to\infty} f_k(x) = 0$

(ii) $\int_{\mathbb{R}} \lim_{k\to\infty} f_k(x)dx = \int_{\mathbb{R}} 0\,dx = 0$

(iii) $\int_{\mathbb{R}} f_k(x)dx = \frac{1}{k}\int_0^k dx = 1$ So $\lim_{k\to\infty} \int_{\mathbb{R}} f_k(x)\,dx = 1.$

(b) The sequence f_k is no monotone

$f_k(k+1/2) = 0 \qquad f_{k+1}(k+1/2) = \frac{1}{k}$ for any $k \in \mathbb{Z}$.

So the M.C.Th does not hold.

(c) Suppose $f \in L^1(\mathbb{R})$ st $\forall k: |f_k(x)| \le f(x)$. (**)

So $f(x) \ge \sum_{k=1}^{\infty} \frac{1}{k}\chi_{[k-1,k]}(x)$

$\int \sum^{\infty} \frac{1}{k}\chi_{[k-1,k]}(x)\,dx = \sum^{\infty} \frac{1}{k} = +\infty$

So $(**)$ cannot hold.

(d) The only hypothesis needed for FATOU is $f_n(x) \geq 0$ which holds in this case. Therefore we have.

$$\liminf f_n(x) = \lim f_n(x) = 0$$

$$\liminf_{n \to \infty} \int_{-\infty}^{\infty} f_n(x)\, dx = 1$$

So

$$\int_{-\infty}^{\infty} \liminf f_n(x)\,dx \leq \liminf \int_{-\infty}^{\infty} f_n(x)\,dx$$

Problem 2 Let $A \subset \mathbb{R}$ be a measurable set with $m(A) = a \in (0, \infty)$.

(a) Prove that there exists $k \in \mathbb{Z}$ such that $m(A \cap [k, k+1]) > 0$.

(b) Prove that for any $\epsilon > 0$ there exists a finite collection of open bounded intervals $\{I_j : j = 1, .., N\}$ such that

$$m(A \triangle (\cup_{j=1}^N I_j)) = m(A - (\cup_{j=1}^N I_j)) + m((\cup_{j=1}^N I_j) - A) < \epsilon$$

(c) Prove that there exists $j_0 \in \{1, .., N\}$ such that

$$m(A \cap I_{j_0}) \geq \frac{9}{10} m(I_{j_0}).$$

Proof (a) Suppose the conclusion does not hold. Then

$\forall k \in \mathbb{Z} \quad m(A \cap [k, k+1]) = 0 \quad$ so

$$\bigcup_{k=-\infty}^{\infty} m(A \cap [k, k+1]) = m\Big(A \cap \Big(\bigcup_{k=-\infty}^{\infty} [k, k+1]\Big)\Big) = m(A) = 0$$

contradiction!

(b) Since A is measurable $\subseteq \mathbb{R}$: $\forall \varepsilon > 0$ $\exists \mathcal{O}$ open st ① $A \subseteq \mathcal{O}$
② $m(\mathcal{O} - A) < \varepsilon/2$

Now $\mathcal{O}$ open in $\mathbb{R}$ Then $\mathcal{O}$ is the union of at most countable many open disjoint intervals i.e. $\mathcal{O} = \bigcup_{j=1}^{\infty} I_j \quad I_j \cap I_k = \phi$ if $k \neq j$

Now

$$m(\mathcal{O}) = m(A) + m(\mathcal{O} - A) < a + \varepsilon/2 \text{ finite.}$$

with $m(\mathcal{O}) = m\Big(\bigcup_{j=1}^{\infty} I_j\Big) = \sum_{j=1}^{\infty} m(I_j) < a + \varepsilon/2$ Hence $\exists N$ st

$$\sum_{j=N+1}^{\infty} m(I_j) < \varepsilon/2.$$

Our finite collection of intervals should be $I_1, \dots, I_N$

Since

$$m\Big(\bigcup_{j=1}^{N} I_j - A\Big) + m\Big(A - \bigcup_{j=1}^{N} I_j\Big) \leq m(\mathcal{O} - A) + m\Big(\mathcal{O} - \bigcup_{j=1}^{N} I_j\Big)$$

$\leq \varepsilon/2 + \varepsilon/2$ and

(c) Suppose the conclusion does not hold i.e

$$\forall j \in \{1, \dots N\} \qquad m(A \cap I_j) < \frac{9}{10} m(I_j)$$

Then:

$$m(A) \leq m\left(A - \bigcup_{j=1}^{N} I_j\right) + m\left(A \cap \left(\bigcup_{j=1}^{N} I_j\right)\right) \underset{\text{part b}}{\leq} \varepsilon + m\left(\bigcup_{j=1}^{N} (A \cap I_j)\right)$$

$$\leq \varepsilon + \sum_{j=1}^{N} m(A \cap I_j) \underset{\text{assump}}{\leq} \varepsilon + \frac{9}{10} \sum_{j=1}^{N} m(I_j)$$

$$= \varepsilon + \frac{9}{10} m\left(\bigcup_{j=1}^{N} I_j\right) \leq \varepsilon + \frac{9}{10}\left(m\left(\bigcup_{j=1}^{N} I_j - A\right) + m(A)\right)$$

$$\underset{\text{part b}}{\leq} \varepsilon + \frac{9}{10}\left(\varepsilon + m(A)\right) \leq 2\varepsilon + \frac{9}{10} m(A).$$

Hence $\frac{1}{10} m(A) \leq 2\varepsilon \Rightarrow m(A) \leq 20\varepsilon$.

taking $\varepsilon < \frac{m(A)}{20}$ we get a <u>contradiction</u>!

Problem 3 Let $g : \mathbb{R}^d \to \mathbb{R}$ be a positive measurable s. t. $g \in L^1(\mathbb{R}^d)$.

(a) Define the sequence

$$g_k(x) = g(x)\, \chi_{\{x: g(x) \le k\}}(x), \qquad k \in \mathbb{Z}^+.$$

Prove that

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |g(x) - g_k(x)| dx = 0$$

(b) Prove given any $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subset \mathbb{R}^d$ is a meaurable set then

$$m(A) < \delta \quad \Rightarrow \quad \int_A g(x) dx < \epsilon.$$

Proof

(a) Notice that $0 \le g_k(x) \le g(x)$. $\forall x \in \mathbb{R}^d$ $\forall k \in \mathbb{Z}^+$

$0 \le g_1(x) \le g_2(x) \le \dots$ with $\lim_{k\to\infty} g_k(x) = g(x)$

With $|g(x) - g_k(x)| \le 2|g(x)| \in L^1(\mathbb{R}^d)$

So using The Lebesgue Dominated Convergence (or The monotone convergence) Th

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |g(x) - g_k(x)| dx = \lim_{k\to\infty} \int_{\mathbb{R}^d} (g(x) - g_k(x)) dx$$

$$= 0 \quad \left(\lim_{k\to\infty} \int_{\mathbb{R}^d} g_k(x) dx = \int_{\mathbb{R}^d} g(x) dx \right)$$

(b)

$$\int_A g(x) dx = \int_A (g(x) - g_k(x) + g_k(x)) dx$$

$$= \int_A (g(x) - g_k(x)) dx + \int_A g_k(x) dx \le \int_{\mathbb{R}^d} |g(x) - g_k(x)| dx + \int_A g_k(x) dx$$

Since $\lim_{k \to \infty} \int_{\mathbb{R}^d} |g(x) - g_k(x)|\,dx = 0$

$\forall \varepsilon > 0 \quad \exists N \quad \left[k \geq N \implies \int_{\mathbb{R}^d} |g(x) - g_k(x)|\,dx < \varepsilon/2 \right]$

<u>Fixing</u> thus <u>$k \geq N$</u> we observe that

$$g_k(x) = g(x)\,\chi_{\{x : g(x) \leq k\}} \leq k$$

Hence

$$\int_A g_k(x)\,dx \leq k\,m(A).$$

Thus $\cancel{***} < \varepsilon/2 + k\,m(A) < \varepsilon$

$\Leftarrow \quad k\,m(A) < \varepsilon/2$

$\Leftarrow \quad m(A) < \varepsilon k/2$

So it suffices to take $\delta < \frac{\varepsilon k}{2}$.