MATH 108 B FALL 2011 FIRST MIDTERM

NAME :

SHOW YOUR WORK CLEARLY. OTHERWISE NO PARTIAL CREDIT.

1) Let V be a vector space such that $dim(V) = n$. Let $T : V \to V$ be a linear map. Assume that $\vec{v}_1, ..., \vec{v}_k$ $(k \leq n)$ are eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, .., \lambda_k$. Prove that $\vec{v}_1, ..., \vec{v}_k$ are linearly independent.

(#4 in the review problems + solved in class.)

We have $T(\vec{v}_\ell) = \lambda_\ell \vec{v}_\ell$ $\ell = 1 \ldots k$, $\lambda_1 \ldots \lambda_k$ distinct.

By induction in k ① Case $k=2$ $T(\vec{v}_1) = \lambda_1 \vec{v}_1$, $T(\vec{v}_2) = \lambda_2 \vec{v}_2$.

Suppose ⊛ $\boxed{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = 0}$. Then applying T to it one gets

$$0 = T(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = \alpha_1 T(\vec{v}_1) + \alpha_2 T(\vec{v}_2) = \alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2 = 0$$

so $\begin{cases} \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = 0 \\ \lambda_1 \alpha_1 \vec{v}_1 + \lambda_2 \alpha_2 \vec{v}_2 = 0 \end{cases}$

multiplying the first equation above by $-\lambda_1$ and adding the result to the second one gets that

$(\lambda_2 - \lambda_1) \alpha_2 \vec{v}_2 = 0$, since $\lambda_2 - \lambda_1 \neq 0 \Rightarrow \alpha_2 \vec{v}_2 = 0$

since $\vec{v}_2$ is an eigenvector $\vec{v}_2 \neq 0 \Rightarrow \alpha_2 = 0$

Returning to ⊛ we have $\alpha_1 \vec{v}_1 = 0 \Rightarrow \alpha_1 = 0$

So $\vec{v}_1, \vec{v}_2$ are L. I.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

② Assuming the result for k
(k eigenvectors corresponding to different eigenvalues of T are LI)
we shall prove it for $k+1$.
So we have that

$$\alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k + \alpha_{k+1} \vec{v}_{k+1} = 0. \quad (1)$$

Applying T is followed that

$$\alpha_1 \lambda_1 \vec{v}_1 + \dots + \alpha_k \lambda_k \vec{v}_k + \alpha_{k+1} \lambda_{k+1} \vec{v}_{k+1} = 0 \quad (2)$$

Multiplying the equation (1) by $-\lambda_{k+1}$ and adding the result to the equation (2) one gets that

$$\alpha_1 (\lambda_1 - \lambda_{k+1}) \vec{v}_1 + \dots + \alpha_k (\lambda_k - \lambda_{k+1}) \vec{v}_k = 0.$$

Now using the induction hypothesis it follows
that $\alpha_1(\lambda_1 - \lambda_{k+1}), \dots, \alpha_k(\lambda_k - \lambda_{k+1}) = 0$, one sees that
since the λ_j's are distinct ($j = 1 \dots k+1$),
$\alpha_1, \dots, \alpha_k = 0$. Then returning to the equation
(1) This gives $\alpha_{k+1} \vec{v}_{k+1} = 0 \implies \alpha_{k+1} = 0$ since $\vec{v}_{k+1} \neq 0$
Therefore from (1) we deduce that $\alpha_1 = \dots = \alpha_k = \alpha_{k+1} = 0$
So $\vec{v}_1 \dots \vec{v}_{k+1}$ are L.I.

2) Given the matrix

$$A = \begin{pmatrix} 6 & 5 \\ 3 & 4 \end{pmatrix}.$$

Find two (different) matrices B_1, B_2 such that

$$B_1^2 = A, \quad B_2^2 = A, \quad (\text{ i.e. } B_1 = +\sqrt{A}, \quad B_2 = -\sqrt{A}).$$

Why can you do this? Explain. (# 7 in the review problems)

First we find the eigenvalues & eigenvectors of A.

$0 = \det(A - \lambda I) = \det\begin{pmatrix} 6-\lambda & 5 \\ 3 & 4-\lambda \end{pmatrix} = (6-\lambda)(4-\lambda) - 15 = \lambda^2 - 10\lambda + 9$

$= (\lambda - 9)(\lambda - 1)$ so $\lambda_1 = 1$, $\lambda_2 = 9$ are the eigenvalues

$\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ eigenvector corresponding to $\lambda_1 = 1$ $\quad A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$\Leftrightarrow \begin{cases} 6x_1 + 5y_1 = x_1 \\ 3x_1 + 4y_1 = y_2 \end{cases} \Leftrightarrow x_1 + y_1 = 0$ we take $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ eigenvector corresponding to $\lambda_2 = 9$ $\quad A\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 9\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$\Leftrightarrow \begin{cases} 6x_2 + 5y_2 = 9x_2 \\ 3x_2 + 4y_2 = 9y_2 \end{cases} \Leftrightarrow -3x_2 + 5y_2 = 0$ we take $\vec{v}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Notice that $\det A = 9 = \lambda_1 \lambda_2$ Trace of $A = 10 = \lambda_1 + \lambda_2$.

so $A\vec{v}_1 = \lambda_1 \vec{v}_1$ $\quad A\vec{v}_2 = \lambda_2 \vec{v}_2$

or $AV = VD$ where $V = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}$

and $D = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$.

Hence $A = VDV^{-1}$ we need to find V^{-1}:

$$\begin{pmatrix} 1 & 5 & \vdots & 1 & 0 \\ -1 & 3 & \vdots & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & \vdots & 1 & 0 \\ 0 & 8 & \vdots & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & 1/8 & 1/8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \vdots & 3/8 & -5/8 \\ 0 & 1 & \vdots & 1/8 & 1/8 \end{pmatrix} \quad \text{so } V^{-1} = \begin{pmatrix} 3/8 & -5/8 \\ 1/8 & 1/8 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 & -5 \\ 1 & 1 \end{pmatrix}$$

check $\begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix} \frac{1}{8}\begin{pmatrix} 3 & -5 \\ 1 & 1 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = I.$

so $A = VDV^{-1} = V\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}V^{-1} = V\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 3 \end{pmatrix}\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 3 \end{pmatrix}V^{-1}$

$= V\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 3 \end{pmatrix}V^{-1}\, V\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 3 \end{pmatrix}V^{-1}.$

Hence $B_1 = \underbrace{V\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}V^{-1} \qquad B_2 = V\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}V^{-1}$

$= V\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 15 \\ -1 & 9 \end{pmatrix}$

$V\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}V^{-1} = \frac{1}{8}\begin{pmatrix} 1 & 15 \\ -1 & 9 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ 1 & 1 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 18 & 10 \\ 6 & 14 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 9 & 5 \\ 3 & 7 \end{pmatrix}$

so

$B_1 . B_1 = \frac{1}{16}\begin{pmatrix} 9 & 5 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 9 & 5 \\ 3 & 7 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 96 & 80 \\ 48 & 64 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 3 & 4 \end{pmatrix}$ qed.

We can do this (as in the real number case) because the eigenvalues are ≥ 0.

3) Let A be a 4×4 real matrix such that $A^2 \neq 0$, and for some $k > 4$, $A^k = 0$.
a) Which are the possible eigenvalues of A?
b) Is A diagonalizable? Explain.
c) For each $j = 1, 2, ..., k$ define

$$E_j = \{\vec{x} \in \mathbb{R}^4 \ : \ A^j \vec{x} = \vec{0}\}.$$

Prove that each E_j is a subspace of $\mathbb{R}^4$ and

$$\{0\} \subsetneq E_1 \subset E_2 \subset ... \subset E_k.$$

d) Prove that $A^4 = 0$. (# 15 in the review list, solved in class).

(a) If $\lambda_1, \vec{v}_1$ are eigenvalues & eigenvectors of A one has $A\vec{v}_1 = \lambda_1 \vec{v}_1$ so $A^k \vec{v}_1 = \lambda_1^k \vec{v}_1$ but $A^k = 0$

so $\lambda_1^k \vec{v}_1 = 0$ (but $\vec{v}_1 \neq 0$) $\Rightarrow \lambda_1^k = 0 \Rightarrow \lambda_1 = 0$.

The only possible eigenvalue of A is 0.

(b) $E_1 = \{x \in \mathbb{R}^4 \ / \ A\vec{x} = 0\} = \text{null}(A) \neq \mathbb{R}^4$ (since $A^2 \neq 0 \Rightarrow A \neq 0$)

so $\dim E_0 \neq 4$ no diagonalizable

There are not 4 linearly independent eigenvectors of 0.

(c) First $E_j \subseteq E_{j+1}$: $x \in E_j \Leftrightarrow A^j x = 0 \Rightarrow A^{j+1} x = A0 = 0$
$\Rightarrow x \in E_{j+1}$.

Clearly E_j's are subspaces $E_j = \text{null}(A^j)$.

Next since 0 is the only possible eigenvalue and the set of eigenvalues can not be empty Then 0 is an eigenvalue so $\exists \vec{v}_1 \neq 0$ st $A\vec{v}_1 = 0\vec{v}_1 = 0 \Rightarrow \vec{v}_1 \in E_1$.

(d) Observe that if
"$E_j = E_{j+1} \Longrightarrow E_j = E_{j+1} = \dots = E_{j+l} \quad \forall l = 1, 2, \dots$"

<u>Proof of This claim</u>

$[E_j = E_{j+1}] \Longleftrightarrow [\; A^j y = 0 \Longleftrightarrow A^{j+1} y = 0 \;]$

Suppose $A^{j+2} x = 0 \Longrightarrow A^{j+1} (\underbrace{Ax}_{=y}) = 0 \Longleftrightarrow A^j (Ax) = 0$

$\Longleftrightarrow A^{j+1} x = 0 \Longleftrightarrow A^j x = 0.$

repeating This argument

$A^{j+3} x = A^{j+2}(Ax) = 0 \Longleftrightarrow A^j (Ax) = 0 \Longleftrightarrow A^{j+1} x = 0 \Longleftrightarrow$

$\Longleftrightarrow A^j = 0.$

we get the result.

so $\dim E_1 \geq 1$, and $E_1 \subseteq E_2$

so $\dim E_1 \leq \dim E_2$: two case:

① = holds $\Rightarrow E_1 = E_2 \Longrightarrow E_1 = E_4 = \dots = E_k = \mathbb{R}^4$

done!

② $<$ $\Rightarrow \dim E_2 \geq 2$.

repeat the argument 2 times more to get

That $E_4 = \mathbb{R}^4$ or $\dim E_4 = 4 \Longrightarrow E_4 = \mathbb{R}^4$

qed.