Math 5-A FALL 2011 MID-TERM PRACTICE PROBLEMS

(1) Find the general solution of

$$\begin{aligned}
&(a)\ \ 2y'' - 5y' - 3y = 0,\\
&(b)\ \ y''' + 3y'' - 4y = 0,\\
&(c)\ \ y^{(4)} + 2y'' + y = 0.
\end{aligned}$$

(2) Find the general solution of

$$\begin{aligned}
&(a)\ \ y'' - 5y' + 4y = 8\,e^{x},\\
&(b)\ \ y'' + 4y = x\ \cos(x),\\
&(c)\ \ y'' - 9y' + 14y = x\,e^{2x} + x,\\
&(d)\ \ y'' + y = \sec(x).
\end{aligned}$$

(3) Find the solution following initial value problems

$$\begin{aligned}
&(a)\ \ y'' + 4y = -2 \qquad y(\pi/8) = 1/2,\ \ y'(\pi/8) = 2,\\
&(b)\ \ y'' + 4y = F_0\ \sin(ax), \qquad y(0) = y'(0) = 0,\\
&(c)\ \ y'' - 9y' + 14y = x\,e^{2x} + x, \quad y(0) = 1,\ \ y'(0) = 0.
\end{aligned}$$

(4) Consider equation

$$\frac{d^2x}{dt^2} + 4\,x = -4\ \sin(2t).$$

(a) Find the solution satisfying the initial conditions $x(0) = 0, \quad x'(0) = 1.$
(b) Sketch the graph of the solution for $0 \le t \le 4\pi$.
(c) If the function $x(t)$ in part (a) is the position at time t of a mass attached to a spring, how do you describe, in your own words, the motion of the mass? (Hint : Your answer may involve words such as amplitude, equilibrium point and pseudo-period)

(5) Given the function $x(t) = Ae^{-t} + Be^{3t} + cos(t) + 3$, find a second order linear constant coefficients differential equation for which this function is the general solution.

ANS : $x'' - 2x' - 3x = -4\cos(t) + 2\sin(t) - 9.$

(6) Consider the initial-value problem

$$\begin{cases} \dfrac{d^2x(t)}{dt^2} + 4\,x(t) = 4\,cos(2t),\\ x(0) = 0, \quad x'(0) = 2. \end{cases}$$

(a) Find its solution $x(t)$ (verify your answer).
(b) Sketch the graph of the solution for $0 \le t \le 20$.

(c) If the function $x(t)$ is the position at time t of a mass attached to a spring under the external force $f(t) = 4\, cos(2t)$, how do you describe the motion of the mass?

(7) Find the solution of the initial value problem

$$x''(t) - x'(t) - x(t) = \sin(t), \qquad x(0) = 1,\ x'(0) = -3.$$

Verify your answer.

(8) Find a 2×2 matrix B having $\lambda_1 = -1,\ \lambda_2 = 1$ as eigenvalues and

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as the corresponding eigenvectors (verify your answer).

(9) For which values of the parameters $a,\ b$ the system below has
(a) no solution (5 pts.)
(b) a unique solution (5 pts.)
(c) infinitely many solutions (5 pts.).

$$\begin{cases} x - y + 2z = 1, \\ 3x + y + 2z = 3, \\ x - 3y + az = b. \end{cases}$$

(10) Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{v}) = A\vec{v}$, where

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

(a) Determine the image under the map T of the square having vertices $(0,0), (1,0), (1,1), (0,1)$.
(b) Repeat part (a) for the tringle with vertices $(0,0), (1,1), (-1,1)$.
(c) Repeat part (a)-(b) with A_1 as

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(d) Compute $det(A)$ and $det(A_1)$. Can you guess a connection?

(11) In each case give an example of 3×3 matrices A such that $T(\vec{v}) = A\vec{v}$ has the following property:
(a) the $Image(T)$ is the plane $2x - 3y + z = 0$.
(b) the $Image(T)$ is the line spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
(c) the $Ker(T)$ is spanned by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(12) Find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix}$$

(13) Given the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

(a) Find A^{1000}.
(b) Find $B = \sqrt{A}$, i.e $B^2 = A$ (verify your answer).
(HINT: Compute the eigenvalues and eigenvectors of A, consider the matrix V whose columns are the eigenvalues of A,....)

(14) Define $T(\vec{v}) = A\vec{v}$, where

$$A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$$

(a) Find the $Ker(T)$ and $Image(T)$.
(b) Find the eigenvalues of A and the corresponding eigenvectors.
(c) Is A diagonalizable?