Math 5-A FALL 2011 FINAL PRACTICE PROBLEMS

(1) Find the general solution of

$$(a)\ \ 2y'' - 5y' - 3y = 0,$$
$$(b)\ \ y''' + 3y'' - 4y = 0,$$

(2) Find the general solution of

$$(a)\ \ y'' + 4y = x\cos(x),$$
$$(b)\ \ y'' - 9y' + 14y = x\,e^{2x} + x,$$
$$(c)\ \ y'' + y = \sec(x).$$

(3) Find the solution following initial value problems

$$(a)\ \ y'' + 4y = -2 \qquad y(\pi/8) = 1/2,\ \ y'(\pi/8) = 2,$$
$$(b)\ \ y'' + 4y = F_0\sin(ax), \qquad y(0) = y'(0) = 0,$$
$$(c)\ \ y'' - 9y' + 14y = x\,e^{2x} + x, \quad y(0) = 1,\ \ y'(0) = 0.$$

(4) Consider equation

$$\frac{d^2x}{dt^2} + 4\,x = -4\sin(2t).$$

(a) Find the solution satisfying the initial conditions $x(0) = 0, \quad x'(0) = 1$.
(b) Sketch the graph of the solution for $0 \le t \le 4\pi$.
(c) If the function $x(t)$ in part (a) is the position at time t of a mass attached to a spring, how do you describe, in your own words, the motion of the mass? (Hint : Your answer may involve words such as amplitude, equilibrium point and pseudo-period. For a similar problem see Solutions First Midterm)

(5) (a) Given the function $x(t) = Ae^{-t} + Be^{3t} + e^t + 3$, find a second order linear constant coefficients differential equation for which this function is the general solution.

(SOLUTION : $x'' - 2x' - 3x = -4e^t - 9$)

(b) If $\cos(2t) - \sin(2t) = A\cos(2(t - \delta/2))$. Find A and δ.

(SOLUTION : $A = \sqrt{2}, \quad \delta = 3\pi/2 + \pi/4$)

(6) Find the solution of the initial value problem

$$x''(t) - x'(t) - x(t) = \sin(t), \qquad x(0) = 1,\ x'(0) = -3.$$

Verify your answer.

(7) Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{v}) = A\vec{v}$, where

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

(a) Determine the image under the map T of the square having vertices $(0,0),(1,0),(1,1),(0,1)$.

(b) Repeat part (a) for the tringle with vertices $(0,0),(1,1),(-1,1)$.

(c) Repeat part (a)-(b) with A_1 as

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(d) Compute $det(A)$ and $det(A_1)$. Can you guess a connection?

(8) In each case give an example of 3×3 matrices A such that $T(\vec{v}) = A\vec{v}$ has the following property:

(a) the $Image(T)$ is the plane $x + y + z = 0$.

(b) the $Image(T)$ is the line spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(c) the $Ker(T)$ is spanned by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(9) (a) Find a 2×2 matrix B having $\lambda_1 = -1,\ \lambda_2 = 1$ as eigenvalues and

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as corresponding eigenvectors. Answer :

$$B = \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

(b) Find a 2×2 matrix B having $\lambda_1 = 1,\ \lambda_2 = 4$ as eigenvalues and

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as corresponding eigenvectors. Answer :

$$B = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

(c) Find a 3×3 matrix B having $\lambda_1 = \lambda_2 = -1,\ \lambda_3 = 2$ as eigenvalues and

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

as corresponding eigenvectors. Answer :

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(10) Find the eigenvalues and the corresponding eigenvectors of the following 3×3 matrices

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix}$$

(11) Given the matrices

$$A_1 = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}.$$

(a) Find A_j^{1000}, $j = 1, 2, 3, 4$.

(HINT: Compute the eigenvalues and eigenvectors of A_j, consider the matrix V whose columns are the eigenvalues of A_j,.... diagonalize A_j))

(b) Find $B = \sqrt{A_1}$ a real matrix such that $B^2 = A_1$ (verify your answer).

Can you do the same for A_2? For A_3? Explain.

(12) Given the matrices

$$A_5 = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 6 & -1 \\ 4 & -2 \end{pmatrix}, \quad A_7 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad A_8 = \begin{pmatrix} -4 & -5 \\ 2 & -6 \end{pmatrix}.$$

(a) Find the general solution of the system

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A_j \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad j = 1, 2, ..., 7, 8,$$

with A_j, $j = 1, 2, 3, 4$ as in problem 11.

(b) In each case draw an approximate phase portrait of the solutions (trajectories).

(c) In each case $j = 1, .., 8$, decide if the origin $(0, 0)$ an isolated stationary (equilibrium) solution $(x(t), y(t)) \equiv (0, 0)$. If this is the case, classify it as a saddle, source (repelling node), sink (attracting source), spiral sink (attracting spiral), repelling spiral (spiral source), or a center.

(c) SOLUTIONS : A_1 eigenvalues $2, 3$, so the origin is a source, A_3 eigenvalues $-1, -6$, so the origin is a sink, A_7 eigenvalues $= i, -i$ so the origin is center with trajectories traveling clockwise.

(13) Given the matrix

$$A \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix},$$

(a) Find the matrix e^{tA}.

(b) Check that

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

is the solution of the initial value problem

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

(14) (a) For the in-homogemeous system

$$\begin{cases} x' = -3x + y - t, \\ y' = x - 3y + 1. \end{cases}$$

(i) Find its general solution.
(ii) Verify your answer.

SOLUTION : $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -3t/8 + 9/32 \\ -t/8 + 15/32 \end{pmatrix}$

(b) Do the same for the system

$$\begin{cases} x' = 6x + y + 6t, \\ y' = 4x + 3y - 10t + 4. \end{cases}$$

(c) Do the same for the system

$$\begin{cases} x' = 2x - y, \\ y' = 3x - 2y + 4t. \end{cases}$$

(15) Define $T(\vec{v}) = A\vec{v}$, where

$$A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$$

(a) Find the $Ker(T)$ and $Image(T)$.
(b) Find the eigenvalues of A and the corresponding eigenvectors.
(c) Is A diagonalizable?

(16) For each system find the general solution

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = B_j \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad j = 1, 2,$$

with

$$B_1 = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}.$$

(17) If the vectors x_1 and x_2 are the columns of S, what are the eigenvalues and eigenvectors of

$$B = S \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} S^{-1}, \qquad C = S \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} S^{-1} ?$$

(18) For the system of equations

$$\begin{cases} x' = 6x - y, \\ y' = 5x + 4y. \end{cases}$$

(a) Find the general form of the solution.
(b) Verify your answer.
(c) Determine whether the origin (the fixed point) is a source, sink, center, or a saddle (in the case of a spiral or a center decide if the solutions move clock-wise or counter clock-wise).

(19) Solve the following problems in the textbook:
Section 7.1 problems : 10-17,
Section 7.2 problems : 1-10.