- Just like the case with curves, $\iint_S \vec{F} \cdot d\vec{S}$ may differ by a $-$ sign depending on the parametrization
  - If $\vec{r}_1$ and $\vec{r}_2$ parametrize the same surface S but have opposite orientations, then

$$\iint_{D_1} \vec{F}(\vec{r}_1(u,v)) \cdot \vec{N}_1(u,v)\,dA$$
$$= -\iint_{D_2} \vec{F}(\vec{r}_2(u,v)) \cdot \vec{N}_2(u,v)\,dA$$

  - $\vec{r}_1$ and $\vec{r}_2$ have opposite orientations if their normal vectors point in opposite directions

Ex: Consider the two parametrizations:

$\vec{r}_1(u,v) = (\cos v \cos u, \cos v \sin u, \sin v) \quad (u,v) \in [0,2\pi] \times [0,\frac{\pi}{2}]$

$\vec{r}_2(u,v) = (\cos u \sin v, \sin u \sin v, \cos v) \quad (u,v) \in [0,2\pi] \times [0,\pi/2]$

These both parametrize the upper hemisphere of radius 1, but the normal vectors point in opposite directions:

$$\cancel{\vec{N}_1(u,v) = (\cos^2 v \cos u, \cos^2 v \sin u, 2\cos v \sin v)}$$

$$\vec{N}_1(u,v) = \cos v(\cos v \cos u, \cos v \sin u, 2\sin v)$$

$$\vec{N}_2(u,v) = -\sin v(\sin v \cos u, \sin v \sin u, 2\cos v)$$

- If we integrate $\vec{F}(x,y,z) = (0,0,1)$ over both parametrizations:

$$\iint_{S_1} \vec{F} \cdot dS_1 = \int_0^{2\pi}\int_0^{\pi/2} (0,0,1) \cdot \left[\cos v(\sim, \sim, 2\sin v)\right] dv\,du$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 2\sin v \cos v \, dv\,du$$

$$= \int_0^{2\pi} \sin^2 v \Big|_{v=0}^{v=\pi/2} du$$

$$= \int_0^{2\pi} 1 \, du = \boxed{2\pi}$$

$$\iint_{S_2} \vec{F} \cdot d\vec{S}_2 = \int_0^{2\pi}\int_0^{\pi/2} (0,0,1) \cdot \left[-\sin v(\sim, \sim, 2\cos v)\right] dv\,du$$

$$= \int_0^{2\pi}\int_0^{\pi/2} -2\sin v \cos v \, dv\,du = \boxed{-2\pi}$$

8.1 Green's Theorem

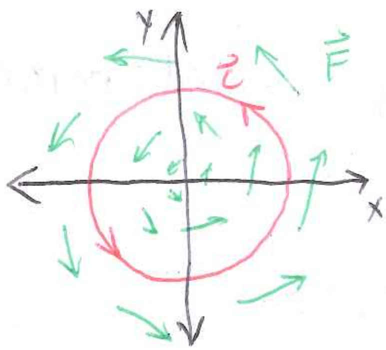
- We've seen two versions of the fundamental theorem of calculus:

$$\int_I \frac{d}{dt} f(t)\,dt = f(b) - f(a),$$

$$\int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

  - Is there an equivalent concept for integrals over regions? Yes!

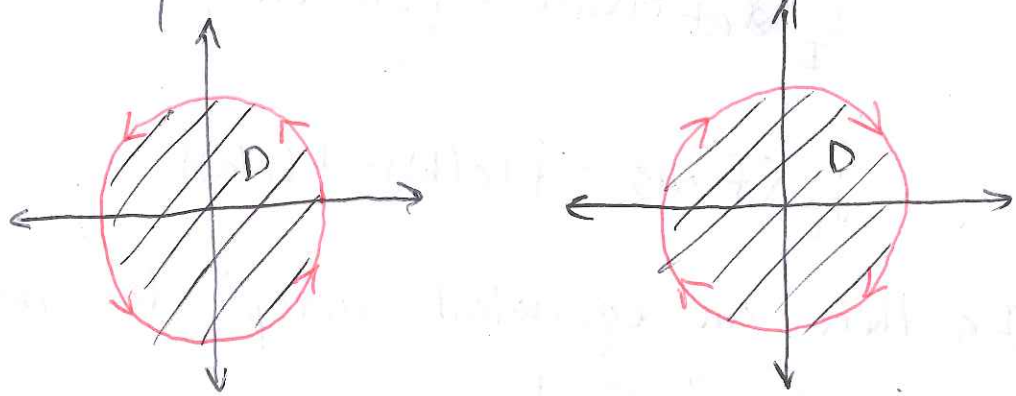
- Recall the <u>circulation</u> of $\vec{F}$ around a closed curve $\vec{c}$: $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$



  - We suspected that the circulation of $\vec{F}$ over $\vec{c}$ had some relationship with the scalar curl of $\vec{F}$

  - There is a relationship! Suppose D is a region whose boundary ∂D is parametrized by a path $\vec{c}(t)$

- We also require that $\vec{c}(t)$ has the <u>positive</u> orientation along the boundary ∂D: the region D is on the left as you walk along the boundary of D:



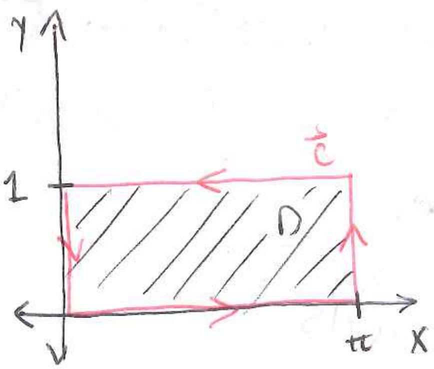
positively oriented boundary

negatively oriented boundary

- <u>Green's Theorem</u>: Let D be a region in $\mathbb{R}^2$, and let $\vec{c}(t)$ be a positively oriented param. of ∂D. Then

$$\boxed{\iint_D \underbrace{\nabla \times \vec{F}}_{\text{scalar curl}}\, dA = \underbrace{\int_{\vec{c}} \vec{F} \cdot d\vec{s}}_{\text{circulation of } \vec{F} \text{ around } \vec{c}}}$$

$\underline{\text{Ex}}$: Compute $\int_{\vec{c}} \vec{F}\cdot d\vec{s}$, where $\vec{F}(x,y) = (e^y, \sin x)$ and $\vec{c}$ is the boundary of the rectangle $[0,\pi]\times[0,1]$.



- We could set up the actual line integrals, but we'd have to parametrize 4 separate lines
- using Green's theorem:

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \iint_D \nabla\times\vec{F}\, dA$$

$$= \iint_D \left(\frac{\partial}{\partial x}\sin x - \frac{\partial}{\partial y} e^y\right) dA$$

$$= \iint_D (\cancel{+}\cos x - e^y)\, dA$$

$$= \int_0^{\pi}\int_0^1 (\cancel{+}\cos x - e^y)\, dy\, dx$$

$$= \int_0^{\pi} (\cancel{+}y\cos x - e^y)\Big|_{y=0}^{y=1} dx$$

$$= \int_0^{\pi} (\cos x - e + 1)\, dx$$

$$= \sin x + (1-e)x\Big|_{x=0}^{x=\pi} = \boxed{\pi(1-e)}$$

8.2 Divergence Theorem

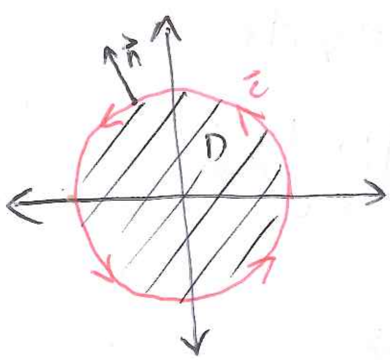
- Similar to Green's theorem, there's a relationship involving divergence:

Divergence Theorem: Let D be a region in $\mathbb{R}^2$, and let $\vec{c}$ be the boundary of D, with the positive orientation. Then

$$\boxed{\iint_D \nabla \cdot \vec{F}\, dA = \int_{\vec{c}} \vec{F} \cdot \vec{n}\, ds}$$

- here $\vec{n}$ is the outward unit normal to D.

Ex: Compute $\int_{\vec{c}} \vec{F} \cdot \vec{n}\, ds$, where $\vec{c}$ is the boundary of the unit circle oriented counterclockwise, and $\vec{F}(x,y) = (x^2, xy)$.



$$\int_{\vec{c}} \vec{F} \cdot \vec{n}\, ds = \iint_D \nabla \cdot \vec{F}\, dA$$

$$= \iint_D \left(\frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} xy\right) dA$$

$$= \iint_D 2x + x\, dA = \iint_D 3x\, dA$$

- using polar coordinates...

$$= \int_0^1 \int_0^{2\pi} (3r\cos\theta)(r\,d\theta\,dr)$$

$$= \int_0^1 \int_0^{2\pi} 3r^2 \cos\theta \, d\theta \, dr$$

$$= \int_0^1 3r^2 \, dr \int_0^{2\pi} \cos\theta \, d\theta$$

$$= \left[r^3 \Big|_{r=0}^{r=1} \right] \left[\sin\theta \Big|_{\theta=0}^{\theta=2\pi} \right] = \boxed{0}$$

8.3 Stoke's Theorem

- Green's and Divergence theorems made the following generalization:

$$\int_I \frac{d}{dt} f(t)\,dt = f(b) - f(a) \;\nearrow\; \iint_D \nabla \times \vec{F}\, dA = \int_{\partial D} \vec{F} \cdot d\vec{s}$$

$$\searrow\; \iint_D \nabla \cdot \vec{F}\, dA = \int_{\partial D} \vec{F} \cdot \vec{n}\, ds$$

- can we generalize the case for curves?

$$\int_{\vec{c}} \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

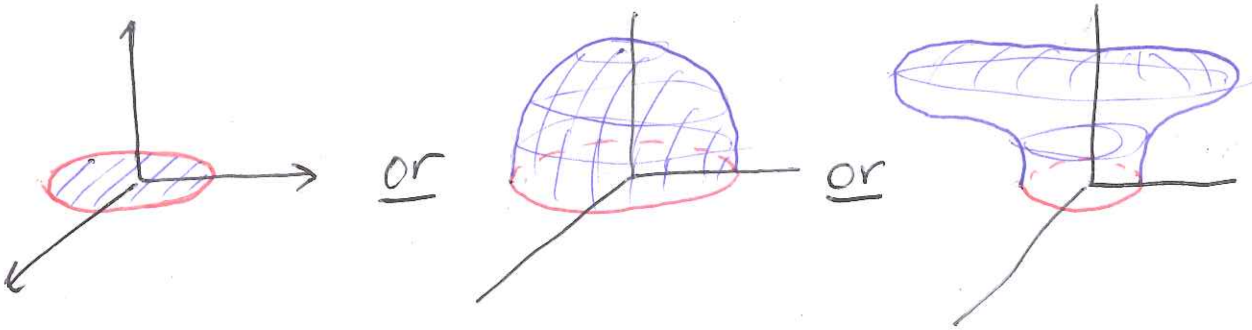
$$\Rightarrow ???$$

<u>Stoke's Theorem</u>: Let S be a surface in $\mathbb{R}^3$, and let ∂S be the positively oriented boundary of S. Then

$$\boxed{\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}}$$

<u>Ex</u>: Evaluate $\int_{\vec{c}} \vec{F} \cdot d\vec{S}$, where $\vec{c}$ is the unit circle in the xy-plane and $\vec{F}(x,y,z) = (2, x, y^2)$

- when using Stoke's theorem, S can be <u>any</u> surface such that $\partial S = \vec{c}$



- Let's keep things simple and use the first surface. Then

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

$$= \iint_S (2y, 0, 1) \cdot dS$$

- Now, we can parametrize S, or we can see that the normal vector to S is always $(0,0,1)$:

$$= \iint_D (2y, 0, 1) \cdot (0,0,1)\, dA$$

$$= \iint_D dA = \text{area of the region } D = \pi(1)^2 = \boxed{\pi}$$