- Since 2-sided surfaces have two sides, we call one side the "outside" and one side the "inside." This choice is called the orientation of the surface.

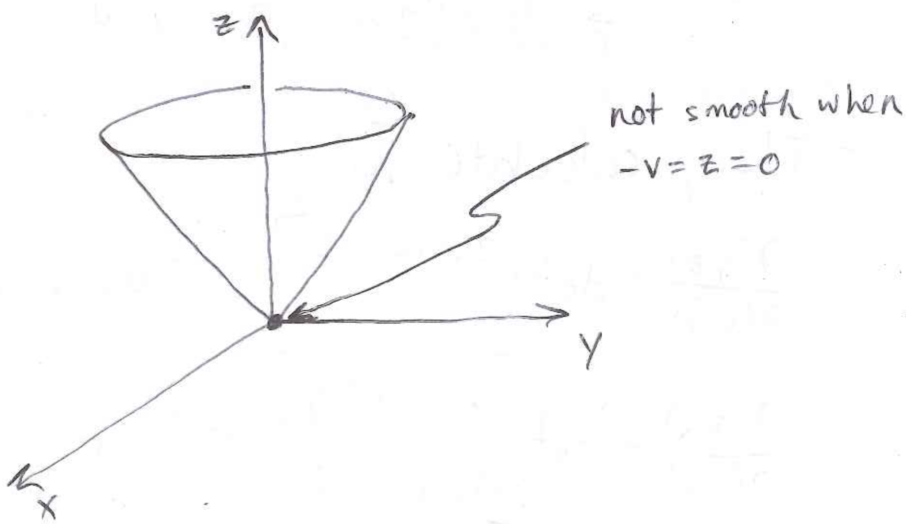
- One last definition: a parametrization $\vec{r}$ is called smooth if $\vec{N}$ exists and $\vec{N} \neq \vec{0}$.

Ex: Is $\vec{r}(u,v) = (v\cos u, v\sin u, v)$ smooth?

- We already calculated $\vec{N}$:

$$\vec{N} = (v\cos u, v\sin u, -v)$$

We can see that $\vec{N}(u,0) = (0,0,0)$, so $\vec{r}$ is not smooth.



7.3 Surface Integrals of Real-Valued Functions

- Integrals over curves are defined by

$$\int_{\vec{c}} f\, ds = \int_a^b f(\vec{c}(t))\, \|\vec{c}\,'(t)\|\, dt$$

Translating this to surfaces:

$$\boxed{\iint_S f\, dS = \iint_D f(\vec{r}(u,v))\, \|\vec{N}(u,v)\|\, dA}$$

Ex: Compute the surface integral $\iint_S xy\, dS$, where S is the cylinder $x^2+y^2=4$, $-1 \le z \le 1$.

- First, we parametrize the cylinder:

$$\vec{r}(u,v) = (2\cos u, 2\sin u, v);\ u \in [0, 2\pi],\ v \in [-1,1]$$

- Then, calculate $\vec{N}$:

$$\frac{\partial(y,z)}{\partial(u,v)} = \det\begin{bmatrix} 2\cos u & 0 \\ 0 & 1 \end{bmatrix} = 2\cos u$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \det\begin{bmatrix} 0 & 1 \\ -2\sin u & 0 \end{bmatrix} = 2\sin u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} -2\sin u & 0 \\ 2\cos u & 0 \end{bmatrix} = 0$$

$$\Rightarrow \vec{N} = (2\cos u, 2\sin u, 0)$$

$$\Rightarrow \|\vec{N}\| = \sqrt{(2\cos u)^2 + (2\sin u)^2 + 0^2} = \sqrt{4} = \boxed{2}$$

- Now the actual integral:

$$\iint_S xy \, dS = \int_0^{2\pi} \int_{-1}^{1} (2\cos u)(2\sin u)\, 2 \, dv \, du$$

$$= 8 \int_0^{2\pi} v \cos u \sin u \Big|_{v=-1}^{v=1} du$$

$$= 8 \int_0^{2\pi} 2 \cos u \sin u \, du$$

$$= 16 \left[\frac{1}{2} \sin^2 u \Big|_{u=0}^{u=2\pi} \right] = \boxed{0}$$

- Just like with curves, $\iint_S f \, dS$ is independent of the parametrization that you use: you get the same answer no matter how the surface is parametrized

Ex: Compute the integral of $f(x,y,z) = \arctan\left(\frac{y}{x}\right)$ over the surface parametrized by

$$\vec{r}(u,v) = (v\cos u, v\sin u, v^2)$$

$$u \in [0, 2\pi], \; v \in [1,2]$$

- Calculating $\vec{N}$:

$$\frac{\partial(y,z)}{\partial(u,v)} = \det\begin{bmatrix} v\cos u & \sin u \\ 0 & 2v \end{bmatrix} = 2v^2\cos u$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \det\begin{bmatrix} 0 & 2v \\ -v\sin u & \cos u \end{bmatrix} = 2v^2\sin u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} -v\sin u & \cos u \\ v\cos u & \sin u \end{bmatrix} = -v$$

$$\Rightarrow \vec{N} = (2v^2\cos u,\ 2v^2\sin u,\ -v)$$

$$\Rightarrow \|\vec{N}\| = \sqrt{(2v^2\cos u)^2 + (2v^2\sin u)^2 + (-v)^2}$$

$$= \sqrt{8v^4 + v^2} = v\sqrt{8v^2+1}$$

- Then the integral:

$$\iint_S \arctan\left(\frac{y}{x}\right) dS = \iint_D \arctan\left(\frac{\cancel{v}\sin u}{\cancel{v}\cos u}\right) v\sqrt{8v^2+1}\, dA$$

$$= \int_1^2\int_0^{2\pi} uv\sqrt{8v^2+1}\, du\, dv$$

heyo!

$$= \int_1^2 v\sqrt{8v^2+1}\, dv \int_0^{2\pi} u\, du$$

$$= \left[\frac{1}{8}\sqrt{8v^2+1} \Big|_{v=1}^{v=2} \right] \left[\frac{1}{2}u^2 \Big|_{u=0}^{u=2\pi} \right]$$

$$= \left[\frac{1}{8}\left(\sqrt{33} - \sqrt{9} \right) \right] 2\pi^2$$

- $\vec{r}(u,v) = (v^2 \cos u^3, v^2 \sin u^3, v^4)$

$$u \in [0, (2\pi)^{1/3}], \quad v \in [1, \sqrt{2}]$$

is a parametrization of the same surface, so we should get the same answer:

$$\vec{N} = (12u^2 v^5 \cos u^3, 12u^2 v^5 \sin u^3, -6u^2 v^3)$$

$$\Rightarrow \|\vec{N}\| = \sqrt{(12u^2v^5)^2 + (12u^2v^5)^2 + (-6u^2v^3)^2}$$

$$= 6u^2v^3\sqrt{(2v^2)^2 + (2v^2)^2 + 1}$$

$$= 6u^2v^3\sqrt{8v^4+1}$$

$$\Rightarrow \iint_S \arctan\left(\frac{y}{x}\right) dS = \iint_D \arctan\left(\frac{v^2 \cos u^3}{v^2 \sin u^3}\right)\left(6u^2v^3\sqrt{8v^4+1}\right) dA$$

$$= \int_1^{\sqrt{2}} \int_0^{(2\pi)^{1/3}} 6u^4 v^3 \sqrt{8v^4+1}\, du\, dv$$

$$\vdots$$

$$= \left[\frac{1}{8}\left(\sqrt{33} - 3 \right) \right] 2\pi^2 \quad \checkmark$$

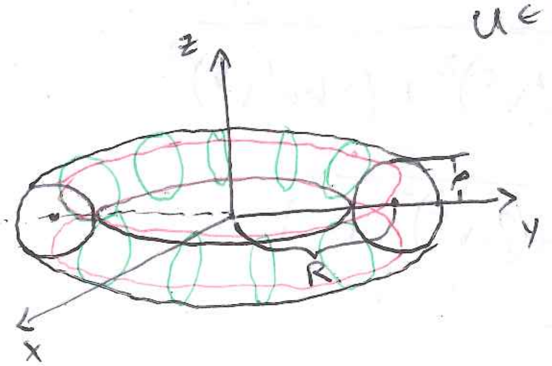
- Just like how we can calculate arclength for a curve, we can calculate surface area for a surface:

$$\text{arclength} = \int_a^b \|\vec{c}\,'(t)\|\,dt$$

$$\Updownarrow$$

$$\text{surface area} = \iint_D \|\vec{N}(u,v)\|\,dA$$

Ex: Find the surface area of the torus param. by $\vec{r}(u,v) = ((R+\rho\cos v)\cos u, (R+\rho\cos v)\sin u, \rho\sin v)$,
$u \in [0, 2\pi]$, $v \in [0, 2\pi]$



- First, calculate the normal vector:

$$\frac{\partial(y,z)}{\partial(u,v)} = \det\begin{bmatrix} (R+\rho\cos v)\cos u & -\rho\sin v\sin u \\ 0 & \rho\cos v \end{bmatrix}$$

$$= \boxed{(R+\rho\cos v)}\,\rho\cos v\cos u$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \det\begin{bmatrix} 0 & \rho\cos v \\ -(R+\rho\cos v)\sin u & -\rho\sin v\cos u \end{bmatrix}$$

$$= \boxed{(R+\rho\cos v)}\,\rho\cos v\sin u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} -(R+\rho\cos v)\sin u & -\rho\sin v\cos u \\ (R+\rho\cos v)\cos u & -\rho\sin v\sin u \end{bmatrix}$$

$$= \boxed{(R+\rho\cos v)}\,\rho\sin v$$

$$\Rightarrow \vec{N}(u,v) = \boxed{(R+\rho\cos v)}\left(\rho\cos v\cos u,\ \rho\cos v\sin u,\ \rho\sin v\right)$$

$$\Rightarrow \|\vec{N}(u,v)\| = (R+\rho\cos v)\sqrt{(\rho\cos v\cos u)^2+(\rho\cos v\sin u)^2+(\rho\sin v)^2}$$

$$\vdots$$

$$= \rho(R+\rho\cos v)$$

$$\Rightarrow \text{surface area} = \iint_D \rho(R+\rho\cos v)\,dA$$

$$= \int_0^{2\pi}\int_0^{2\pi} \rho(R+\rho\cos v)\,du\,dv$$

$$= \int_0^{2\pi} \rho(R+\rho\cos v)\,dv \int_0^{2\pi} du$$

$$= \left[\rho(Rv+\rho\sin v)\Big|_{v=0}^{v=2\pi}\right]\left[u\Big|_{u=0}^{u=2\pi}\right]$$

$$= \boxed{4\pi^2 R\rho}$$

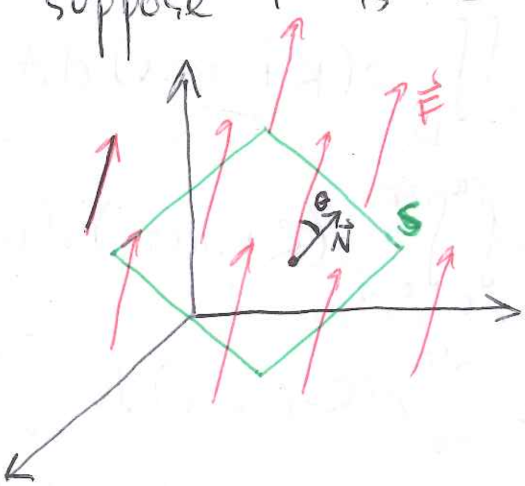
7.4 Surface Integrals of Vector Fields

- Generalize from curves to surfaces:

for curves: $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$

for surfaces: $\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\,dA}$

- What does this mean? Take S to be a plane, and suppose $\vec{F}$ is constant:



- We can assume that $\|\vec{F}\| = C$ and that $\|\vec{N}\| = 1$. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\,dA$$

$$= \iint_D \|\vec{F}(\vec{r})\|\,\|\vec{N}\| \cos\theta\,dA$$

$$= C\cos\theta \iint_D \|\vec{N}\|\,dA$$

$$= C\cos\theta \cdot [\text{surface area of } S]$$

- If $\vec{F}$ represents the velocity of a fluid, then $C\cos\theta\, A(S)$ is exactly the volume of fluid that passes through S in a unit of time.

- For this reason, sometimes $\iint_S \vec{F}\cdot d\vec{S}$ is called the flux of $\vec{F}$ through the surface S.

Ex: Compute the flux of $\vec{F}(x,y,z) = (y^3, x^3, 3z^2)$ over the surface parametrized by

$$\vec{F}(u,v) = (u, v, u^2+v^2), \quad u^2+v^2 \le 4$$

- First: compute the normal

$$\frac{\partial(y,z)}{\partial(u,v)} = \det\begin{bmatrix} 0 & 1 \\ 2u & 2v \end{bmatrix} = -2u$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \det\begin{bmatrix} 2u & 2v \\ 1 & 0 \end{bmatrix} = -2v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

$$\Rightarrow \vec{N} = (-2u, -2v, 1)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\, dA$$

$$= \iint_D (v^3, u^3, 3(u^2+v^2)^2) \cdot (-2u, -2v, 1)\, dA$$

$$= \iint_D -2uv^3 - 2u^3v + 3(u^2+v^2)^2\, dA$$

$$= \iint_D -2uv(v^2+u^2) + 3(u^2+v^2)^2\, dA$$

$$= \iint_D (u^2+v^2)\left[3(u^2+v^2) - 2uv\right] dA$$

- D is a circular region ($u^2+v^2 \leq 4$), and there are lots of u^2+v^2 terms in the integrand, so let's switch to polar coordinates:

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad dA = r\,dr\,d\theta$$

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$\rightarrow \int_0^{2\pi}\int_0^2 r^2\left[3r^2 - 2r^2\sin\theta\cos\theta\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^5\left[3 - 2\sin\theta\cos\theta\right] dr\,d\theta$$

$$= \int_0^{2\pi}(3 - 2\sin\theta\cos\theta)\,d\theta \int_0^2 r^5\,dr$$

$$\vdots$$

$$= 64\pi$$