- Then our volume becomes

$$V = \int_1^2 \int_x^{2x} e^{2x+y}\, dy\, dx$$

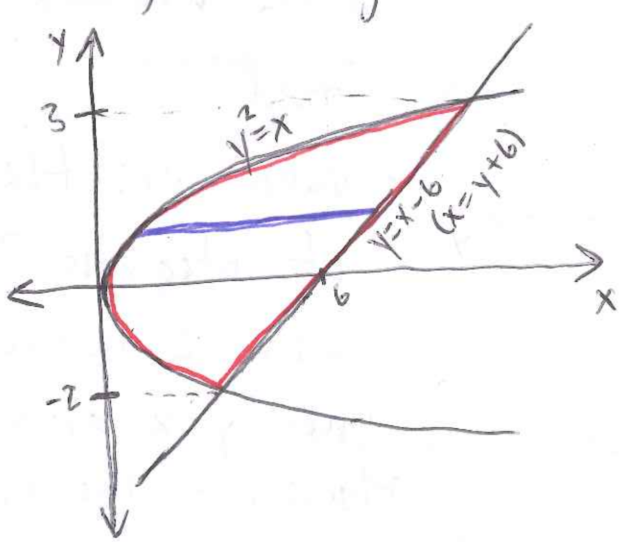
$$= \int_1^2 \left[e^{2x+y} \right] \Big|_{y=x}^{y=2x} dx$$

$$= \int_1^2 \left(e^{4x} - e^{3x} \right) dx$$

$$= \frac{1}{4} e^{4x} - \frac{1}{3} e^{3x} \Big|_1^2 = \frac{1}{4}(e^8 - e^4) - \frac{1}{3}(e^6 - e^3)$$

<u>Ex</u>: Evaluate $\iint_D 2y\, dA$, where D is the region bounded by $y = x - 6$ and $y^2 = x$

- Drawing the region:



- How should we slice the region? If we slice vertically (holding x fixed to find areas), the limits of integration won't be "nice"

- If we slice holding y fixed, we find the area

$$A(y) = \int_{y^2}^{y+6} 2y \, dx$$

- How do we determine the limits of the outside integral?

$$\int_{?}^{?} \int_{y^2}^{y+6} 2y \, dx \, dy$$

- Use $y^2 = x$ and $y = x - 6$ to solve for y:

$$y = y^2 - 6 \rightarrow y^2 - y - 6 = 0$$

$$\rightarrow (y-3)(y+2) = 0$$

$$\rightarrow y = -2, 3$$

- Thus,

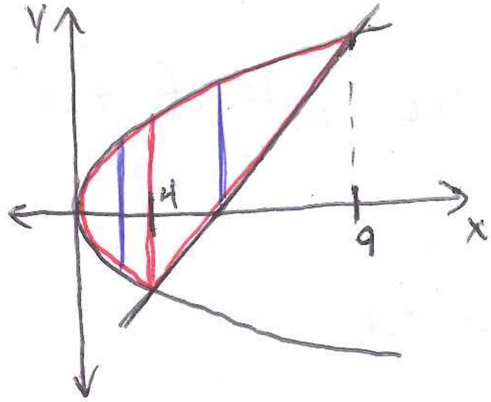
$$\iint_D 2y \, dA = \int_{-2}^{3} \int_{y^2}^{y+6} 2y \, dx \, dy$$

$$= \int_{-2}^{3} 2yx \Big|_{x=y^2}^{x=y+6} dy$$

$$= \int_{-2}^{3} \left(2y(y+6) - 2y^3\right) dy$$

$$= \boxed{\frac{125}{6}}$$

- We can still slice it vertically, and get the same answer. We need to split the region into 2 regions though



- For the first region, the area of the slices are

$$A(x) = \int_{-\sqrt{x}}^{\sqrt{x}} 2y\,dy$$

- For the second area,

$$A(x) = \int_{x-6}^{\sqrt{x}} 2y\,dy$$

- Now the integral is

$$\iint_D 2y\,dA = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} 2y\,dy\,dx + \int_4^9 \int_{x-6}^{\sqrt{x}} 2y\,dy\,dx$$

$$= \int_0^4 y^2 \Big|_{-\sqrt{x}}^{\sqrt{x}} dx + \int_4^9 y^2 \Big|_{y=x-6}^{y=\sqrt{x}} dx$$

$$= \int_0^4 (\sqrt{x})^2 - (-\sqrt{x})^2\,dx + \int_4^9 (\sqrt{x})^2 - (x-6)^2\,dx$$

$$= \int_0^4 0\,dx + \int_4^9 (-x^2 + 13x - 36)\,dx$$

$$= \frac{125}{6}$$

6.3 Examples and Techniques for Double Integrals

- The order of integration can make things easier or harder:

<u>Ex</u>: Evaluate $\iint_R y\sin(xy)\,dA$, where R is the rectangle $[1,3]\times[0,\frac{\pi}{2}]$.

- One way: $\int_0^{\pi/2}\int_1^3 y\sin(xy)\,dx\,dy$

$$= \int_0^{\pi/2} -\cos(xy)\Big|_{x=1}^{x=3}\,dy$$

$$= \int_0^{\pi/2}\left[\cos(y)-\cos(3y)\right]dy$$

$$= \sin(y)-\frac{1}{3}\sin(3y)\Big|_0^{\pi/2} = \boxed{\frac{4}{3}}$$

- The other way:

$$\int_1^3\int_0^{\pi/2} y\sin(xy)\,dy\,dx \qquad \text{(int. by parts, ugh)}$$

$$u=y \qquad dv=\sin(xy)\,dy$$
$$du=dy \qquad v=-\frac{1}{x}\cos(xy)$$

$$= \int_1^3\left[-\frac{y}{x}\cos(xy)\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{1}{x}\cos(xy)\,dy\right]dx$$

$$= \int_1^3\left[-\frac{\pi}{2x}\cos\left(\frac{\pi}{2}x\right) + \underbrace{\int_0^{\pi/2}\frac{1}{x}\cos(xy)\,dy}_{u=xy \quad du=x\,dy \;\Rightarrow\; \frac{du}{x^2}=\frac{1}{x}dy}\right]dx$$

$$= \int_1^3 \left[-\frac{\pi}{2x}\cos\left(\tfrac{\pi}{2}x\right) + \frac{1}{x^2}\sin(xy)\Big|_{y=0}^{y=\pi/2} \right] dx$$

$$= \int_1^3 \left[-\frac{\pi}{2x}\cos\left(\tfrac{\pi}{2}x\right) + \frac{1}{x^2}\sin\left(\tfrac{\pi}{2}x\right) \right] dx$$

(int. by parts on first term:

$u = \frac{\pi}{2x}$ $\qquad dv = \cos\left(\frac{\pi}{2}x\right)$

$du = -\frac{\pi}{2x^2}dx$ $\qquad v = \frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)dx$)

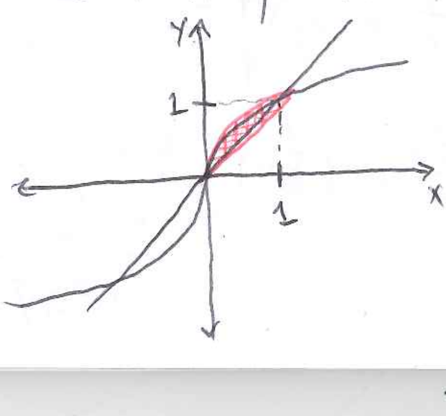
$$= -\left(\frac{1}{x}\sin\left(\tfrac{\pi}{2}x\right)\Big|_1^3 + \cancel{\int_1^3 \frac{1}{x^2}\sin\left(\tfrac{\pi}{2}x\right)dx} \right) + \cancel{\int_1^3 \frac{1}{x^2}\sin\left(\tfrac{\pi}{2}x\right)dx}$$

$$= -\frac{1}{x}\sin\left(\tfrac{\pi}{2}x\right)\Big|_1^3 = \boxed{\frac{4}{3}}$$

- Way way <u>way</u> harder!

- Sometimes it's actually impossible to reverse the order of integration:

<u>Ex</u>: Evaluate $\iint_D e^{x/y}\,dA$, where D is the region between $x=y$ and $x=y^3$ in the first quadrant.



- Holding ~~x~~ fixed, the integral is

$$\iint_D e^{x/y}\,dA = \int_0^1 \int_{y^3}^{y} e^{x/y}\,dx\,dy$$

$$= \int_0^1 \left[y e^{x/y} \Big|_{x=y^3}^{x=y} \right] dy$$

$$= \int_0^1 \Big[y e^{1} - \underbrace{y e^{y^2}}_{u=y^2 \quad du = 2y\,dy} \Big] dy$$

$$= \frac{y^2}{2} e^1 - \frac{1}{2} e^{y^2} \Big|_0^1 = \boxed{\frac{1}{2}}$$

- Reversing the order, we see that $x \le y \le x^{1/3}$:

$$\iint_D e^{x/y}\,dA = \int_0^1 \left(\int_x^{x^{1/3}} e^{x/y}\,dy \right) dx$$

There is no exact solution to this integral!

- Under certain conditions, we can separate a double integral into two single integrals:
  - S'pose $f(x,y) = g(x)\,h(y)$. Then

$$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_a^b \int_c^d g(x)\,h(y)\,dx\,dy$$

$h(y)$ is a constant in the inner integral
$\rightarrow$ factor it out

$$= \int_a^b h(y) \int_c^d g(x)\,dx\,dy$$

$\int_c^d g(x)\,dx$ is a constant in the outer integral
→ factor it out:

$$\Rightarrow \boxed{\int_a^b \int_c^d g(x)h(y)\,dx\,dy = \int_c^d g(x)\,dx \int_a^b h(y)\,dy}$$

- This is called separation of variables, and usually makes things easier!

Ex: Evaluate $\iint_D e^{x+y}\,dA$, where $D = [0,1]\times[0,1]$.

$$\iint_D e^{x+y}\,dA = \int_0^1 \int_0^1 e^{x+y}\,dx\,dy$$

$$= \int_0^1 \int_0^1 e^x e^y\,dx\,dy$$

$$= \left(\int_0^1 e^x\,dx\right)\left(\int_0^1 e^y\,dy\right) = \boxed{(e-1)^2}$$

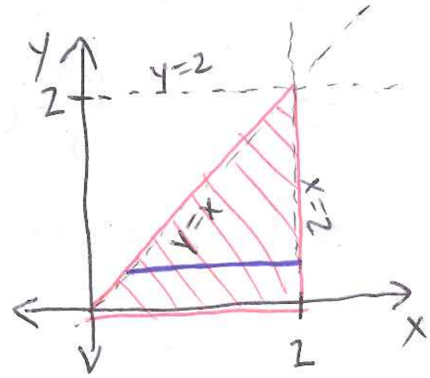
- Sometimes you are forced to change the order of integration:

Ex: Evaluate $\int_0^2 \int_y^2 e^{x^2}\,dx\,dy$

- There is no closed form for $\int e^{x^2} dx$, so we must swith the order of integration and hope for the best:

- We see that

$$\int_0^2 \int_y^2 e^{x^2}\, dx\, dy \implies$$



$0 \le y \le 2$ $\qquad$ $y \le x \le 2$

$y=0$ $\quad$ $y=2$ $\quad$ $y=x$ $\quad$ $x=2$

- To switch the order of integration, we see that $0 \le x \le 2$ and $0 \le y \le x$, which gives

$$\int_0^2 \int_0^x e^{x^2}\, dy\, dx$$

$$= \int_0^2 e^{x^2} y \Big|_{y=0}^{y=x} dx$$

$$= \int_0^2 x e^{x^2} dx \qquad u = x^2 \quad du = 2x\,dx$$

$$= \frac{1}{2} e^{x^2} \Big|_{x=0}^{x=2} = \boxed{\frac{1}{2} e^4 - \frac{1}{2}}$$

6.4 Change of Variables in Double Integrals

- Let's look at how we do change of variables (i.e. u-substitution) in 1-D first:

Ex: $\int_1^2 e^{5x}\,dx \qquad u = 5x \rightarrow x = \frac{1}{5}u$

$\qquad du = 5dx \rightarrow dx = \frac{1}{5}du$

$$= \int_5^{10} e^u \tfrac{1}{5}\,du$$

- viewing ~~[illegible]~~ $x = \frac{1}{5}u = x(u)$ as a function of u, we see:

$$\int_1^2 e^{5x}\,dx = \int_{x^{-1}(1)}^{x^{-1}(2)} e^{5x(u)}\,x'(u)\,du$$

- More generally,

$$\int_I f(x)\,dx = \int_{I^*} f(x(u))\left|\frac{\partial x}{\partial u}\right| du$$

- Here, I is the interval of integration (i.e. $I = [a,b]$).
- $x(u)$ is a transformation which maps I^* to I: $x : I^* \rightarrow I$