<u>Ex</u>: Compute $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$ for $\vec{F}(x,y) = \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right)$

and $\vec{c}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

$$\vec{F}(\vec{c}(t)) = (-\sin t, \cos t), \quad \vec{c}\,'(t) = (-\sin t, \cos t)$$

$$\Rightarrow \int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} dt = 2\pi \neq 0$$

- Since $\vec{c}$ is a simple closed curve and $\int_{\vec{c}} \vec{F} \cdot d\vec{s} \neq 0$, we know that $\vec{F}$ <u>cannot</u> be a gradient vector field

- Weirdly:

$$\nabla \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

scalar curl

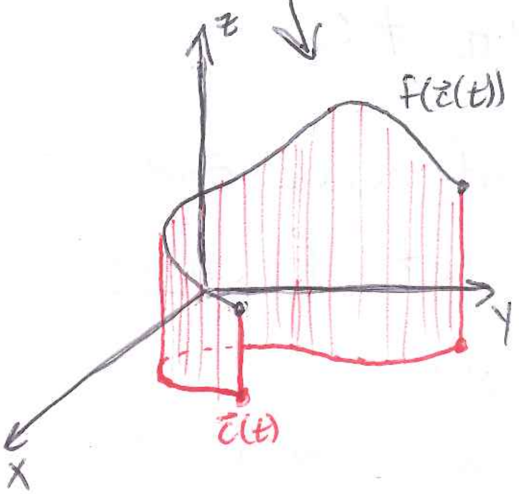
$$= \frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2} = 0$$

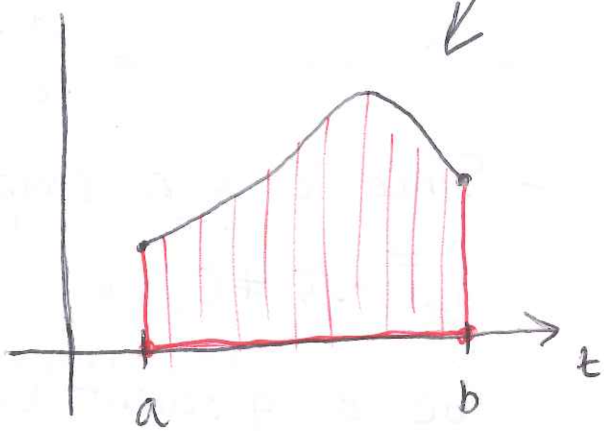
- Not so weird though, since $\vec{F}$ is defined on all $(x,y) \in \mathbb{R}^2$ <u>except</u> $(0,0)$, i.e. $\mathbb{R}^2 - \{(0,0)\}$, which is <u>not</u> simply connected.

6.2 Double Integrals Over General Regions

- Up to now, all the integrals we've done have been computing areas:

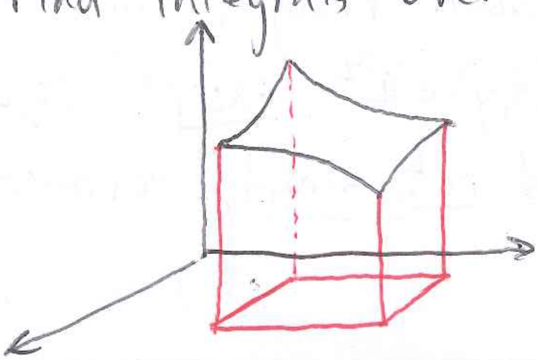
$$\int_{\vec{c}} f\,ds = \int_a^b f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$



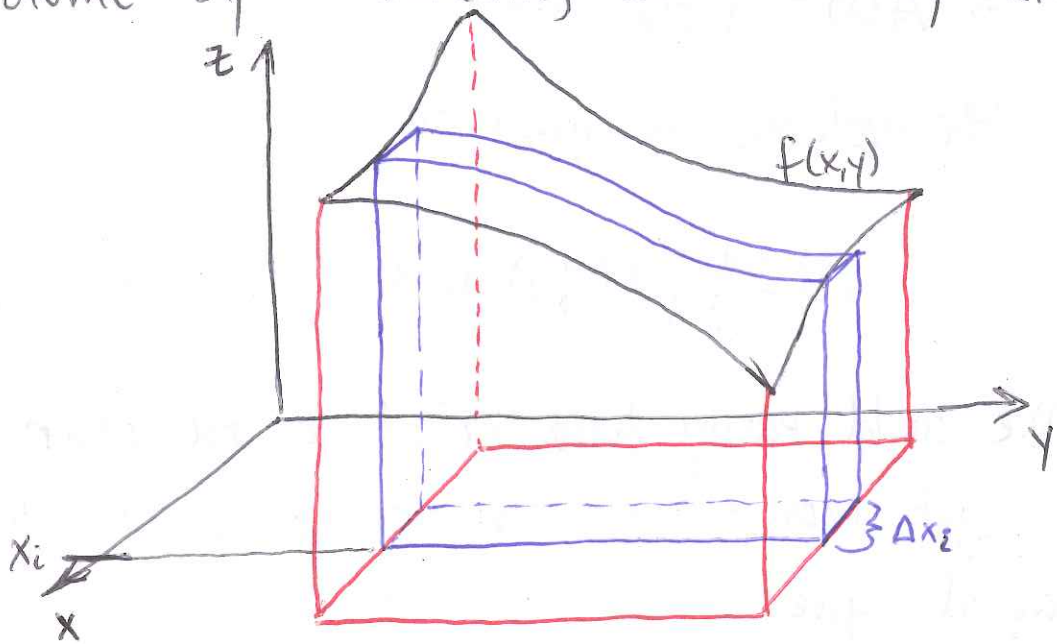


- Integral over a curve (1D object) gives an <u>area</u>
- Integral over a region (2D object) gives a <u>volume</u>

- How do we find integrals over rectangles?



We do Riemann sums, of course! Slice the solid vertically, and approximate the volume by "thickening" each slice by Δx



- The area of the side of this slice is some fn of x — call it $A(x)$
- Then the volume is $A(x_i)\Delta x_i$
- The approximate volume is the sum of the volume of the slices:

$$V \approx \sum_{i=1}^{n} A(x_i)\,\Delta x_i$$

- Turning this into an integral by $\lim_{n\to\infty}$:

$$V = \int A(x)\,dx$$

- All that remains is to find $A(x)$, an area
  → another integral
  → $A(x) = \int f(x,y)\,dy$

- So, the volume is given by

$$V = \iint f(x,y)\,dy\,dx$$

  - We could easily have sliced it the other way, to get cross-sectional volumes $A(y)\,\Delta y$, which would give

$$V = \iint f(x,y)\,dx\,dy$$

  - This is called Fubini's theorem — the order of integration does not matter.

Ex: Find the volume of the solid below the surface defined by $f(x,y) = 6x^2y$ and above the rectangle $R = \underbrace{[-1,1]}_{x\text{-range}} \times \underbrace{[0,4]}_{y\text{-range}}$.

  - For a fixed x, the area of the cross-section is

$$A(x) = \int_0^4 6x^2 y\, dy$$

$$= 3x^2y^2 \Big|_{y=0}^{y=4}$$

$$= 3x^2(4^2 - 0^2) = 48x^2$$

- Then the volume is given by

$$V = \int_{-1}^{1} A(x)\, dx$$

$$= \int_{-1}^{1} 48x^2\, dx$$

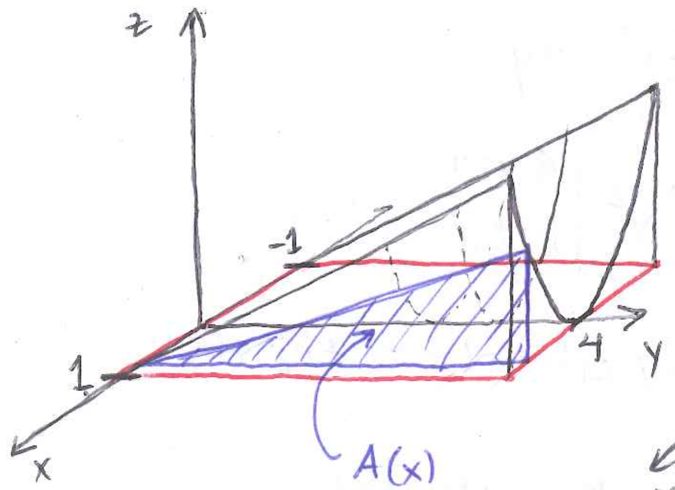
$$= 16x^3 \Big|_{x=-1}^{x=1} = \boxed{32}$$

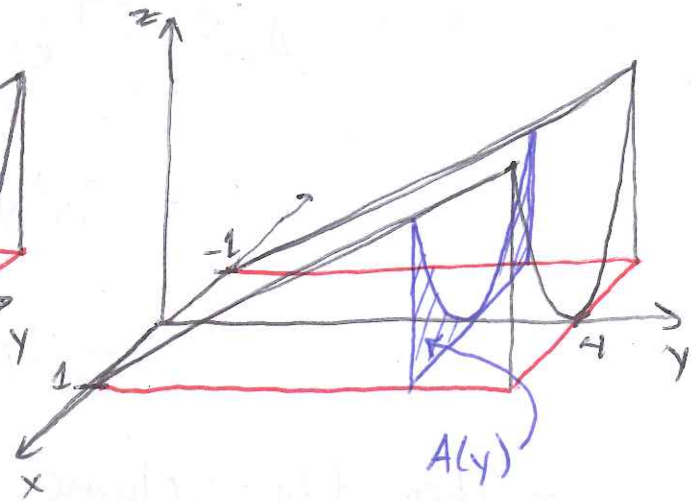
- Alternatively, for a fixed y, the area of a cross-section is

$$A(y) = \int_{-1}^{1} 6x^2 y\, dx$$

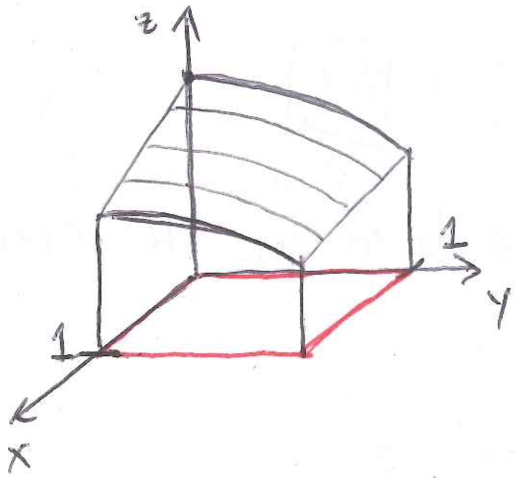
$$= 2x^3 y \Big|_{x=-1}^{x=1} = 4y$$

$$\rightarrow V = \int_0^4 4y\, dy = 2y^2 \Big|_{y=0}^{y=4} = \boxed{32}$$





<u>Ex</u>: Find the volume of the solid bounded by the surface $z = 5 - 2x - y^2$, the 3 coordinate planes, and the planes $x = 1$ and $y = 1$



$$V = \int_0^1 \int_0^1 (5 - 2x - y^2)\,dy\,dx$$

$$= \int_0^1 \left[(5-2x)y - \tfrac{1}{3}y^3 \right]\Big|_{y=0}^{y=1} dx$$

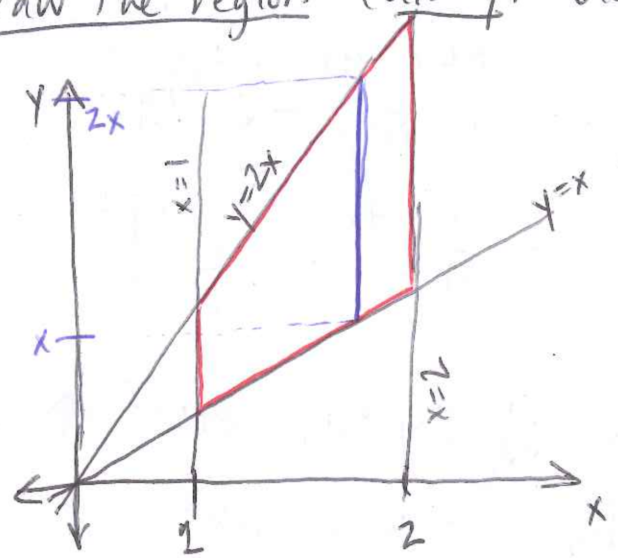
$$= \int_0^1 \left[(5-2x) - \tfrac{1}{3} \right] dx$$

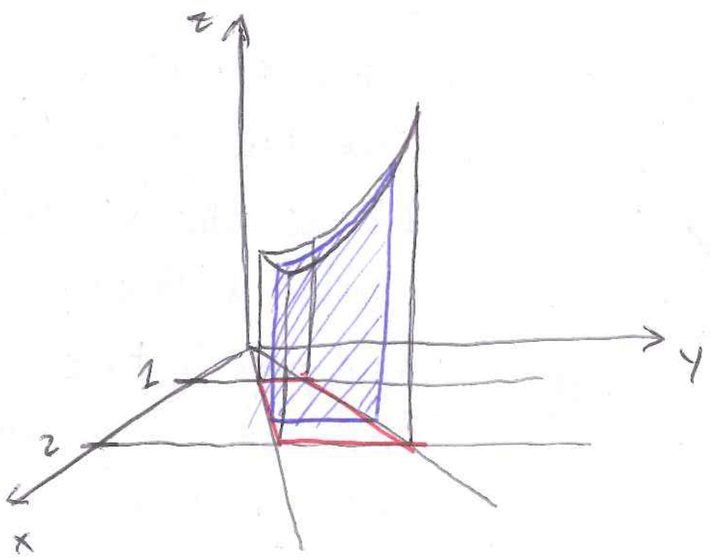
$$= \frac{14}{3}x - x^2 \Big|_{x=0}^{x=1} = \boxed{\frac{11}{3}}$$

- What if we want to integrate over a region that isn't a rectangle?

Ex: Evaluate $\iint_D e^{2x+y}\,dA$, where D is the region bounded by the lines $y=2x$, $y=x$, $x=1$, and $x=2$. (here dA basically means $dx\,dy$)

- First, <u>draw the region</u> (<u>always</u> do this first)





- Let's find cross-section areas where x is held fixed
- What are the limits of integration? Because we're between the lines $y=x$ and $y=2x$, those are the limits:

$$A(x) = \int_x^{2x} e^{2x+y}\,dy$$

- Then our volume becomes

$$V = \int_1^2 \int_x^{2x} e^{2x+y}\, dy\, dx$$

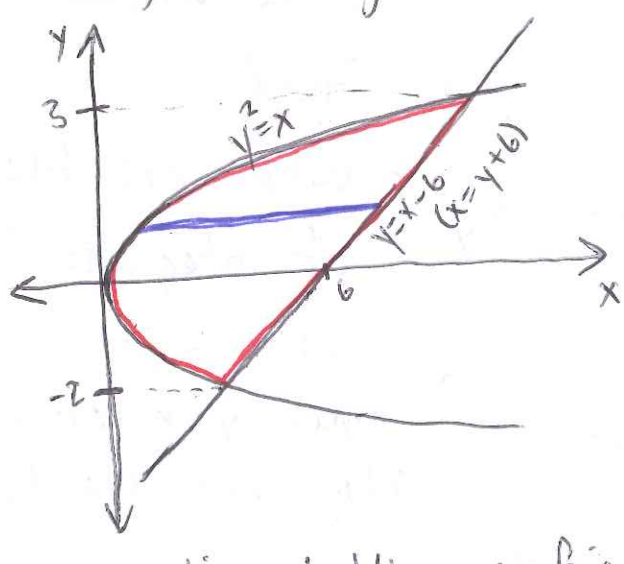
$$= \int_1^2 \left[e^{2x+y} \right] \Big|_{y=x}^{y=2x} dx$$

$$= \int_1^2 \left(e^{4x} - e^{3x} \right) dx$$

$$= \frac{1}{4} e^{4x} - \frac{1}{3} e^{3x} \Big|_1^2 = \frac{1}{4}\left(e^8 - e^4\right) - \frac{1}{3}\left(e^6 - e^3\right)$$

<u>Ex</u>: Evaluate $\iint_D 2y\, dA$, where D is the region bounded by $y = x - 6$ and $y^2 = x$

- Drawing the region:



- How should we slice the region? If we slice vertically (holding x fixed to find areas), the limits of integration won't be "nice"

- If we slice holding y fixed, we find the area