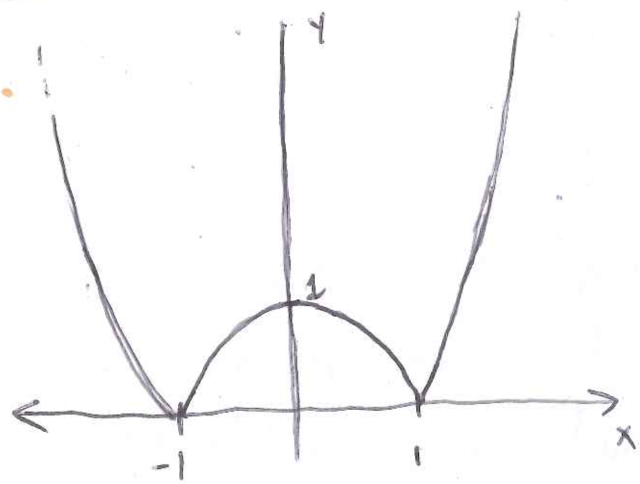
5.1 Path and Parametrizations

- Recall: A <u>path</u> is a function $\vec{c}:[a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$), where $[a,b]$ is an interval in $\mathbb{R}$.
  - The image of $\vec{c}$ is called a <u>curve</u>, and $\vec{c}(t)$ is a <u>parametrization</u> of that curve.

- A path is a <u>C^1 path</u> if each of its components ~~are~~ have at least one continuous derivative.
  - A path is a <u>piecewise C^1 path</u> if the domain $[a,b]$ can be broken into subintervals such that the path is C^1 on each subinterval.

<u>Ex</u>: Let $\vec{c}(t) = (t, |t^2-1|)$, $t \in [-2,3]$.



- $\vec{c}(t)$ is <u>not</u> a C^1 path, since $\vec{c}\,'(-1)$ and $\vec{c}\,'(1)$ do not exist.

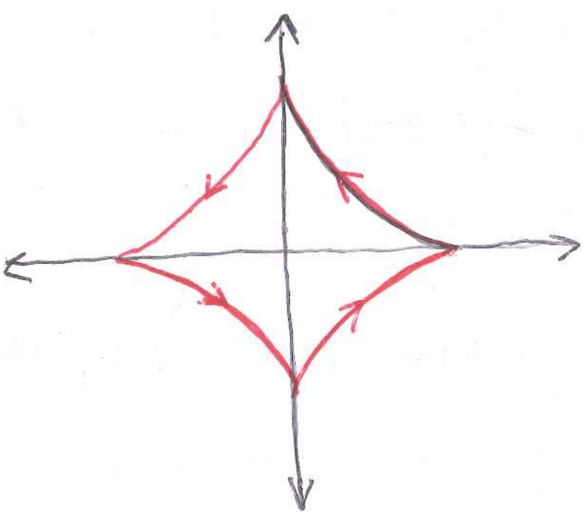
- It is <u>piecewise</u> C^1, however, since

$$\vec{c}(t)\big|_{[-2,-1]}, \quad \vec{c}(t)\big|_{[-1,1]}, \quad \vec{c}(t)\big|_{[1,3]}$$

are all C^1 (here $\vec{c}(t)\big|_{[-2,-1]}$ means "the path $\vec{c}(t)$ restricted to the interval $[-2,-1]$")

- Just because a curve is "pointy," doesn't mean its parametrization is not C^1:

<u>Ex</u>: Consider $\vec{c}(t) = (\cos^3 t, \sin^3 t)$, $t \in [0, 2\pi]$.



The image of $\vec{c}$ has cusps (i.e. "points"), but $\vec{c}(t)$ is still C^1.

- Let's formalize reparametrizations: Suppose you have a C^1 path $\vec{c} : [a,b] \to \mathbb{R}^2$, and a function $\varphi : [\alpha, \beta] \to [a,b]$ which is <u>bijective</u> and C^1.

- Here, φ is a "normal" function of only 1 variable. Recall that <u>bijective</u> means that φ is both <u>1-1</u> and <u>onto</u> (or, that φ^{-1} exists)

- We can compose $\vec{c}$ and φ:

$$\vec{\gamma} = \vec{c} \circ \varphi : [\alpha, \beta] \to \mathbb{R}^2$$

is called a <u>reparametrization</u> of $\vec{c}$

<u>Ex</u>: Let $\vec{c}(t) = (\cos t, \sin t, t)$, $t \in [0, 2\pi]$, and let $\varphi : [0, \pi/2] \to [0, 2\pi]$ be defined by

$$\varphi(t) = 4t$$

$\varphi(t)$ is a C^1 bijection (since $\varphi^{-1}(t) = \frac{t}{4}$), the path

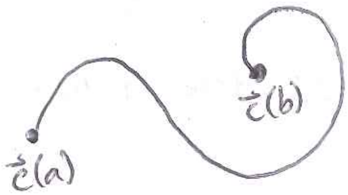
$$\gamma(t) = \vec{c}(\varphi(t)) = (\cos 4t, \sin 4t, 4t)$$

is a <u>reparametrization</u> of $\vec{c}$.

- Note: $\|\vec{c}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = \sqrt{2}$, and since $\vec{\gamma}\,'(t) = \vec{c}\,'(\varphi(t))\,\varphi'(t)$,

$$\|\vec{\gamma}\,'(t)\| = \|\vec{c}\,'(\varphi(t))\,\varphi'(t)\| = \|\vec{c}\,'(\varphi(t))\|\,|\varphi'(t)|$$
$$= 4\sqrt{2}$$

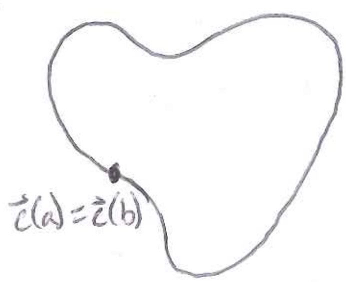
- Let $\vec{c}:[a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) be a 1-1, piecewise C^1 path. Then the image of $\vec{c}$ is called a simple curve.
  - The 1-1 condition means simple curves do not cross themselves:



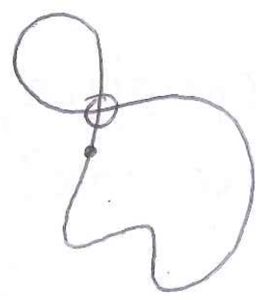
simple

not simple

- A 1-1, piecewise C^1 path $\vec{c}$ such that $\vec{c}(a) = \vec{c}(b)$ is called a simple closed curve.



simple closed curve



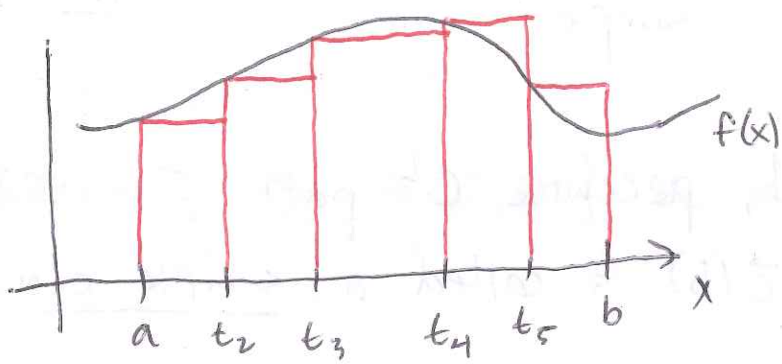
not simple closed curve

5.2 Path Integrals of Real-Valued Functions

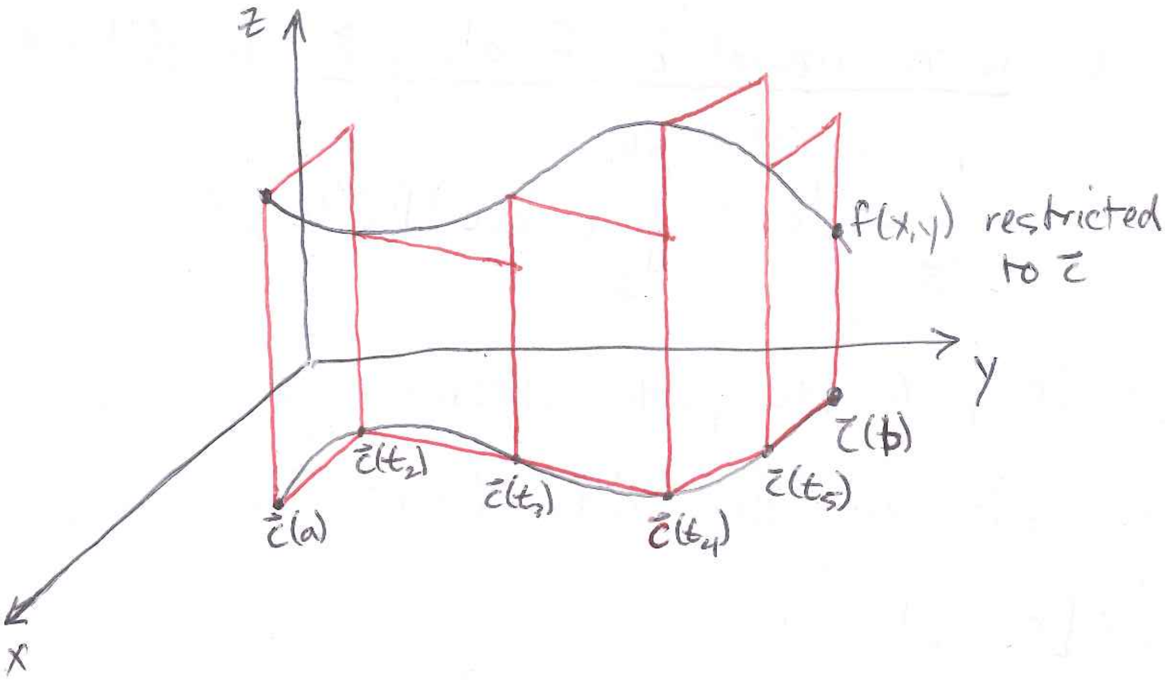
- Recall: the definite integral of a function $f(x)$ on the interval $[a,b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\,\Delta t_i$$

  - Here, the sum on the r.h.s. is a Riemann sum:



- Can we generalize this to functions of more than one variable? Yes!
  - Suppose f is a scalar function of 2 vars, and instead of integrating along the line segment $[a,b]$, we want to integrate along the <u>curve</u> $\vec{c}$



- How do we find the areas of all those rectangles? Recall that from our derivation of arclength,

$$\|\vec{c}(t_{i+1}) - \vec{c}(t_i)\| \approx \|\vec{c}\,'(t_i)\|\,\Delta t_i$$

- Thus, our Riemann sum becomes

$$\sum_{i=1}^{n} \underbrace{f(\vec{c}(t_i))}_{\text{height}}\, \underbrace{\|\vec{c}\,'(t_i)\|\,\Delta t_i}_{\text{approx. width}},$$

- This motivates our definition: let $\vec{c} : [a,b] \to \mathbb{R}^n$ be a C^1 path and let $f : \mathbb{R}^n \to \mathbb{R}$ be a fn. such that $f(\vec{c}(t))$ is continuous. Then the path integral of f along $\vec{c}$ is defined by

$$\int_{\vec{c}} f\,ds = \int_a^b f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$

Ex: Compute the path integral $\int_c f\,ds$ for $f(x,y,z) = xyz$ and $\vec{c}(t) = (-\sin t, \sqrt{2}\cos t, \sin t)$, $t \in [0, \pi/2]$

- First calculate $f(\vec{c}(t))$:

$$f(\vec{c}(t)) = f(-\sin t, \sqrt{2}\cos t, \sin t)$$

$$= (-\sin t)(\sqrt{2}\cos t)(\sin t)$$

$$= -\sqrt{2}\sin^2 t\cos t$$

- Then calculate $\|\vec{c}\,'(t)\|$:

$$\vec{c}\,'(t) = (-\cos t, -\sqrt{2}\sin t, \cos t)$$

$$\Rightarrow \|\vec{c}\,'(t)\| = \sqrt{(-\cos t)^2 + (-\sqrt{2}\sin t)^2 + (\cos t)^2}$$

$$= \sqrt{2\cos^2 t + 2\sin^2 t}$$

$$= \sqrt{2}$$

by

- Finally, the actual integral:

$$\int_{\vec{c}} f\,ds = \int_0^{\pi/2} f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$

$$= \sqrt{2}\int_0^{\pi/2} (-\sqrt{2}\sin^2 t\cos t)\,dt$$

$$= -2\int_0^{\pi/2} \sin^2 t\cos t\,dt$$

$u = \sin t \quad du = \cos t\,dt$

$$= -2\int_0^{1} u^2\,du$$

$$= -2\left[\tfrac{1}{3}u^3\right]\Big|_0^1 = \boxed{-\tfrac{2}{3}}$$

<u>Ex</u>: Find the average temperature of a wire whose shape is given by

$$\vec{c}(t) = (\cos t,\ t/10,\ \sin t),\quad t \in [0, 10\pi]$$

and temperature is given by

$$T(x,y,z) = x^2 + y^2 + z^2$$

- First, $T(\vec{c}(t))$:

$$T(\cos t,\ t/10,\ \sin t) = \cos^2 t + \frac{t^2}{100} + \sin^2 t$$

$$= \frac{t^2}{100} + 1$$

- Then, $\|\vec{c}\,'(t)\|$:

$$\vec{c}\,'(t) = \left(-\sin t, \tfrac{1}{10}, \cos t\right)$$

$$\Rightarrow \|\vec{c}\,'(t)\| = \sqrt{(-\sin t)^2 + \left(\tfrac{1}{10}\right)^2 + (\cos t)^2}$$

$$= \sqrt{\frac{1}{100} + 1} = \frac{\sqrt{101}}{10}$$

- Finally, the integral:

$$\int_{\vec{c}} T\,ds = \int_0^{10\pi} \left(\frac{t^2}{100} + 1\right)\left(\frac{\sqrt{101}}{10}\right) dt$$

$$= \frac{\sqrt{101}}{10} \int_0^{10\pi} \left(\frac{t^2}{100} + 1\right) dt$$

$$= \frac{\sqrt{101}}{10} \left[\frac{t^3}{300} + t\right]\Bigg|_0^{10\pi}$$

$$= \frac{\sqrt{101}}{10} \left[\frac{10}{3}\pi^3 + 10\pi\right]$$

- But wait! The problem asked for the <u>average</u> temperature; $\int_{\vec{c}} T\,ds$ is the "total" temperature – to get the average, we divide by the length of the wire:

$$\ell(\vec{c}) = \int_0^{10\pi} \|\vec{c}\,'(t)\|\,dt = \int_0^{10\pi} \frac{\sqrt{101}}{10}\,dt = \sqrt{101}\,\pi$$

So the average temperature is

$$\frac{\int_{\vec{c}} T\, ds}{\ell(\vec{c})} = \frac{\frac{\sqrt{101}}{10}\left[\frac{10}{3}\pi^3 + 10\pi\right]}{\sqrt{101}\,\pi}$$

$$= \frac{1}{3}\pi^2 + 1$$

- It turns out that path integrals are independent of the parametrization that you use: let f be a scalar function, $\vec{c}$ be a parametrization of some curve, and suppose we have a reparametrization

$$\vec{\gamma}(t) = \vec{c}(\varphi(t))$$

such that $\varphi : [\alpha, \beta] \to [a, b]$, $\varphi(\alpha) = a$, $\varphi(\beta) = b$.

Then:

$$\int_{\vec{\gamma}} f\, ds = \int_\alpha^\beta f(\vec{\gamma}(t))\,\|\vec{\gamma}'(t)\|\,dt$$

$$= \int_\alpha^\beta f(\vec{c}(\varphi(t)))\,\|\vec{c}'(\varphi(t))\,\varphi'(t)\|\,dt$$

$$= \int_\alpha^\beta f(\vec{c}(\varphi(t)))\,\|\vec{c}'(\varphi(t))\|\,|\varphi'(t)|\,dt$$

letting $u = \varphi(t) \rightarrow du = \varphi'(t)\,dt$,

$$= \int_{\varphi(\alpha)}^{\varphi(\beta)} f(\vec{c}(u)) \, \|\vec{c}\,'(u)\| \, du$$

$$= \int_a^b f(\vec{c}(u)) \, \|\vec{c}\,'(u)\| \, du$$

$$\Rightarrow \quad \int_{\vec{\gamma}} f \, ds = \int_{\vec{c}} f \, ds$$

- This says that no matter what parametrization of a curve we use, the path integral will always be the same. Cool!