- Then:

$$\ell(\vec{c}) = \int_0^1 \|\vec{c}\,'(t)\|\,dt$$

$$= \int_0^1 \|(1,t)\|\,dt$$

$$= \int_0^1 \sqrt{1^2+t^2}\,dt$$

$$= \frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)\Big|_0^1$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2}) \approx 1.148$$

- The arclength function $s(t)$ for a path $\vec{c}(t)$ is given by

$$\boxed{s(t) = \int_a^t \|\vec{c}\,'(\tau)\|\,d\tau}$$

  - intuitively, $s(t)$ tells you how far along the path you've traveled in the time $a \le \tau \le t$.
  - one nice property of the arclength function:

$$\frac{d}{dt}s(t) = \frac{d}{dt}\int_a^t \|\vec{c}\,'(\tau)\|\,d\tau = \|\vec{c}\,'(t)\|$$

- We can reparametrize any path using its arclength function: instead of $\vec{c}(t)$, we : solve for t in $s(t)=s$ to get $\vec{c}(s)$

<u>Ex</u>: Let $\vec{c}(t)=(a\cos t, a\sin t, bt)$, $t\in[0,2\pi]$

Then

$$\vec{c}\,'(t)=(-a\sin t, a\cos t, b)$$

$$\Rightarrow \|\vec{c}\,'(t)\|=\sqrt{a^2+b^2}$$

So the arclength function is:

$$s(t)=\int_0^t \|\vec{c}\,'(\tau)\|\,d\tau$$

$$=\int_0^t \sqrt{a^2+b^2}\,d\tau=\sqrt{a^2+b^2}\,t$$

Solving for t in $s(t)=s$:

$$s=s(t)\Rightarrow s=\sqrt{a^2+b^2}\,t$$

$$\Rightarrow t=\frac{s}{\sqrt{a^2+b^2}}$$

Then the reparametrization is

$$\vec{c}(s)=\left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), b\frac{s}{\sqrt{a^2+b^2}}\right)$$

- Why bother with reparametrizing by arclength?
One nice property:

$$\vec{c}\,'(t) = \frac{d\vec{c}(t)}{dt} = \frac{d\vec{c}(s)}{ds} \cdot \frac{ds}{dt} = \vec{c}\,'(s) \cdot \|\vec{c}\,'(t)\|$$

$$\Rightarrow \vec{c}\,'(s) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}$$

- this says a curve parametrized by its arclength is traversed with constant speed 1

- The vector $\boxed{\vec{T}(t) = \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}}$ is called the unit tangent vector to $\vec{c}$ at $\vec{c}(t)$

3.4 Acceleration and Curvature

- Facts we know: $s'(t) = \|\vec{c}\,'(t)\|$, $\vec{T}(t) = \dfrac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}$.
  - Using these facts, we can express the velocity as

$$\vec{v}(t) = \vec{c}\,'(t) = \|\vec{c}\,'(t)\| \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|} = \frac{ds}{dt}\vec{T}(t)$$

  - Now we can re-express acceleration:

$$\vec{a}(t) = \vec{v}\,'(t) = \frac{d}{dt}\left[\frac{ds}{dt} \cdot \vec{T}(t)\right]$$

$$= \frac{d^2 s}{dt^2}\vec{T}(t) + \frac{ds}{dt}\vec{T}\,'(t)$$

- We know $\vec{T}(t)$ is tangent to the curve $\vec{c}(t)$, but what direction does $\vec{T}\,'(t)$ point relative to $\vec{c}(t)$?

$$\|\vec{T}(t)\| = 1 \Rightarrow \frac{d}{dt}\left[\|\vec{T}(t)\|^2\right] = 0$$

however: $\|\vec{T}(t)\|^2 = \vec{T}(t) \cdot \vec{T}(t)$, so

$$\frac{d}{dt}\left[\|\vec{T}(t)\|^2\right] = \frac{d}{dt}\left[\vec{T}(t) \cdot \vec{T}(t)\right]$$

$$= \vec{T}\,'(t) \cdot \vec{T}(t) + \vec{T}(t) \cdot \vec{T}\,'(t)$$

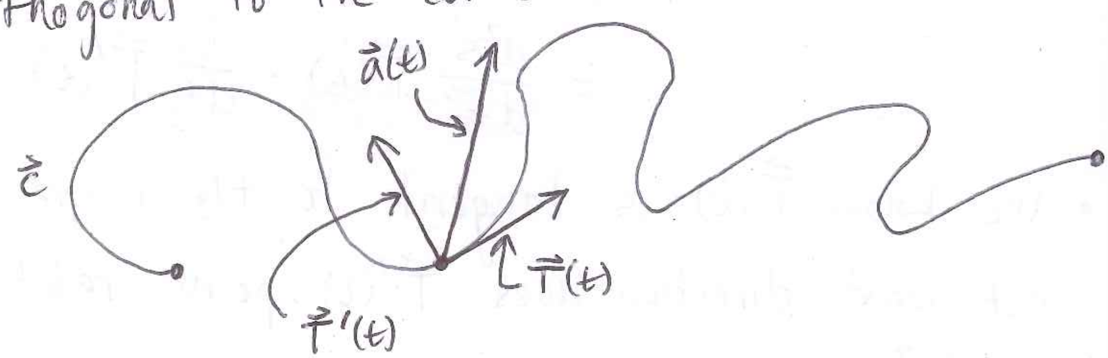
$$= 2\vec{T}\,'(t) \cdot \vec{T}(t)$$

combining these gives:

$$2\vec{T}'(t) \cdot \vec{T}(t) = 0 \Rightarrow \vec{T}'(t) \cdot \vec{T}(t) = 0$$

which tells us that $\vec{T}(t)$ and $\vec{T}'(t)$ are orthogonal $\rightarrow$ $\vec{T}'(t)$ is orthogonal to the curve $\vec{c}(t)$.

- Since $\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T}(t) + \frac{ds}{dt}\vec{T}'(t)$, we see that the acceleration has two components: one in the direction of the tangent, and one orthogonal to the curve:



- The component in the tangential direction is called the <u>tangential acceleration</u>:

$$\boxed{\vec{a}_T = \frac{d^2s}{dt^2}\vec{T}(t)}$$

and the component in the orthogonal direction is called the <u>normal acceleration</u>:

$$\boxed{\vec{a}_N = \frac{ds}{dt}\,\vec{T}\,'(t)}$$

- If you don't want to mess directly with the arclength function $s(t)$, always remember:

$$\boxed{s'(t) = \frac{ds}{dt} = \|\vec{c}\,'(t)\|}$$

Ex: Express the acceleration of the motion $\vec{c}(t) = (t^2, t, t^2)$ in terms of its normal and tangential components.

- Quantities we need to find:

$$\vec{a}_T = \frac{d^2s}{dt^2}\,\vec{T}(t)$$

$$\frac{d^2s}{dt^2}: \quad \boxed{\frac{d}{dt}\|\vec{c}\,'(t)\|} \ ③ \qquad \vec{T}(t): \quad \frac{\boxed{\vec{c}\,'(t)} \ ①}{\boxed{\|\vec{c}\,'(t)\|} \ ②}$$

$$\vec{a}_N = \frac{ds}{dt}\,\vec{T}\,'(t)$$

$$\frac{ds}{dt}: \quad \|\vec{c}\,'(t)\| \qquad \vec{T}\,'(t): \quad \boxed{\frac{d}{dt}\left[\frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}\right]} \ ④$$

① Calculate $\vec{c}\,'(t)$: $\vec{c}\,'(t) = (2t, 1, 2t)$

② Calculate $\|\vec{c}\,'(t)\|$:

$$\|\vec{c}\,'(t)\| = \sqrt{(2t)^2 + 1^2 + (2t)^2}$$

$$= \sqrt{8t^2 + 1}$$

③ Calculate $\frac{d}{dt}\|\vec{c}\,'(t)\|$:

$$\frac{d}{dt}\|\vec{c}\,'(t)\| = \frac{d}{dt}\left(\sqrt{8t^2+1}\right)$$

$$= \frac{1}{2}(8t^2+1)^{-1/2} \cdot 16t$$

$$= 8t(8t^2+1)^{-1/2}$$

④ Calculate $\frac{d}{dt}\left[\frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}\right]$:

$$\frac{d}{dt}\left[\frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}\right] = \frac{d}{dt}\left[\frac{(2t, 1, 2t)}{\sqrt{8t^2+1}}\right]$$

$$= \frac{d}{dt}\left(2t(8t^2+1)^{-1/2},\ (8t^2+1)^{-1/2},\ 2t(8t^2+1)^{-1/2}\right)$$

$$= \Big(2(8t^2+1)^{-1/2} + 2t \cdot -\tfrac{1}{2}(8t^2+1)^{-3/2} \cdot 16t,$$

$$-\tfrac{1}{2}(8t^2+1)^{-3/2} \cdot 16t,$$

same as component 1 $\Big)$

$$= \left(\frac{2}{\sqrt{8t^2+1}} - \frac{16t^2}{(8t^2+1)^{3/2}}, -\frac{8t}{(8t^2+1)^{3/2}}, \right.$$

same as component 1)

$$= \left(\frac{2}{(8t^2+1)^{3/2}}, -\frac{8t}{(8t^2+1)^{3/2}}, \frac{2}{(8t^2+1)^{3/2}} \right)$$

Putting everything together:

$$\vec{a}_T = \frac{d^2s}{dt^2}\vec{T}(t) = \frac{d}{dt}\left[\|\vec{c}\,'(t)\|\right] \cdot \frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}$$

$$= 8t(8t^2+1)^{-1/2} \cdot \frac{(2t, 1, 2t)}{\sqrt{8t^2+1}}$$

$$\boxed{= \left(\frac{16t^2}{8t^2+1}, \frac{8t}{8t^2+1}, \frac{16t^2}{8t^2+1} \right)}$$

$$\vec{a}_N = \frac{ds}{dt} \cdot \vec{T}\,'(t) = \|\vec{c}\,'(t)\| \cdot \frac{d}{dt}\left[\frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}\right]$$

$$= \sqrt{8t^2+1} \left(\frac{2}{(8t^2+1)^{3/2}}, -\frac{8t}{(8t^2+1)^{3/2}}, \frac{2}{(8t^2+1)^{3/2}} \right)$$

$$\boxed{= \left(\frac{2}{8t^2+1}, -\frac{8t}{8t^2+1}, \frac{2}{8t^2+1} \right)}$$

- There is another way of finding $\vec{a}_N$, if you don't want to calculate $\frac{d}{dt}\left[\frac{\vec{c}\,'(t)}{\|\vec{c}\,'(t)\|}\right]$: $\vec{a}_N = \vec{a} - \vec{a}_T$

- From our previous example:

$$\vec{a}_N = \vec{a} - \vec{a}_T$$

$$= (2,0,2) - \left(\frac{16t^2}{8t^2+1}, \frac{8t}{8t^2+1}, \frac{16t^2}{8t^2+1}\right)$$

$$= \left(\frac{2}{8t^2+1}, -\frac{8t}{8t^2+1}, \frac{2}{8t^2+1}\right) \checkmark$$

- How "curvy" is a curve? Given a path $\vec{c}$, the <u>curvature</u> of $\vec{c}$ at a point $\vec{c}(s)$ is

$$\boxed{K(s) = \left\| \frac{d\vec{T}(s)}{ds} \right\|}$$

  - this is for curves parametrized by arclength.

  In general:

$$\frac{d\vec{T}(s)}{ds} = \frac{d\vec{T}(t)}{dt} \cdot \frac{dt}{ds} = \frac{T'(t)}{\|\vec{c}'(t)\|}$$

$$\Rightarrow \boxed{K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{c}'(t)\|}}$$

<u>Ex</u>: Consider $\vec{c}(t) = (r\cos t, r\sin t)$, the circle of radius r.

$$\Rightarrow \vec{c}'(t) = (-r\sin t, r\cos t)$$

$$\Rightarrow \|\vec{c}'(t)\| = \sqrt{(-r\sin t)^2 + (r\cos t)^2} = r$$