3.2 Tangents, Velocity, and Acceleration

- Recall: if $\vec{c}(t)$ gives the position of an object, then:

  $\vec{v}(t) = \vec{c}\,'(t) = $ <u>velocity</u>

  $\|\vec{v}(t)\| = $ <u>speed</u>

  $\vec{a}(t) = \vec{v}\,'(t) = $ <u>acceleration</u>

- <u>Ex</u>: A force field moves a particle according to $\vec{c}(t) = (t, 2t^2, t^3)$. At $t=1$, the field is turned off. What is the position of the particle at $t=3$?
  - Once the field is turned off, the particle moves in whatever direction it was travelling.
  - At $t=1$, the velocity:

    $$\vec{v}(t) = \vec{c}\,'(t) = (1, 4t, 3t^2)$$

    $$\Rightarrow \vec{v}(1) = (1, 4, 3)$$

  - So after $t=1$, the particle is at $\vec{c}(1) = (1,2,1)$ and moves in the direction $\vec{v}(1) = (1,4,3)$:

    $$\vec{l}(s) = (1,2,1) + s(1,4,3)$$

- So at $t=3$ (i.e. $s=2$), the position is

$$\vec{\ell}(2) = (1,2,1) + (2)(1,4,3)$$
$$= (3,10,7)$$

- We know that parametrizations are not unique, but sometimes it is possible to specify the <u>speed</u> along the path:

<u>Ex</u>: Find a parametrization of a particle which moves around the circle of radius R with constant speed s.

- One parametrization is $\vec{c}(t) = (R\cos t, R\sin t)$, but that doesn't have a variable for speed.
- We need to use:

$$\vec{c}(t) = (R\cos(\omega t), R\sin(\omega t))$$

where ω controls the speed of the particle

- We want $s = \|\vec{c}\,'(t)\|$, so

$$\|\vec{c}\,'(t)\| = \|(-R\omega\sin(\omega t), R\omega\cos(\omega t))\|$$
$$= \sqrt{(-R\omega\sin(\omega t))^2 + (R\omega\cos(\omega t))^2}$$
$$= R\omega\sqrt{\sin^2\omega t + \cos^2\omega t}$$
$$= R\omega$$

$$\Rightarrow s = R\omega \Rightarrow \omega = \frac{s}{R}$$

- So the correct parametrization is:

$$\vec{c}(t) = \left(R\cos\left(\tfrac{s}{R}t\right), R\sin\left(\tfrac{s}{R}t\right)\right)$$

- Suppose we have some curve $\vec{c}(t)$ and some vector field $F: \mathbb{R}^2 \to \mathbb{R}^2$. F maps $\vec{c}(t)$ onto a new curve:

$$\vec{d}(t) = F(\vec{c}(t))$$

  - it turns out that DF also maps $\vec{c}\,'(t)$ onto $\vec{d}\,'(t)$, i.e. DF maps tangent vectors to tangent vectors

<u>Ex</u>: Let $\vec{c}(t) = (t, 0)$, and define

$$F(x,y) = (x\cos x - y\sin x,\ x\sin x + y\cos x)$$

Then $\vec{d}(t) = F(\vec{c}(t)) = (t\cos t,\ t\sin t)$

- We can compute $\vec{d}\,'(t)$ in 2 ways:

① Directly:

$$\vec{d}\,'(t) = (\cos t - t\sin t,\ \sin t + t\cos t)$$

② Using DF:

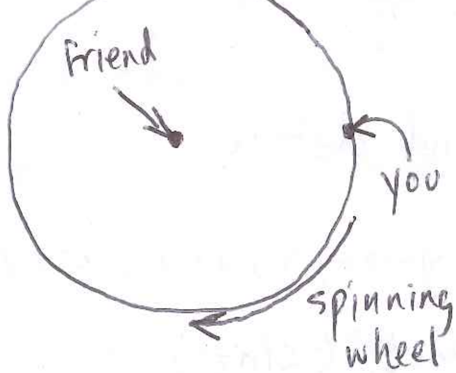
$$DF(x,y) = \begin{bmatrix} \cos x - x\sin x - y\cos x & -\sin x \\ \sin x + x\cos x - y\sin x & \cos x \end{bmatrix}$$

$$\Rightarrow DF(\vec{c}(t)) = \begin{bmatrix} \cos t - t\sin t & -\sin t \\ \sin t + t\cos t & \cos t \end{bmatrix}$$

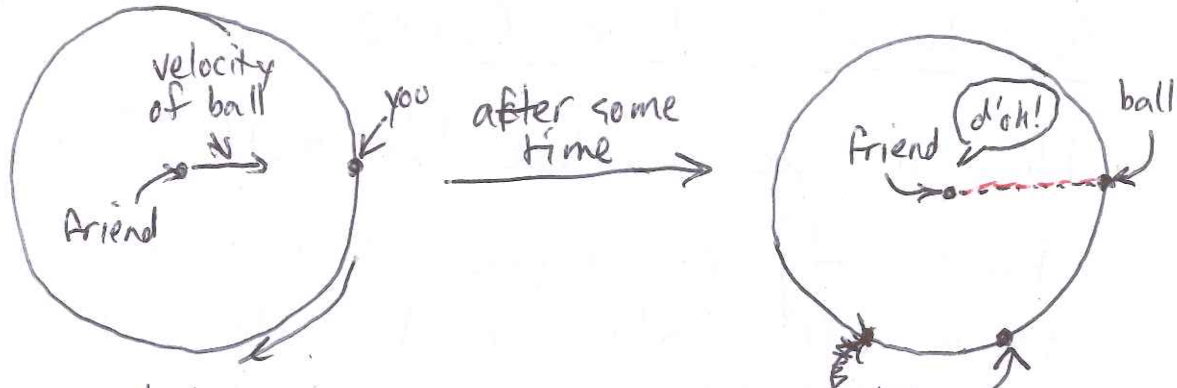
$$\Rightarrow \vec{d}\,'(t) = DF(\vec{c}(t)) \cdot \vec{c}\,'(t)$$

$$= (\cos t - t\sin t, \sin t + t\cos t)$$

- this particular example is an example of Coriolis acceleration: Imagine you and a friend are in a giant spinning wheel in space: your friend is floating at the exact center of the wheel, and you are at the edge.

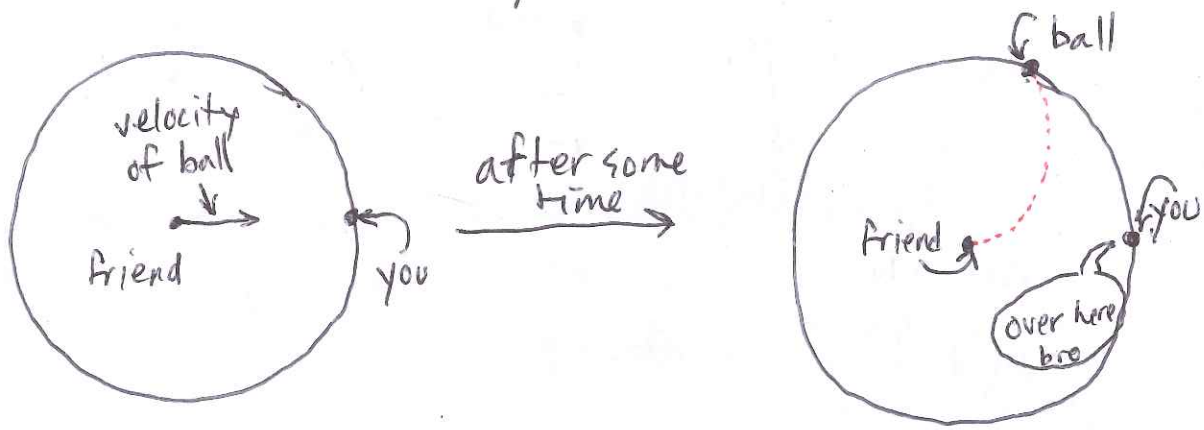


your friend tosses a ball to you at $t=0$. what happens? From the perspective of your friend, the ball travels in a straight line (like $\vec{c}(t)$), but misses you since you're on the wheel



at $t=0$

From <u>your</u> perspective, the ball takes a curved trajectory, like $\vec{d}(t)$:

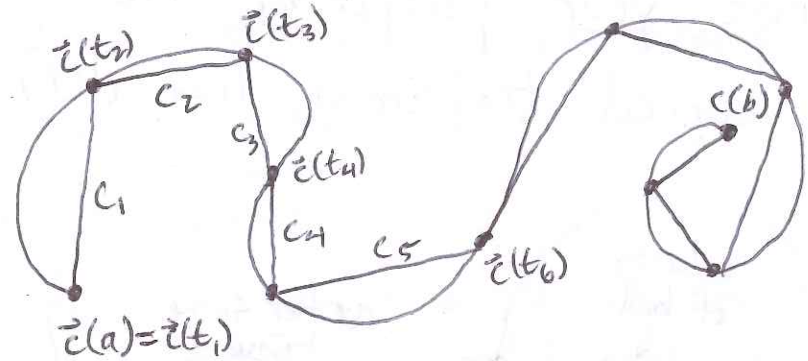


at $t=0$

The curvature of the ball's path is due to Coriolis acceleration. Here, F is acting as a kind of "change of perspective" transformation

3.3 Length of a Curve

- How do we calculate the length of a curve?
  - First, try an approximation: given a ~~curve~~ path $\vec{c}(t)$ (where $c(t)$ is defined on some interval $[a,b]$), pick some points along the path and connect them with line segments:



  - call the line segments c_1, c_2, etc, and the initial points $\vec{c}(t_i)$
  - The length of each line segment is

$$l_i = \|\vec{c}(t_{i+1}) - \vec{c}(t_i)\|$$

  - We can write l_i differently:

$$\vec{c}\,'(t_i) \approx \frac{\vec{c}(t_{i+1}) - \vec{c}(t_i)}{t_{i+1} - t_i}$$

$$\Rightarrow l_i = \|\vec{c}(t_{i+1}) - \vec{c}(t_i)\| \approx \|\vec{c}\,'(t_i)\|(t_{i+1} - t_i)$$

  - Letting $t_{i+1} - t_i = \Delta t_i$, we get that

$$\text{length of curve} \approx \sum_i l_i$$

$$\approx \sum_i \|\vec{c}(t_i)\| \, \Delta t_i$$

- this is a Riemann sum (remember those?), so if we take the number of segments to infinity, we get

$$\boxed{\text{length of curve} = \int_a^b \|\vec{c}\,'(t)\| \, dt}$$

~~Ex: Compute the length of the graph of~~

- We use the term <u>arclength</u> and length of a curve interchangeably, and we usually denote the arclength of a path $\vec{c}(t)$ by $l(\vec{c})$.

<u>Ex:</u> Compute the arclength of the graph of $y = \frac{x^2}{2}$ for $0 \leq x \leq 1$.

- First, we parametrize the curve:

$$\vec{c}(t) = \left(t, \frac{t^2}{2}\right), \quad t \in [0,1]$$

- Then:

$$\ell(\vec{c}) = \int_0^1 \|\vec{c}\,'(t)\| \, dt$$

$$= \int_0^1 \|(1,t)\| \, dt$$

$$= \int_0^1 \sqrt{1^2+t^2} \, dt$$

$$= \frac{1}{2} t\sqrt{1+t^2} + \frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right) \Big|_0^1$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2}) \approx 1.148$$

- The arclength function $s(t)$ for a path $\vec{c}(t)$ is given by

$$\boxed{s(t) = \int_a^t \|\vec{c}\,'(\tau)\| \, d\tau}$$

  - intuitively, $s(t)$ tells you how far along the path you've traveled in the time $a \leq \tau \leq t$.
  - one nice property of the arclength function:

$$\frac{d}{dt} s(t) = \frac{d}{dt} \int_a^t \|\vec{c}\,'(\tau)\| \, d\tau = \|\vec{c}\,'(t)\|$$