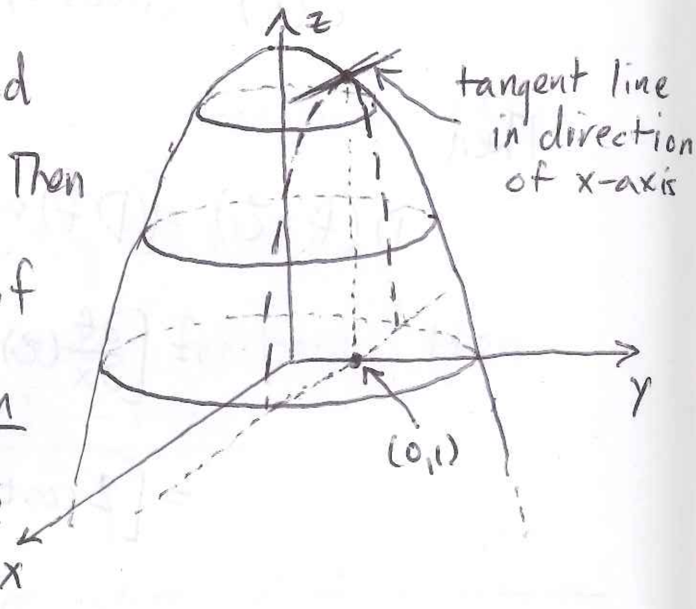
2.7 Gradient and Directional Derivative

- Recall the definition of the gradient: given a function $f: \mathbb{R}^3 \to \mathbb{R}$, the gradient of f is

$$\nabla f = \left[\frac{\partial f}{\partial x} \ \frac{\partial f}{\partial y} \ \frac{\partial f}{\partial z}\right]$$

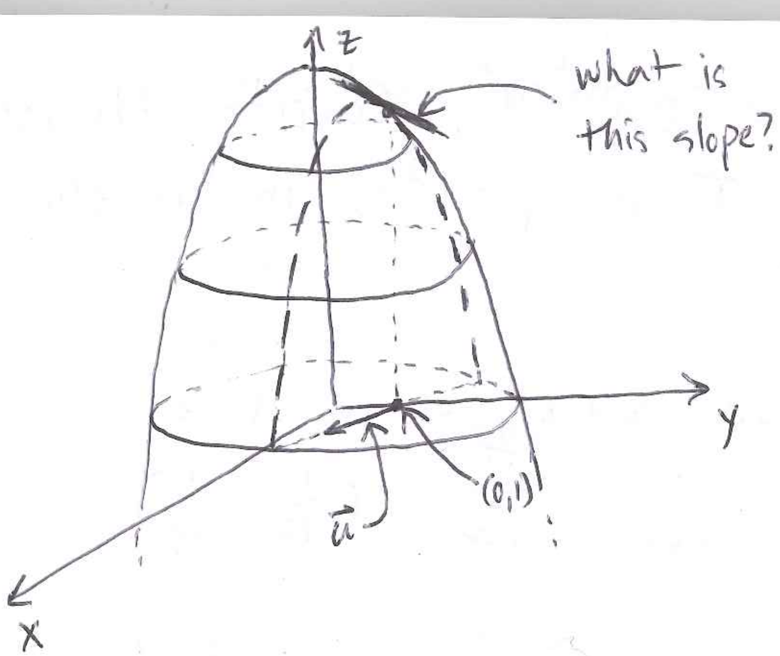
- The partial derivatives give the rate of change of f in the direction of that axis: consider

$f(x,y) = 9 - x^2 - y^2$ and the point $(x,y) = (0,1)$. Then $\frac{\partial f}{\partial x}(0,1)$ gives the rate of change of f at $(0,1)$ in the direction of the x-axis



$$\frac{\partial f}{\partial x} = -2x \Rightarrow \frac{\partial f}{\partial x}(0,1) = 0$$

- What if you wanted to know the rate of change in an arbitrary direction?



We can parametrize the line in the xy-plane in the direction of $\vec{u}$ by

$$(0,1) + t\vec{u}$$

Then the slope of f in the direction of $\vec{u}$ at (0,1) is $\frac{d}{dt} f((0,1) + t\vec{u})\Big|_{t=0}$

- one caveat: $\vec{u}$ must be a <u>unit vector</u>, i.e. $\|\vec{u}\| = 1$

- If $f : \mathbb{R}^m \to \mathbb{R}$ is a scalar function, then the <u>directional derivative</u> of f at the point $\vec{p}$ in the direction of the unit vector $\vec{u}$ is

$$D_{\vec{u}} f(\vec{p}) = \frac{d}{dt} f(\vec{p} + t\vec{u})\Big|_{t=0}$$

- Sometimes this is difficult to calculate, but it turns out that

$$\boxed{D_{\vec{u}} f(\vec{p}) = \nabla f(p) \cdot \vec{u}}$$

Ex: Let $T(x,y) = 30e^{-x^2-y^2}$. Compute the rate of change of T at $\vec{p} = (0,1)$ in the dir. $\vec{v} = (1,-1)$.

$\vec{v}$ is not a unit vector, so we take

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Then since

$$\nabla T = \left(-60xe^{-x^2-y^2}, -60ye^{-x^2-y^2}\right)$$

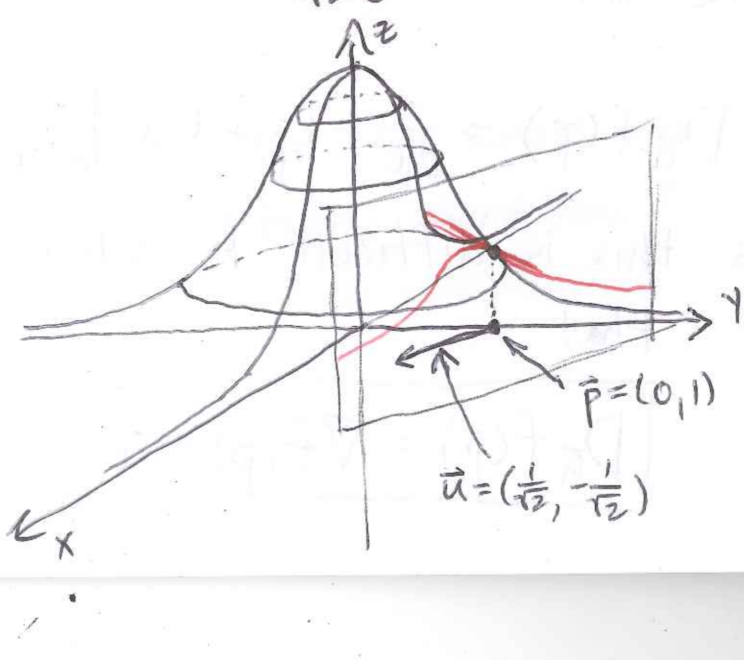
$$\Rightarrow \nabla T(\vec{p}) = \nabla T(0,1) = \left(0, -60e^{-1}\right)$$

we have that

$$D_{\vec{u}}T(\vec{p}) = \nabla T(0,1) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$= \left(0, -60e^{-1}\right) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$= \frac{60}{\sqrt{2}e}$$



- What direction produces the largest slope? Since any dot product is biggest when the two vectors point in the same direction, we see that

$$\nabla f(\vec{p}) \cdot \vec{u}$$

is largest when $\vec{u} = \dfrac{\nabla f(\vec{p})}{\|\nabla f(p)\|}$

 - the gradient points in the direction of largest increase
 - the gradient always points "uphill"

- Gradients and level curves are also related: Let $f(x,y) = e^{-x^2-y^2}$, and consider a level curve of f: $e^{-x^2-y^2} = k \Rightarrow x^2+y^2 = -\ln(k)$
 - We can parametrize this curve (it's a circle of radius $r = \sqrt{\ln(k^{-1})}$):

$$\vec{c}(t) = (r\cos t, r\sin t)$$

 - Because $\vec{c}$ parametrizes the level curve of value k, we know that

$$f(\vec{c}(t)) = k.$$

Then

$$D[f(\vec{c}(t))] = D[k]$$

$$\Rightarrow D[f(\vec{c}(t))] = 0$$

$$\Rightarrow (Df)(\vec{c}(t)) \cdot D\vec{c}(t) = 0$$

$$\Rightarrow \nabla f(\vec{c}(t)) \cdot \vec{c}\,'(t) = 0$$

- Since $\vec{c}\,'(t)$ is tangent to the level curve, we see that $\nabla f(\vec{c}(t))$ is perpendicular to the level curve.

- In our example:

$$\nabla f(x,y) = (-2xe^{-x^2-y^2}, -2ye^{-x^2-y^2})$$

$$\vec{c}\,'(t) = (-r\sin t, r\cos t)$$

$$\Rightarrow \nabla f(\vec{c}(t)) = (-2(r\cos t)e^{-r^2}, -2(r\sin t)e^{-r^2})$$

$$\Rightarrow \nabla f(\vec{c}(t)) \cdot \vec{c}\,'(t)$$

$$= 2r^2 \sin t \cos t e^{-r^2} - 2r^2 \sin t \cos t e^{-r^2}$$

$$= 0$$

- To sum up:
  - The directional derivative of f at $\vec{p}$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(\vec{p}) = \nabla f(\vec{p}) \cdot \vec{u}$$

(where $\vec{u}$ is a unit vector)

  - The direction of greatest <u>increase</u> of a scalar function f is given by ∇f
  - The direction of greatest <u>decrease</u> is given by $-\nabla f$
  - ∇f is always perpendicular to level sets

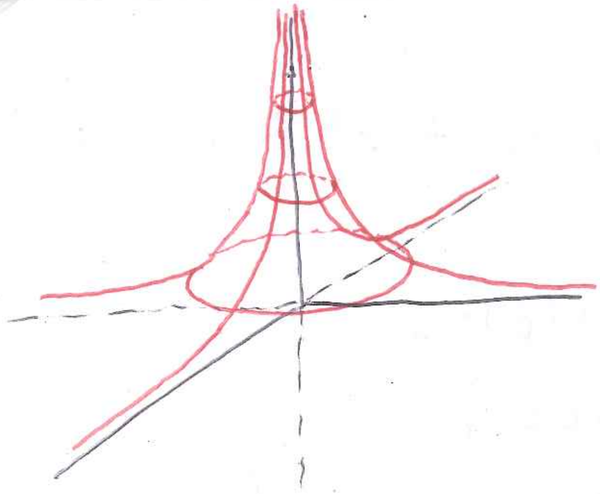
<u>Ex</u>: The electrostatic potential of a point charge at the origin is

$$\varphi(x,y) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{\sqrt{x^2+y^2}}$$

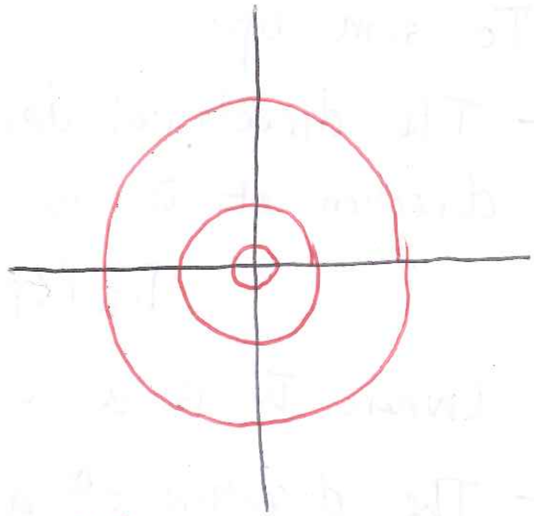
The electrostatic field $\vec{E}$ from this potential is given by

$$\vec{E}(x,y) = -\nabla \varphi(x,y)$$

$\vec{E}$ describes the force acting on a charged particle at (x,y) from the point charge at the origin

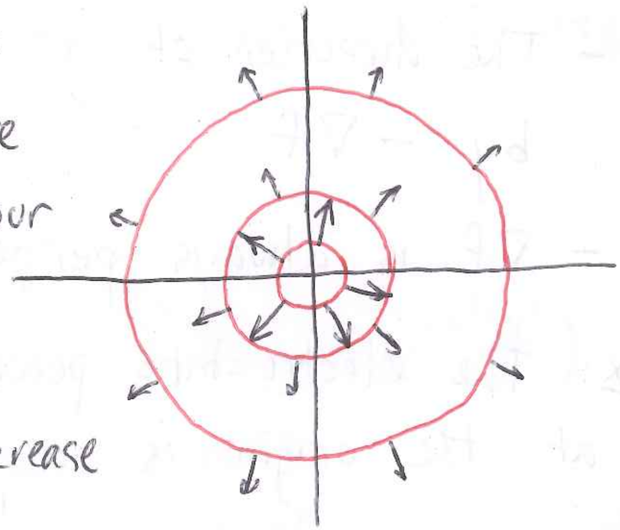


$\varphi(x,y)$ surface



$\varphi(x,y)$, contour curves

- Since $\vec{E} = -\nabla\varphi$, we know that the vectors of $\vec{E}$ are perpendicular to the contour curves of φ.
- Also, $-\nabla\varphi$ points in the direction of greatest decrease (i.e. "downhill")



The field E

- We can also calculate $\vec{E}$:

$$\vec{E}(x,y) = -\nabla\varphi(x,y)$$
$$= -\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{-2qx}{x^2+y^2}, \frac{1}{4\pi\varepsilon_0}\cdot\frac{-2qy}{x^2+y^2}\right)$$
$$= \frac{q}{2\pi\varepsilon_0}\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

- written in vector notation:

$$\vec{E}(\vec{x}) = \frac{q}{2\pi\varepsilon_0} \cdot \frac{\vec{x}}{\|\vec{x}\|}$$

- $\vec{E}$ is an example of a conservative vector field: in general, a vector field $\vec{F}$ is conservative if there is some scalar function V such that $\vec{F} = -\nabla V$
  - If $\vec{F}$ is conservative, the corresponding scalar fn V is called the potential function
  - This kind of function is pretty fundamental to all branches of physics

Ex: The field

$$\vec{F}(x,y,z) = (yz, zx-1, xy)$$

is conservative. Find its corresponding potential function.

- We want to find V such that $\nabla V = -\vec{F}$, i.e.

$$\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) = (-yz, -zx+1, -xy)$$

We know $\frac{\partial V}{\partial x} = -yz$, so

$$V = -yzx + C(y,z)$$

- Differentiating w.r.t. y:

$$\Rightarrow \frac{\partial V}{\partial y} = -xz + \frac{\partial C(y,z)}{\partial y} = \underbrace{-xz+1}_{F_2}$$

$$\Rightarrow \frac{\partial C(y,z)}{\partial y} = 1$$

$$\Rightarrow C(y,z) = y + D(z)$$

$$\Rightarrow V = -xyz + y + D(z)$$

- Differentiating w.r.t. z:

$$\Rightarrow \frac{\partial V}{\partial z} = -xy + \frac{\partial D(z)}{\partial z} = -xy$$

$$\Rightarrow \frac{\partial D(z)}{\partial z} = 0 \Rightarrow D(z) = \text{constant}$$

$$\Rightarrow \boxed{V = -xyz + y + \text{constant}}$$