Math 5A - Midterm - 7/16

Name: ______________________________ Perm: ______________

- Read all directions carefully.
- Show all your work. Problems without work shown will receive no credit.
- No calculators or notecards.
- Good luck!

Problem 1 (*10 points*): Frank the mad scientist is in his lab, working on his Doomsday Device$^{\text{TM}}$. It just so happens that the device relies on a precisely calibrated mass-spring system. Frank determines that the system requires a mass of 1 kilogram, a spring with spring constant 1 $\frac{\text{Newtons}}{\text{meter}}$, and a dashpot which produces a damping constant of 2 $\frac{\text{Newtons}\cdot\text{second}}{\text{meter}}$.

In order to start the device, the mass is stretched 0.5 meters beyond its resting length, and let go without imparting any initial velocity. There are no external forces acting on the mass. Help Frank complete the plans for his Doomsday Device$^{\text{TM}}$ by determining the position of the mass as a function of time.

Solution: This is an initial value problem:

$$\ddot{x} + 2\dot{x} + x = 0, \qquad x(0) = 0.5, \qquad \dot{x}(0) = 0.$$

The characteristic equation for this differential equation is

$$r^2 + 2r + 1 = 0 \qquad \Rightarrow \qquad r = -1.$$

Since there's only one real root, the general solution is

$$x(t) = c_1 e^{-t} + c_2 t e^{-t} \quad \Rightarrow \quad \dot{x}(t) = -c_1 e^{-t} + c_2(1-t)e^{-t}.$$

Substituting the initial conditions, we get

$$\begin{aligned} 0.5 &= x(0) = c_1, \\ 0 &= \dot{x}(0) = -c_1 + c_2, \end{aligned}$$

which has solution $c_1 = 0.5, c_2 = 0.5$. Thus, the motion of the mass is governed by the equation

$$\boxed{x(t) = \frac{1}{2}e^{-t} + \frac{1}{2}te^{-t}}.$$

Problem 2(a) (*5 points*): Frank's lab assistant, Joe, decided to "help" Frank by putting in a much more "efficient" spring into the Doomsday Device$^{\text{TM}}$. The new spring has a spring constant of $2\frac{\text{N}}{\text{m}}$. What is the general solution to the motion of the new and "improved" mass-spring system?

Solution: We must find the general solution to the DE

$$\ddot{x} + 2\dot{x} + 2x = 0.$$

The characteristic equation:

$$r^2 + 2r + 2 = 0 \quad \Rightarrow \quad r = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i.$$

Since the roots are complex-valued, the general solution is

$$\boxed{x(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)}.$$

Problem 2(b) (*5 points*): After the device malfunctioned (and after he fired Joe), Frank discovers that the spring had fused with the device, making it unable to be replaced. Frank brilliantly determines that the problem can be fixed by applying a periodic magnetic field to the mass. The resulting external forcing function is $f(t) = \cos(2t)$. What is the general solution to the motion of the corrected mass-spring system?

Solution: We must find the general solution to the DE

$$\ddot{x} + 2\dot{x} + 2x = \cos(2t).$$

We already know the homogeneous solution from part (a), so we must find the particular solution. Using the method of undetermined coefficients, we guess

$$\begin{aligned} x_p &= A\cos(2t) + B\sin(2t), \\ \dot{x}_p &= -2A\sin(2t) + 2B\cos(2t), \\ \ddot{x}_p &= -4A\cos(2t) - 4B\sin(2t). \end{aligned}$$

Substituting into the DE gives

$$\begin{aligned} (-2A + 4B)\cos(2t) + (-4A - 2B)\sin(2t) = \cos(2t) \\ \Rightarrow -2A + 4B = 1, \quad -4A - 2B = 0, \end{aligned}$$

which has solution $A = -\frac{1}{10}$, $B = \frac{2}{10}$. Then the general solution is

$$\boxed{x(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) - \frac{1}{10}\cos(2t) + \frac{1}{5}\sin(2t)}.$$

Problem 3 (*10 points*): While Frank is merrily wreaking havoc in the local town using his Doomsday Device$^{\text{TM}}$, he notices that the mass-spring system settles into a periodic motion after a long time. Frank determines that this is from the steady-state solution to the motion of the mass-spring system. What is the steady state solution? Express your answer in phase-amplitude form.

Solution: Since the homogeneous part of the general solution has e^{-t} terms, the steady-state solution simplifies to the particular solution. In phase-amplitude form, we get that the amplitude is

$$A = \sqrt{\left(-\frac{1}{10}\right)^2 + \left(\frac{2}{10}\right)^2} = \frac{\sqrt{5}}{10},$$

and the phase angle is

$$\delta = \tan^{-1}\left(\frac{\frac{2}{10}}{-\frac{1}{10}}\right) = \tan^{-1}(-2).$$

Note: Technically, the phase angle should be $\delta = \tan^{-1}(-2) + \pi$ *in order for it to be in the correct quadrant, but I accepted both answers.* Thus, the steady-state solution, in phase-amplitude form, is

$$\boxed{x_{ss}(t) = \frac{\sqrt{5}}{10}\cos\left(2t - \tan^{-1}(-2)\right)}.$$

Problem 4 (*10 points*): Before Frank could do very much damage, his arch-nemesis Resonance Man determined the resonant frequency of the Doomsday Device$^{\text{TM}}$ and destroyed it. Frank was thrown into a high-security jail, but as he was thrown into his cell, he noticed the guard using a keypad next to the door. It looks like this:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 3 \\ 4 & 3 & 5 \end{bmatrix}.$$

Frank determines that if he can calculate the image of the linear transformation $T(\mathbf{v}) = A\mathbf{v}$, he can short-circuit the locking mechanism. Help Frank escape prison by calculating the basis for $\mathrm{Im}(T)$.

Solution: The image of T, by definition, is

$$\mathrm{Im}(t) = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right\}.$$

In order to obtain a basis, we need to determine which vectors are linearly independent. Row reducing A,

$$\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 3 \\ 4 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

we see that the third column vector (since it doesn't correspond to a pivot column) is a linear combination of the other two. Thus, a basis for $\mathrm{Im}(T)$ is the first two columns of A:

$$\boxed{B = \left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}}.$$

Problem 5(a) (*8 points*): Frank made a mistake in his calculations, and realizes that he actually needs to determine the *kernel* of T. Help Frank escape prison once and for all by calculating the basis for $\text{Ker}(T)$.

Solution: In order to find the kernel, we row reduce A (which luckily we already did):

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

. Then v_3 is a free variable, so setting $v_3 = r$, we get the solution

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}r \\ -r \\ r \end{bmatrix} = r \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}.$$

Setting $r = 2$ gives the single vector in the basis for $\text{Ker}(T)$:

$$\boxed{B = \left\{ \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}}.$$

Problem 5(b) (*2 points*): What is the nullity of T? What is the rank of T?

Solution: Since there are two vectors in the basis for $\text{Im}(T)$,

$$\boxed{\text{rank}(t) = 2},$$

and since there is only one vector in the basis for $\text{Ker}(T)$,

$$\boxed{\text{nullity}(T) = 1}.$$