Math 5B - HW4 (Written Portion)

Solutions

4.5.2 Show that the curve $\mathbf{c}(t) = (\frac{3}{5}\cos t + \frac{4}{5}\sin t, -\frac{3}{5}\sin t + \frac{4}{5}\cos t)$ is a flow line of the vector field $\mathbf{F}(x,y) = (\frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}})$ going through the point $(\frac{4}{5}, -\frac{3}{5})$.

We show that $\mathbf{c}(t)$ satisfies the equation $F(\mathbf{c}(t)) = \mathbf{c}'(t)$:

$$\begin{aligned}
\mathbf{F}(\mathbf{c}(t)) &= \mathbf{F}\left(\frac{3}{5}\cos t + \frac{4}{5}\sin t, -\frac{3}{5}\sin t + \frac{4}{5}\cos t\right) \\
&= \left(\frac{-\frac{3}{5}\sin t + \frac{4}{5}\cos t}{\sqrt{(\frac{3}{5}\cos t + \frac{4}{5}\sin t)^2 + (-\frac{3}{5}\sin t + \frac{4}{5}\cos t)^2}},\right. \\
&\qquad \left. -\frac{\frac{3}{5}\cos t + \frac{4}{5}\sin t}{\sqrt{(\frac{3}{5}\cos t + \frac{4}{5}\sin t)^2 + (-\frac{3}{5}\sin t + \frac{4}{5}\cos t)^2}}\right) \\
&= \left(-\frac{3}{5}\sin t + \frac{4}{5}\cos t, -\frac{3}{5}\cos t - \frac{4}{5}\sin t\right),
\end{aligned}$$

$$\mathbf{c}'(t) = \left(-\frac{3}{5}\sin t + \frac{4}{5}\cos t, -\frac{3}{5}\cos t - \frac{4}{5}\sin t\right),$$

so $\mathbf{c}(t)$ is a flow line for $\mathbf{F}$. Finally, since $\mathbf{c}(\frac{\pi}{2}) = (\frac{4}{5}, -\frac{3}{5})$, we also see that $\mathbf{c}$ goes through the correct point.

4.5.5 Find the flow line of the constant vector field $\mathbf{F}(x,y) = (a,b)$ (a and b are real numbers with $a \neq 0$ and/or $b \neq 0$) that goes through the origin.

We want to find a curve $\mathbf{c}(t)$ such that $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$. Denoting $\mathbf{c}(t) = (x(t), y(t))$, we find that

$$\begin{aligned}
x'(t) &= a & y'(t) &= b \\
\Rightarrow x(t) &= at + C_1 & \Rightarrow y(t) &= bt + C_2
\end{aligned}$$

In order for $\mathbf{c}(t)$ to pass through the origin, $C_1 = 0$ and $C_2 = 0$, so our flowline is $\mathbf{c}(t) = (at, bt)$.

4.5.12 Show that the curve $\mathbf{c}(t) = (e^t, 2\ln t, t^{-1}),\ t > 0$ is a flow line of the vector field $\mathbf{F}(x,y,z) = (x, 2z, -z^2)$.

We show that $\mathbf{c}(t)$ satisfies the equation $F(\mathbf{c}(t)) = \mathbf{c}'(t)$:

$$\begin{aligned}\mathbf{F}(\mathbf{c}(t)) &= \mathbf{F}(e^t, 2\ln t, t^{-1}) \\ &= \left(e^t, 2t^{-1}, -t^{-2}\right)\end{aligned}$$

$$\mathbf{c}'(t) = \left(e^t, 2t^{-1}, -t^{-2}\right),$$

so $\mathbf{c}(t)$ is a flow line for $\mathbf{F}$.

4.6.13 Find the curl and divergence of the vector field $\mathbf{F}(x, y, z) = (y^2 z, -xz, xyz)$.

The curl:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial y}xyz - \frac{\partial}{\partial z}(-xz), \frac{\partial}{\partial z}y^2 z - \frac{\partial}{\partial x}xyz, \frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial y}y^2 z\right) \\ &= \left(xz + x, y^2 - yz, -z - 2yz\right),\end{aligned}$$

and the divergence:

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}y^2 z + \frac{\partial}{\partial y}(-xz) + \frac{\partial}{\partial z}xyz \\ &= 0 + 0 + xy = xy.\end{aligned}$$

4.6.14 Find the curl and divergence of the vector field $\mathbf{F}(x, y, z) = (0, 0, \ln z + xy)$.

The curl:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial y}(\ln z + xy) - \frac{\partial}{\partial z}0, \frac{\partial}{\partial z}0 - \frac{\partial}{\partial x}(\ln z + xy), \frac{\partial}{\partial x}0 - \frac{\partial}{\partial y}0\right) \\ &= (x + 0, 0 - y, 0) \\ &= (x, -y, 0),\end{aligned}$$

and the divergence:

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}0 + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}(\ln z + xy) \\ &= 0 + 0 + \frac{1}{z} = \frac{1}{z}.\end{aligned}$$

4.6.25 Consider $\mathbf{F}(x, y, z) = (-y, -x, -3)$. Is $\mathbf{F}$ a conservative vector field? If so, find a real-valued function V such that $\mathbf{F} = -\nabla V$.

$\mathbf{F}$ is defined on all of $\mathbb{R}^3$, which is a simply connected set, and since

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial y}(-3) - \frac{\partial}{\partial z}(-x), \frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(-3), \frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(-y)\right) \\ &= (0 - 0, 0 - 0, -1 + 1) = (0, 0, 0),\end{aligned}$$

we see that $\mathbf{F}$ is a conservative vector field. Now we find V such that $\mathbf{F} = -\nabla V$:

$$\frac{\partial V}{\partial x} = y \qquad\qquad \frac{\partial V}{\partial y} = x \qquad\qquad \frac{\partial V}{\partial x} = -3$$

$$\begin{aligned}
&\Rightarrow \quad V = xy + C(y,z) \\
&\Rightarrow \quad \frac{\partial V}{\partial y} = x + \frac{\partial C}{\partial y}(y,z) \quad \Rightarrow \quad x + \frac{\partial C}{\partial y}(y,z) = x \\
&\qquad\qquad\qquad\qquad\qquad\quad\; \Rightarrow \quad \frac{\partial C}{\partial y}(y,z) = 0 \\
&\qquad\qquad\qquad\qquad\qquad\quad\; \Rightarrow \quad C(y,z) = C(z) \\
&\qquad\qquad\qquad\qquad\qquad\quad\; \Rightarrow \quad V = xy + C(z) \\
&\qquad\qquad\qquad\qquad\qquad\quad\; \Rightarrow \quad \frac{\partial V}{\partial z} = \frac{\partial C}{\partial z}(z) \quad \Rightarrow \quad \frac{\partial C}{\partial z}(z) = -3 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \Rightarrow \quad C(z) = -3z + C \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \Rightarrow \quad V = xy - 3z + C.
\end{aligned}$$